

The M-Homomorphism and M-Anti Homomorphism of an M-Fuzzy Subgroup and its Level M-Subgroups

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Abstract

In this paper, we introduce the concept of an M-fuzzy subgroup of an M-group and discussed some of its properties.

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Keywords

M-group, fuzzy set, fuzzy subgroup, M-fuzzy subgroup of an M-group, level subset, level M-subgroups, M-homomorphism, M-anti homomorphism.

Introduction

The concept of fuzzy sets was initiated by Zadeh. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld gave the idea of fuzzy subgroups. Author N. Jacobson introduced the concept of M-group, M-subgroup.

1. Preliminaries

This section contains some definitions and results to be used in the sequel.

1.1 Definition

Let S be a set. A fuzzy subset A of S is a function $A: S \rightarrow [0,1]$.

1.2 Definition

Let G be a group. A fuzzy subset A of G is called a fuzzy subgroup if for $x, y \in G$,

- (i) $A(xy) \geq \min \{ A(x), A(y) \}$,
- (ii) $A(x^{-1}) = A(x)$.

1.3 Definition

A group with operators is an algebraic system consisting of a group G , a set M and a function defined in the product set $M \times G$ and having values in G such that, if ma denotes the element in G determined by the element a of G and the element m of M , then $m(ab) = (ma)(mb)$ holds for all $a, b \in G$ and $m \in M$. We

shall use the phrases “ G is an M-group” to a group with operators.

A subgroup H of an M-group G is said to be an M-subgroup if $mx \in H$ for all $m \in M$ and $x \in H$.

1.4 Definition

Let G be an M-group and A be a fuzzy subgroup of G . Then A is called an M-fuzzy subgroup of G if for all $x \in G$ and $m \in M$, then $A(mx) \geq A(x)$.

1.5 Definition

Let A be a fuzzy subset of S . For $t \in [0, 1]$, the level subset of A is the set, $A_t = \{ x \in S : A(x) \geq t \}$.

1.6 Definition

Let G be a finite group of order n and A be a fuzzy subgroup of G . Let $\text{Im}(A) = \{ t_i : A(x) = t_i \text{ for some } x \in G \}$. Then $\{ A_{t_i} \}$ are the only level subgroups of A .

1.1 Example

Let A be a fuzzy subset of an M-group G , then A is defined by

$$A(x) = \begin{cases} 0.7 & \text{if } x \in G \\ 0.2 & \text{otherwise.} \end{cases}$$

Then it is easy to verify that A is an M-fuzzy subgroup of G .

1.7 Definition

Let G and G' be any two M-groups. Then the function $f: G \rightarrow G'$ is said to be an M-homomorphism if

- (i) $f(xy) = f(x)f(y)$ for all x, y in G .
- (ii) $f(mx) = mf(x)$ for all m in M and x in G .

1.8 Definition

Let G and G' be any two M -groups (not necessarily commutative). Then the function $f: G \rightarrow G'$ is said to be an M -anti homomorphism if

- (i) $f(xy) = f(y)f(x)$ for all $x, y \in G$.
- (ii) $f(mx) = mf(x)$ for all m in M and x in G .

2. M-fuzzy subgroups of an M-group G under M-homomorphism and M-anti homomorphism

2.1 Theorem

Let f be a M -homomorphism from an M -group G onto an M -group G' . If A is an M -fuzzy subgroup of G and A is f -invariant, then $f(A)$, the image of A under f , is an M -fuzzy subgroup of G' .

Proof

Let $\alpha \in \text{Image } f(A)$.

Then for some $y \in G'$, $(f(A))(y) = \sup_{x \in f^{-1}(y)} A(x) = \alpha$,
where $\alpha \leq A(e)$.

Clearly A_α is an M -subgroup of G .

If $\alpha = 1$, then $(f(A))_\alpha = G'$.

If $0 < \alpha < 1$, then $(f(A))_\alpha = f(A_\alpha)$, because ,

$$\begin{aligned} z \in (f(A))_\alpha &\Leftrightarrow (f(A))(z) \geq \alpha \\ &\Leftrightarrow \sup_{x \in f^{-1}(z)} A(x) \geq \alpha \text{ (since } 0 < \alpha < 1) \\ &\Leftrightarrow \text{there exists } x \text{ in } G \text{ such that } f(x) = z \\ &\quad \text{and } A(x) \geq \alpha \\ &\Leftrightarrow z \in (f(A_\alpha)). \end{aligned}$$

Hence, $(f(A))_\alpha = (f(A_\alpha))$.

Since f is an M -homomorphism, $(f(A_\alpha))$ is an M -subgroup of G' .

Hence $(f(A))_\alpha$ is an M -subgroup of G' .

Hence $f(A)$ is an M -fuzzy subgroup of G' .

2.2 Theorem

The M -homomorphic pre-image of an M -fuzzy subgroup of an M -group G' is an M -fuzzy subgroup of an M -group G .

Proof

Let $f: G \rightarrow G'$ be an M -homomorphism. Let the fuzzy set V on G' be an M -fuzzy subgroup.

We have to prove that any fuzzy set A on G is an M -fuzzy subgroup, where $V = f(A)$.

$$\begin{aligned} \text{Now, } A(xy) &= V(f(xy)) \\ &= V(f(x)f(y)) \text{ as } f \text{ is an } M\text{-homomorphism.} \\ &\geq \min \{ V(f(x)), V(f(y)) \} \\ &\quad \text{as } V \text{ is an } M\text{-fuzzy subgroup of } G'. \\ &= \min \{ A(x), A(y) \}. \end{aligned}$$

That is, $A(xy) \geq \min \{ A(x), A(y) \}$.

For $x \in G$,

$$\begin{aligned} A(x^{-1}) &= V(f(x^{-1})) \\ &= V((f(x))^{-1}) \text{ as } f \text{ is an } M\text{-homomorphism} \\ &= V(f(x)) \text{ as } V \text{ is an } M\text{-fuzzy subgroup of } G' \\ &= A(x). \end{aligned}$$

That is, $A(x^{-1}) = A(x)$.

$$\begin{aligned} \text{Clearly, } A(mx) &= V(f(mx)) \\ &= V(mf(x)), \text{ as } f \text{ is an } M\text{-homomorphism} \\ &\geq V(f(x)) \text{ as } V \text{ is an } M\text{-fuzzy subgroup of } G' \\ &= A(x). \end{aligned}$$

That is, $A(mx) \geq A(x)$.

Hence A is an M -fuzzy subgroup of G .

2.3 Theorem

Let f be an M -anti homomorphism from an M -group G onto an M -group G' . If A is an M -fuzzy subgroup of G and A is f -invariant, then $f(A)$, the image of A under f , is an M -fuzzy subgroup of G' .

Proof

Let $\alpha \in \text{Image } f(A)$.

Then for some $y \in G'$, $(f(A))(y) = \sup_{x \in f^{-1}(y)} A(x) = \alpha$,
where $\alpha \leq A(e)$.

Clearly A_α is an M -subgroup of G .

If $\alpha = 1$, then $(f(A))_\alpha = G'$.

If $0 < \alpha < 1$, then $(f(A))_\alpha = f(A_\alpha)$, because ,

$$\begin{aligned} z \in (f(A))_\alpha &\Leftrightarrow (f(A))(z) \geq \alpha \\ &\Leftrightarrow \sup_{x \in f^{-1}(z)} A(x) \geq \alpha \text{ (since } 0 < \alpha < 1) \\ &\Leftrightarrow \text{there exists } x \text{ in } G \text{ such that } f(x) = z \end{aligned}$$

and $A(x) \geq \alpha$.

$$\Leftrightarrow z \in (f(A_\alpha)).$$

Hence, $(f(A))_\alpha = (f(A_\alpha))$.

Since f is an M -anti homomorphism, $(f(A_\alpha))$ is an M -subgroup of G' .

Hence $(f(A))_\alpha$ is an M -subgroup of G' .

Hence $f(A)$ is an M -fuzzy subgroup of G' .

2.4 Theorem

The M -anti homomorphic pre-image of an M -fuzzy subgroup of an M -group G' is an M -fuzzy subgroup of an M -group G .

Proof

Let $f: G \rightarrow G'$ be an M -anti homomorphism. Let the fuzzy set V on G' be an M -fuzzy subgroup.

We have to prove that any fuzzy set A on G is an M -fuzzy subgroup, where $V = f(A)$.

Now, $A(xy) = V(f(xy))$.

$$= V(f(x)f(y))$$

as f is an M -anti homomorphism.

$$\geq \min \{ V(f(x)), V(f(y)) \}$$

as V is an M -fuzzy subgroup of G' .

$$= \min \{ A(x), A(y) \}.$$

That is, $A(xy) \geq \min \{ A(x), A(y) \}$.

For $x \in G$,

$$A(x^{-1}) = V(f(x^{-1}))$$

$$= V((f(x))^{-1}) \text{ as } f \text{ is an } M\text{-anti homomorphism}$$

$$= V(f(x)) \text{ as } V \text{ is an } M\text{-fuzzy subgroup of } G'$$

$$= A(x).$$

That is, $A(x^{-1}) = A(x)$.

Clearly, $A(mx) = V(f(mx))$

$$= V(mf(x)), \text{ as } f \text{ is an } M\text{-anti homomorphism}$$

$$\geq V(f(x)) \text{ as } V \text{ is an } M\text{-fuzzy subgroup of } G'$$

$$= A(x).$$

That is, $A(mx) \geq A(x)$.

Hence A is an M -fuzzy subgroup of G .

3. Properties of level subsets of an M -fuzzy subgroup of an

M -group:

3.1 Theorem

Let A be a fuzzy subset of an M -group G . If A is an M -

fuzzy subgroup of G , then the level subsets A_t , $t \in \text{Im}(A)$ are M -subgroups of G .

Proof

Let $t \in \text{Im}(A)$ and $x, y \in A_t$.

Then $A(x) = t$ and $A(y) = t$.

Given that A is an M -fuzzy subgroup of G .

Therefore, A is a fuzzy subgroup of G .

Hence $A(xy) \geq \min \{ A(x), A(y) \} = t$.

That is, $A(xy) \geq t$.

That is, $xy \in A_t$.

Moreover, if $x \in A_t$, then $A(x^{-1}) = A(x) \geq t$.

Hence $x^{-1} \in A_t$.

Hence A_t is a subgroup of G .

Now, for any $x \in A_t$ and $m \in M$, then

$$A(mx) \geq A(x) \geq t.$$

Hence $mx \in A_t$.

Hence A_t is an M -subgroup of G .

3.2 Theorem

Let A be a fuzzy subset of an M -group G . If the level subsets A_t , $t \in \text{Im}(A)$ are M -subgroups of G , then A is an M -fuzzy subgroup of G .

Proof

Let the level subsets A_t , $t \in \text{Im}(A)$ are M -subgroups of G .

If there exist $x_0, y_0 \in G$ such that $A(x_0y_0) < \min \{ A(x_0), A(y_0) \}$.

Let $t_0 = (A(x_0y_0) + \min \{ A(x_0), A(y_0) \}) / 2$, we have $A(x_0y_0) < t_0 < \min \{ A(x_0), A(y_0) \}$.

It follows that $x_0, y_0 \in A_{t_0}$, but $x_0y_0 \notin A_{t_0}$.

Which is a contradiction.

Hence $A(xy) \geq \min \{ A(x), A(y) \}$.

Similarly, we have $A(x^{-1}) \geq A(x)$.

Hence A is a fuzzy subgroup of G .

Now, suppose, for $m \in M$ and $x \in G$, $A(mx) < A(x)$.

Let $t_0 = (A(mx) + A(x)) / 2$.

Then, $A(mx) < t_0 < A(x)$.

That is, for $m \in M$ and $x \in G$, then $x \in A_{t_0}$, but $mx \notin A_{t_0}$.

Which is a contradiction to A_{t_0} is a M -subgroup of G .

Hence $A(mx) \geq A(x)$.

Hence A is an M -fuzzy subgroup of G .

3.1 Definition

Let A be an M -fuzzy subgroup of an M -group G . Then the M -subgroups A_t , for $t \in [0,1]$ and $t \geq A(e)$, are called level M -subgroups of A .

4. Level M -subgroups of M -fuzzy subgroups of an M -group G under M -homomorphism and M -anti homomorphism

4.1 Theorem

The M -homomorphic image of a level M -subgroup of an M -fuzzy subgroup A of an M -group G is a level M -subgroup of an M -fuzzy subgroup $f(A)$ of an M -group G' where A is f -invariant.

Proof

Let G and G' be any two M -groups.

Let $f: G \rightarrow G'$ be an M -homomorphism.

Let A be an M -fuzzy subgroup of G .

Clearly, $f(A)$ is an M -fuzzy subgroup of G' .

Let A_α be a level M -subgroup of an M -fuzzy subgroup A of G .

Since f is an M -homomorphism, $f(A_\alpha)$ is an M -subgroup $f(A)$ of G' and $f(A_\alpha) = (f(A))_\alpha$.

Hence $(f(A))_\alpha$ is a level M -subgroup $f(A)$ of G' .

4.2 Theorem

The M -homomorphic pre-image of a level M -subgroup of an M -fuzzy subgroup V of an M -group G' is a level M -subgroup of an M -fuzzy subgroup $f^{-1}(V)$ of an M -group G .

Proof

Let G and G' be any two M -groups.

Let $f: G \rightarrow G'$ be an M -homomorphism.

Let V be an M -fuzzy subgroup of G' .

Clearly $f^{-1}(V)$ is an M -fuzzy subgroup of G .

Let V_t be a level M -subgroup of an M -fuzzy subgroup V of G' .

Since, f is an M -homomorphism, $f^{-1}(V_t)$ is an M -subgroup of $f^{-1}(V)$ of G

and $f^{-1}(V_t) = (f^{-1}(V))_t$, is an M -subgroup of an M -fuzzy subgroup $f^{-1}(V)$ of G .

That is, $(f^{-1}(V))_t$ is a level M -subgroup of an M -fuzzy subgroup $f^{-1}(V)$ of G .

4.3 Theorem

The M -anti homomorphic image of a level M -subgroup of an M -fuzzy subgroup A of an M -group G is a level M -subgroup of an M -fuzzy subgroup $f(A)$ of an M -group G' where A is f -invariant.

Proof

Let G and G' be any two M -groups.

Let $f: G \rightarrow G'$ be an M -anti homomorphism.

Let A be an M -fuzzy subgroup of G .

Clearly, $f(A)$ is an M -fuzzy subgroup of G' .

Let A_α be a level M -subgroup of an M -fuzzy subgroup A of G .

Since f is an M -anti homomorphism, $f(A_\alpha)$ is an M -subgroup $f(A)$ of G' and $f(A_\alpha) = (f(A))_\alpha$.

Hence $(f(A))_\alpha$ is a level M -subgroup $f(A)$ of G' .

4.4 Theorem

The M -anti homomorphic pre-image of a level M -subgroup of an M -fuzzy subgroup V of an M -group G' is a level M -subgroup of an M -fuzzy subgroup $f^{-1}(V)$ of an M -group G .

Proof

Let G and G' be any two M -groups.

Let $f: G \rightarrow G'$ be an M -anti homomorphism.

Let V be an M -fuzzy subgroup of G' .

Clearly $f^{-1}(V)$ is an M -fuzzy subgroup of G .

Let V_t be a level M -subgroup of an M -fuzzy subgroup V of G' .

Since, f is an M -anti homomorphism, $f^{-1}(V_t)$ is an M -subgroup of $f^{-1}(V)$ of G

and $f^{-1}(V_t) = (f^{-1}(V))_t$, is an M -subgroup of an M -fuzzy subgroup $f^{-1}(V)$ of G .

That is , $(f^{-1}(V))_t$ is a level M-subgroup of an M-fuzzy subgroup $f^{-1}(V)$ of G.

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