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EIGEN DECOMPOSITION OF REED MULLER TRANSFORM USING KRONECKER METHOD

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Abstract

Spectral methods have been applied to many areas of digital system design. Reed-Muller Transform (RMT) is a spectral transform which is self inverse in nature. In this paper, eigen-decomposition of Reed-Muller Transform using Kronecker Product method is introduced. The properties of eigenvectors and eigenvalues of RMT are also illustrated.

Keywords–eigenvalue, eigenvector, Kronecker Product, Reed Muller Transform.

I. INTRODUCTION

Many spectral transforms have been extensively studied in literature such as Walsh, Reed-Muller, Arithmetic, Haar transforms. The Reed-Muller Transform (RMT) has been motivated by the work of Reed and Muller [1], [2]. Reed-Muller is an important spectral transform that has found its application in areas like fault detection, testing [3] and image processing [4]. This led to considerable interest in the AND-XOR expansion of Boolean functions [5], [6]. Since Reed-Muller transform is so widely used, it will be useful to find algorithms to perform its eigen-decomposition. Eigenfunctions of Reed-Muller Transform were introduced by Sasao and Butler in [7]. The work identified three symmetric functions with certain special properties and showed that these symmetric functions can be found among the eigenfunctions of the Reed-Muller transform. The eigenvectors of the Reed-Muller Transform were evaluated using canonical sum of products. Tseng [8] proposed eigen decomposition of Hadamard transform using kronecker product method. Other orthogonal transforms can also be decomposed using similar technique. In this paper, we propose a method to find the eigenvectors and eigenvalues of Reed-Muller transform using the Kronecker Product method. Also, the properties of kronecker product, eigenvalues and eigenvectors are discussed and illustrated for RMT.

The paper is organized as: In Section-II, Kronecker product and its properties are discussed. Section-III describes the Reed-Muller expansion and its properties. Reed-Muller transform is also defined in same section. The eigen decomposition of RMT using Kronecker product method is described in section-IV. An example is illustrated for RMT with three variables.

II. KRONECKER PRODUCT

Kronecker product representations lead to efficient implementations of numerous discrete orthogonal transforms. The kronecker products can be defined in terms of matrix factorizations [9].

If **A** is an $m \times m$ matrix and **B** is a $n \times n$ matrix, then the Kronecker product (or Tensor product) of **A** and **B** is $\boldsymbol{A} \otimes \boldsymbol{B}$which is an $mn \times mn$ matrix given by

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1m}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & \cdots & a_{mm}\boldsymbol{B} \end{bmatrix} \qquad (1)$$

here a_{ij} are the elements of matrix $\boldsymbol{A}$ of row i and column j.

Kronecker product satisfies the following properties:

Property 1: For all $\boldsymbol{T} \in R^{n\times n} and\ \boldsymbol{S} \in R^{m\times m}$, $(\boldsymbol{T} \otimes \boldsymbol{S})^T = T^T \otimes S^T$, here $(.)^T$ represents transpose of the matrix.

Property 2: If **T** and **S** are real symmetric matrices of size $n \times n$ and $m \times m$ then $\boldsymbol{T} \otimes \boldsymbol{S}$ is also a symmetric matrix of size $nm \times nm$

Property 3: If **T** and **S** are real orthogonal matrices then $\boldsymbol{T} \otimes \boldsymbol{S}$ will also be an orthogonal matrix.

Property 4: Let **T** be a matrix of size $n \times n$ with eigenvectors $(t_1, \dots, t_n)$ and eigenvalues $(\lambda_1, \dots, \lambda_n)$. Another matrix **S** of size $m \times m$ has eigenvectors $(s_1, \dots, s_m)$ and eigenvalues $(\mu_1, \dots, \mu_m)$. Then $\lambda_i \mu_j$ and $\boldsymbol{t}_i \otimes \boldsymbol{s}_j$ will be eigenvalues and eigenvectors of matrix $\boldsymbol{T} \otimes \boldsymbol{S}$ respectively, where $i = 1,2, \dots, n\ and\ j = 1,2, \dots, m$.

III. Reed-Muller Transform

A. Reed-Muller expansion

An expression which can represent any boolean function as a modulo-2 sum of products, is known as the compliment-free ring-sum or Reed-Muller expression (RME). Reed-Muller expansions are advantageous for many important boolean functions in terms of area, speed and testability point of view [10]. A function f(.) with *n* variables can be defined by 2^n terms in a sum of products form as in equation

$$f(x_0, x_1, \dots, x_{n-1}) = d_0\, \tilde{x}_{n-1}\tilde{x}_{n-2} \dots \tilde{x}_0 + d_1\, \tilde{x}_{n-1}\tilde{x}_{n-2} \dots \tilde{x}_1 x_0 + \dots + d_{2^n-1}\, x_{n-1}\, x_{n-2} \dots x_0 \quad (2)$$

here $(d_0, d_1, \dots, d_{2^n-1})$ represents the values in the output column of the truth table of the function. These coefficients can be represented in vector form **D**, called truth vector. Using the positive davio expansion [10], as given in equation

$$f(x_0, x_1, \dots, x_{n-1}) = f_0 \oplus x_0 f_2 \quad (3)$$

Here $\oplus$ denotes modulo-2 addition. $f_0 = f(0, x_1, \dots, x_{n-1})$, $f_1 = f(1, x_1, \dots, x_{n-1})$ and $f_2 = f_0 \oplus f_1$. The Reed-Muller expansion can be written as

$$f(x_0, x_1, \dots, x_{n-1}) = a_0 \oplus a_1 x_0 \oplus \dots \oplus a_{1,2,\dots,n}\, x_0 x_1 \dots x_{n-1} \quad (4)$$

This is positive polarity Reed-Muller expansion (PPRM). The negative davio expansion, is given as

$$f(x_0, x_1, \dots, x_{n-1}) = \tilde{x}_0 f_2 \oplus f_1 \quad (5)$$

Using (5), the Reed Muller expansion for the function can be obtained

$$f(x_0, x_1, \dots, x_{n-1}) = a_0 \oplus a_1 \tilde{x}_0 \oplus \dots \oplus a_{1,2,\dots,n}\, \tilde{x}_0 \tilde{x}_1 \dots \tilde{x}_{n-1} \quad (6)$$

This is negative polarity Reed-Muller Expansion (NPRM). From (4) we observe that in PPRM all the variables are in positive form. The expression can be converted to NPRM by substituting variable x_i with its negation $\tilde{x}_i$ as shown in (6). In this way the canonical form of the Reed-

Muller expansion is retained. If each variable is restricted to retain the same polarity in all the terms of the expansion, that is, either positive or negative but not both, the canonical form is called the Fixed Polarity Reed Muller (FPRM) form. For a function with *n* variables, the number of possible arrangements of polarities is 2^n.

B. Reed-Muller Transform

A Reed-Muller Transformation matrix of *n* variables **R**(n) can be recursively written as

$$\mathbf{R}(n) = \begin{bmatrix} \boldsymbol{R}(n-1) & 0 \\ \boldsymbol{R}(n-1) & \boldsymbol{R}(n-1) \end{bmatrix} \; for\ n \geq 1 \tag{7}$$

and $R(0) = 1$. Thus, the Reed-Muller transformation matrix of one variable, **R**(1) is

$$\mathbf{R}(1) = \begin{bmatrix} \mathbf{1} & 0 \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \tag{8}$$

In this case the matrix multiplication is over the field GF(2). The Reed-Muller transform matrix can be recursively defined using kronecker product [11], [12] as

$$\begin{aligned} R(n) &= R(1) \otimes R(1) \otimes \ldots \otimes R(1) \\ &= \otimes^n R(1) \end{aligned} \tag{9}$$

The eigen decomposition of the matrix **R**(1) is given as

$$R(1) = V(1)D(1)V(1)^{-1} \tag{10}$$

where $V(1) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ is the eigenvector matrix for **R**(1). The columns of matrix represent eigenvectors $\boldsymbol{v}_{10}$ and $\boldsymbol{v}_{11}$. Similarly matrix $D(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is eigenvalue matrix for **R**(1). The columns of eigenvalue matrix are named as λ_{10} and λ_{11}.

IV. EIGEN VECTOR DECOMPOSITION OF REED-MULLER TRANSFORM (RMT)

We propose a method to compute the eigenvectors and eigenvalues of the RMT matrix, **R**(*n*), by using the Kronecker product method. The eigenvector **x** for any matrix **M** is calculated using the characteristic equation $Mx = \lambda x$, where λ is the eigenvalue for matrix **M**. The eigenvectors and eigenvalues of RMT follow the definition given below, as mentioned in [7].

Definition 4.1: Let **v** be a binary vector of 2^n elements and **R**(*n*) be the Reed-Muller Transformation matrix of n variables. Then, a vector **v** satisfying $\boldsymbol{R}(n)\boldsymbol{v} = \boldsymbol{v}$ is an eigenvector of Reed-Muller transform. Eigenvalue λ exhibits only 1 as its nonzero value for RMT.

The RMT matrix **R**(n) can be decomposed using the kronecker product as given by (9). The matrix **R**(*n*) has 2^n eigenvectors which can be computed using the kronecker product method. Let an index $p = 0,1,\ldots,2^n - 1$ whose binary representation is given as $d_{n-1}d_{n-2}\ldots d_1d_0$ where d_i are the binary bits 0 and 1. We can show that Reed-Muller Transform exhibits following properties:

- Property 1: The eigenvectors of RMT matrix can be obtained using the equation
  $$\boldsymbol{v}_{n,p} = v_{1d_{n-1}} \otimes v_{1d_{n-2}} \otimes \ldots \otimes v_{1d_1} \otimes v_{1d_0} \qquad (11)$$
  using the above expansion it can be shown that the eigenvectors of RMT are always a combination of 0 and 1 or 0 and -1.

- Property 2: The matrix **R**(n) has 2^n eigenvalues whose column matrix can be obtained as
  $$\boldsymbol{\lambda}_{n,i} = \lambda_{1d_{n-1}} \otimes \lambda_{1d_{n-2}} \otimes \ldots \otimes \lambda_{1d_1} \otimes \lambda_{1d_0} \qquad (12)$$
  for $0 \le i \le 2^n - 1$. Using (12) it can be shown that the eigenvalues of RMT matrix are always 1.

- Property 3: The norm of eigenvector $\boldsymbol{v}_{n,p}$ is given as
  $$\|\boldsymbol{v}_{n,p}\| = 1 \qquad for\ p = 0,1,\ldots,2^n - 1 \qquad (13)$$
  Thus the eigenvectors for RMT matrix are orthonormal.

As an example, let us consider the Reed-Muller transform matrix **R**(3) for *n*=3. The desired transform can be obtained from kronecker product of **R**(1) i.e.$\boldsymbol{R}(3) = \boldsymbol{R}(1) \otimes \boldsymbol{R}(1) \otimes \boldsymbol{R}(1)$. By doing this we get

$$\mathbf{R}(3) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Table-I shows the eigenvectors of R(3), computed using (11). These eigenvectors satisfy equation $\boldsymbol{R}(n)\boldsymbol{v} = \boldsymbol{v}$, as given by definition 4.1. Also the eigenvalues ¸ are evaluated using (12). The values of λ are found to be 1 as stated in property 2 (section IV)

V. CONCLUSION

In this paper, the eigenvector decomposition of Reed-Muller transform is investigated. A kronecker product method for finding eigenvalues and eigenvectors of Reed-Muller transform is described. Also, the properties of kronecker product, eigenvalues and eigenvectors are discussed for the given transform.

v	*Kronecker Product*	*Eigenvectors of **R**(3)*
$\boldsymbol{v}_{30}$	$\boldsymbol{v}_{10} \otimes \boldsymbol{v}_{10} \otimes \boldsymbol{v}_{10}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$
$\boldsymbol{v}_{31}$	$\boldsymbol{v}_{10} \otimes \boldsymbol{v}_{10} \otimes \boldsymbol{v}_{11}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1]$
$\boldsymbol{v}_{32}$	$\boldsymbol{v}_{10} \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{10}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1]$
$\boldsymbol{v}_{33}$	$\boldsymbol{v}_{10} \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{11}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$
$\boldsymbol{v}_{34}$	$\boldsymbol{v}_{11} \otimes \boldsymbol{v}_{10} \otimes \boldsymbol{v}_{10}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1]$
$\boldsymbol{v}_{35}$	$\boldsymbol{v}_{11} \otimes \boldsymbol{v}_{10} \otimes \boldsymbol{v}_{11}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$
$\boldsymbol{v}_{36}$	$\boldsymbol{v}_{11} \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{10}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]$
$\boldsymbol{v}_{37}$	$\boldsymbol{v}_{11} \otimes \boldsymbol{v}_{11} \otimes \boldsymbol{v}_{11}$	$[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1]$

TABLE 1 KRONECKER PRODUCT OF EIGENVECTOR v OF R(3) WITH SIZE 8x8

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