

A STUDY ON ANTI FUZZY SUB-BIGROUP

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Abstract

In this paper, we made an attempt to study the algebraic nature of anti fuzzy sub-bigroup of a bigroup and anti fuzzy sub-bigroup of a group and discussed some of its properties with counter example.

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Keywords

Fuzzy set , anti fuzzy subgroup , anti fuzzy sub-bigroup of a bigroup, anti fuzzy sub-bigroup of a group, bi-lower level subset, homomorphism, anti-homomorphism.

Introduction

The concept of fuzzy sets was initiated by Zadeh. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld gave the idea of fuzzy subgroups and Ranjith Biswas gave the idea of anti fuzzy subgroups.

The notion of bigroup was first introduced by P.L.Maggu in 1994. W.B. Vasantha Kandasamy and D.Meiyappan introduced concept of fuzzy sub-bigroup of a bigroup and fuzzy sub-bigroup of a group.

In this paper, we introduce the concept of anti fuzzy sub-bigroup of a bigroup and bi-lower level subset of an anti fuzzy sub-bigroup and anti fuzzy sub-bigroup of a group and prove some results with counter examples. Also we establish the relationship between anti fuzzy sub-bigroup of a group and anti fuzzy subgroup of a group.

1. Preliminaries

This section contains some definitions and results to be used in the sequel.

1.1 Definition

Let S be a set. A fuzzy subset A of S is a function $A: S \rightarrow [0,1]$.

1.2 Definition

Let G be a group. A fuzzy subset A of G is called an anti fuzzy subgroup if for $x, y \in G$,

$$(i) \quad A(xy) \leq \max \{ A(x), A(y) \},$$

$$(ii) \quad A(x^{-1}) = A(x).$$

1.3 Definition

Let A be a fuzzy subset of S . For $t \in [0,1]$, the lower level subsets of A is the set,

$$\bar{A}_t = \{ x \in S : A(x) \leq t \}.$$

1.4 Definition

Let G be a finite group of order n . and A be an anti fuzzy subgroup of G .

Let $\text{Im}(A) = \{ t_i : A(x) = t_i \text{ for some } x \in G \}$. Then $\{ \bar{A}_{t_i} \}$ are the only lower level subgroups of A .

1.5 Definition

A set $(G, +, \bullet)$ with two binary operation $+$ and \bullet is called a bigroup if there exist two proper subsets G_1 and G_2 of G such that

$$i. \quad G = G_1 \cup G_2$$

$$ii. \quad (G_1, +) \text{ is a group.}$$

$$iii. \quad (G_2, \bullet) \text{ is a group.}$$

A non-empty subset H of a bigroup $(G, +, \bullet)$ is called a sub-bigroup, if H itself is a bigroup under $+$ and \bullet operations defined on G .

1.6 Definition

Let $G = (G, +, \bullet)$ be a bigroup. Then $A: G \rightarrow [0,1]$ is said to be an anti fuzzy sub-bigroup of the bigroup G if there exists two fuzzy subsets A_1 of G_1 and A_2 of G_2 such that

$$i. \quad A = A_1 \cup A_2$$

$$ii. \quad (A_1, +) \text{ is an anti fuzzy subgroup of } (G_1, +).$$

$$iii. \quad (A_2, \bullet) \text{ is an anti fuzzy subgroup } (G_2, \bullet).$$

1.1 Example

Let $G_1 = \{ 0, a, b, a+b \}$ be a group under the operation $+$ with $a+a = b+b = 0$ and $a+b = b+a$

Let $G_2 = \{ 1, -1, i, -i \}$ be a group under the operation \bullet .

Define $A: G \rightarrow [0,1]$ by

$$A(x) = \begin{cases} 0.2 & \text{for } x = 0 \\ 0.3 & \text{for } x = a, 1 \\ 0.4 & \text{for } x = b, a+b, -1 \\ 0.5 & \text{for } x = i, -i \end{cases}$$

It is easy to verify that A is an anti fuzzy sub-bigroup of the bigroup G, for we can find $A_1 : G_1 \rightarrow [0, 1]$ by

$$A_1(x) = \begin{cases} 0.2 & \text{for } x = 0 \\ 0.3 & \text{for } x = a \\ 0.4 & \text{for } x = b, a+b \end{cases}$$

and $A_2 : G_2 \rightarrow [0, 1]$ is given by

$$A_2(x) = \begin{cases} 0.3 & \text{for } x = 1 \\ 0.4 & \text{for } x = -1 \\ 0.5 & \text{for } x = i, -i \end{cases}$$

Clearly A is an anti fuzzy sub-bigroup of the bigroup G.

1.1 Theorem

Every t – lower level subsets of an anti fuzzy sub-bigroup A of a bigroup G need not in general be a sub-bigroup of the bigroup G.

Proof: We prove this by an example.

Take $G = \{-1, 0, 1\}$ to be a bigroup under the operation '+' and '•' where $G_1 = \{0\}$ and $G_2 = \{-1, 1\}$ are groups respectively with respect to usual addition and usual multiplication.

Define $A : G \rightarrow [0, 1]$

$$A(x) = \begin{cases} 0.5 & \text{for } x = -1, 1 \\ 0.75 & \text{for } x = 0 \end{cases}$$

Clearly $(A, +, \bullet)$ is an anti fuzzy sub-bigroup of the bigroup $(G, +, \bullet)$. Now consider the lower level subset \bar{A}_t for $t = 0.5$ of the anti fuzzy sub-bigroup A

For $t = 0.5$,

$$\bar{A}_t = \{x \in G : A(x) \leq 0.5\}.$$

$$\bar{A}_t = \{-1, 1\}.$$

It is easy to verify that $\{-1, 1\}$ is not a sub-bigroup of the bigroup $(G, +, \bullet)$. Hence the lower level subset \bar{A}_t for $t = 0.5$ of an anti fuzzy sub-bigroup A is not a sub-bigroup of the bigroup $(G, +, \bullet)$.

We introduce the notion of bi-lower level subset of an anti fuzzy sub-bigroup.

1.7 Definition

Let $G = (G_1 \cup G_2, +, \bullet)$ be a bigroup and $A = (A_1 \cup A_2)$ be an anti fuzzy sub-bigroup of the bigroup G. The bi-lower level subset of the anti fuzzy sub-bigroup A of the bigroup G is defined as

$$\bar{A}_t = \bar{A}_1 t \cup \bar{A}_2 t \text{ for every } t \in [\max\{A_1(e_1), A_2(e_2)\}, 1].$$

Where e_1 denotes the identity element of the group $(G_1, +)$ and e_2 denotes the identity element of the group (G_2, \bullet) .

Remark

The condition $t \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$ is essential for the bi-lower level subset to be a sub-bigroup, for if $t \notin [\max\{A_1(e_1), A_2(e_2)\}, 1]$, the bi-lower level subset need not in general be a sub-bigroup of the bigroup G which is evident from the following example.

1.2 Example

Consider Example 1.1, the bi-lower level subset \bar{A}_t for $t = 0.2$ of the anti fuzzy sub-bigroup A is given by $\bar{A}_t = \{0\}$ which is not a sub-bigroup of the bigroup G. Therefore the bi-lower level subset \bar{A}_t for $t = 0.2$ is not a sub-bigroup of the bigroup G.

1.2 Theorem

Every bi-lower level subset of an anti fuzzy sub-bigroup A of a bigroup G is a sub-bigroup of the bigroup G.

Proof

Let $A = (A_1 \cup A_2)$ be an anti fuzzy sub-bigroup of a bigroup $G = (G_1 \cup G_2, +, \bullet)$. Consider the bi-lower level subset \bar{A}_t of an anti fuzzy sub-bigroup A for every $t \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$, where e_1 denotes the identity element of the group $(G_1, +)$ and e_2 denotes the identity element of the group (G_2, \bullet) . Then $\bar{A}_t = \bar{A}_1 t \cup \bar{A}_2 t$ where $\bar{A}_1 t$ and $\bar{A}_2 t$ are subgroups of G_1 and G_2 respectively. Hence by the definition of sub-bigroup \bar{A}_t is a sub-bigroup of the bigroup $(G, +, \bullet)$.

We illustrate the following by an example.

1.3 Example

Let $G = \{0, \pm 1, \pm i\}$ is a bigroup with respect to addition modulo 2 and multiplication. Clearly $G_1 = \{0, 1\}$ and $G_2 = \{\pm 1, \pm i\}$ are groups with respect to addition modulo 2 and multiplication respectively. Define $A : G \rightarrow [0, 1]$ by

$$A(x) = \begin{cases} 0.1 & \text{for } x = 0 \\ 0.5 & \text{for } x = \pm 1 \\ 0.7 & \text{for } x = \pm i \end{cases}$$

It is easy to verify that A is an anti fuzzy sub-bigroup of the bigroup G s there exist two anti fuzzy subgroups $A_1 : G_1 \rightarrow [0, 1]$, $A_2 : G_2 \rightarrow [0, 1]$ such that $A = (A_1 \cup A_2)$ where

$$A_1(x) = \begin{cases} 0.1 & \text{for } x = 0 \\ 0.4 & \text{for } x = 1 \end{cases}$$

and

$$A_2(x) = \begin{cases} 0.5 & \text{for } x = \pm 1 \\ 0.7 & \text{for } x = \pm i \end{cases}$$

We now calculate the bi-lower level subset \bar{A}_t for $t = 0.5$,
 $\bar{A}_t = \bar{A}_{1t} \cup \bar{A}_{2t} = \{x \in G_1 : A_1(x) \leq 0.5\} \cup \{x \in G_2 : A_2(x) \leq 0.5\}$

$$= \{0, 1\} \cup \{-1, 1\}$$

$$= \{0, \pm 1\}$$

$$\bar{A}_t = \{0, \pm 1\}$$

It is easily verified that \bar{A}_t is a sub-bigroup of the bigroup G .

1.3 Theorem

Let G be a bigroup and A_1, A_2 be fuzzy subsets of A such that $A = (A_1 \cup A_2)$. The bi-lower level subset \bar{A}_t of A is a sub-bigroup of G , $t \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$, where e_1 denotes the identity element of G_1 and e_2 denotes the identity element of G_2 respectively. Then A is an anti fuzzy sub-bigroup of a bigroup G .

Proof

Let $G = (G_1 \cup G_2)$ be a bigroup.

Given that the bi-lower level subset $\bar{A}_t = \bar{A}_{1t} \cup \bar{A}_{2t}$ is a sub-bigroup of G .

Clearly \bar{A}_{1t} is a subgroup of G_1 , A_1 is an anti fuzzy subgroup of G_1 .

Clearly \bar{A}_{2t} is a subgroup of G_2 , A_2 is an anti fuzzy subgroup of G_2 .

Hence $A = (A_1 \cup A_2)$ and hence A is an anti fuzzy sub-bigroup of G .

1.8 Definition

Let $A = (A_1 \cup A_2)$ be an anti fuzzy sub-bigroup of a bigroup $G = (G_1 \cup G_2)$. The sub-bigroups \bar{A}_t , for $t \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$, where e_1 denotes the identity element of G_1 and e_2 denotes the identity element of G_2 respectively, are called bi-lower level sub-bigroups of A .

1.4 Theorem

Let $A = (A_1 \cup A_2)$ be an anti fuzzy sub-bigroup of a bigroup $G = (G_1 \cup G_2)$. Two bi-lower level subgroups $\bar{A}_\alpha, \bar{A}_\beta$, $\alpha, \beta \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$, where e_1 denotes the identity element of G_1 and e_2 denotes the identity element of G_2 respectively with $\alpha < \beta$ are equal iff there is no x in G such that $\alpha < A(x) \leq \beta$.

Proof

Let $A = (A_1 \cup A_2)$ be an anti fuzzy sub-bigroup of a bigroup $G = (G_1 \cup G_2)$.

Consider the two bi-lower level subgroups $\bar{A}_\alpha, \bar{A}_\beta$, $\alpha, \beta \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$ where e_1 denotes the identity element of G_1 and e_2 denotes the identity element of G_2 respectively with $\alpha < \beta$.

Let $\bar{A}_\alpha = \bar{A}_\beta$.

We have to prove that there is no x in G such that

$$\alpha < A(x) \leq \beta.$$

Suppose that there is an x in G such that $\alpha < A(x) < \beta$ then $x \in \bar{A}_\beta$ and $x \notin \bar{A}_\alpha$.

This implies $\bar{A}_\alpha \subset \bar{A}_\beta$, which contradicts the assumption that $\bar{A}_\alpha = \bar{A}_\beta$.

Hence there is no x in G such that $\alpha < A(x) \leq \beta$.

Conversely, suppose that there is no x in G such that

$$\alpha < A(x) \leq \beta.$$

Then, by definition, $\bar{A}_\alpha \subset \bar{A}_\beta$.

Let $x \in \bar{A}_\beta$ and there is no x in G such that $\alpha < A(x) \leq \beta$.

Hence $x \in \bar{A}_\alpha$.

$$(i.e) \quad \bar{A}\beta \subset \bar{A}\alpha.$$

$$\text{Hence } \bar{A}\alpha = \bar{A}\beta.$$

1.5 Theorem

A fuzzy subset A of G is an anti fuzzy sub-bigroup of G iff the bi-lower level subsets $\bar{A}t$, $t \in \text{Image } A$, are sub-bigroups of G.

Proof

It is clear.

1.6 Theorem

Any sub-bigroup H of a bigroup G can be realized as a bi-lower level sub-bigroup of some anti fuzzy sub-bigroup of a bigroup G.

Proof: Let $G = (G_1 \cup G_2, +, \bullet)$ be a bigroup.

Let $H = (H_1 \cup H_2, +, \bullet)$ be a sub-bigroup of G.

Let A_1 and A_2 be a fuzzy subsets of A defined by

$$A_1(x) = \begin{cases} 0 & \text{if } x \in H_1 \\ t & \text{if } x \notin H_1 \end{cases}; \quad A_2(x) = \begin{cases} 0 & \text{if } x \in H_2 \\ t & \text{if } x \notin H_2 \end{cases}$$

where $t \in [\max\{A_1(e_1), A_2(e_2)\}, 1]$, and e_1 denotes the identity element of G_1 and e_2 denotes the identity element of G_2 respectively.

We shall prove that $A = (A_1 \cup A_2)$ is an anti fuzzy sub-bigroup of a bigroup G.

Suppose $x, y \in H$, then

$$i. \quad x, y \in H_1 \implies x + y \in H_1 \text{ and } x + (-y) \in H_1.$$

$$A_1(x) = 0, A_1(y) = 0 \text{ and } A_1(x + (-y)) = 0, \text{ then}$$

$$A_1(x + (-y)) \leq \max\{A_1(x), A_1(y)\}.$$

$$ii. \quad x, y \in H_2 \implies xy \in H_2 \text{ and } xy^{-1} \in H_2.$$

$$A_2(x) = 0, A_2(y) = 0 \text{ and } A_2(xy^{-1}) = 0,$$

$$\text{Then } A_2(xy^{-1}) \leq \max\{A_2(x), A_2(y)\}.$$

$$iii. \quad x \in H_1 \text{ and } y \notin H_1 \implies x + y \notin H_1$$

$$\text{and } x + (-y) \notin H_1.$$

$$A_1(x) = 0, A_1(y) = t \text{ and } A_1(x + (-y)) = t, \text{ then}$$

$$A_1(x + (-y)) \leq \max\{A_1(x), A_1(y)\}.$$

$$iv. \quad x \in H_2 \text{ and } y \notin H_2 \implies xy \notin H_2 \text{ and } xy^{-1} \notin H_2.$$

$$A_2(x) = 0, A_2(y) = t \text{ and } A_2(xy^{-1}) = t,$$

$$\text{Then } A_2(xy^{-1}) \leq \max\{A_2(x), A_2(y)\}.$$

Suppose $x, y \notin H$, then

$$i. \quad x, y \notin H_1 \text{ then } x + y \in H_1 \text{ or } x + y \notin H_1.$$

$$x, y \notin H_1 \text{ then } x + (-y) \in H_1 \text{ or } x + (-y) \notin H_1.$$

$$A_1(x) = t, A_1(y) = t \text{ and } A_1(x + (-y)) = 0 \text{ or } t,$$

$$\text{then } A_1(x + (-y)) \leq \max\{A_1(x), A_1(y)\}.$$

$$ii. \quad x, y \notin H_2 \implies xy \in H_2 \text{ or } xy \notin H_2.$$

$$x, y \notin H_2 \implies xy^{-1} \in H_2 \text{ or } xy^{-1} \notin H_2.$$

$$A_2(x) = t, A_2(y) = t \text{ and } A_2(xy^{-1}) = 0 \text{ or } t,$$

$$\text{then } A_2(xy^{-1}) \leq \max\{A_2(x), A_2(y)\}.$$

Thus in all cases,

$$\begin{cases} (A_1, +) \text{ is an anti fuzzy subgroup of } (G_1, +) \text{ and} \\ (A_2, \bullet) \text{ is an anti fuzzy subgroup of } (G_2, \bullet). \end{cases}$$

Hence $A = (A_1 \cup A_2)$ is an anti fuzzy sub-bigroup of a bigroup G. For this anti fuzzy sub-bigroup, $\bar{A}t = \bar{A}_1t \cup \bar{A}_2t = H$.

Remark

As a consequence of the above Theorems, the bi-lower level sub-bigroups of an anti fuzzy sub-bigroup A form a chain. Since $A(e_1) \leq A(x)$ for all x in G_1 or $A(e_2) \leq A(x)$ for all x in G_2 , therefore $\bar{A}t_0$, where $t_0 = \max\{A_1(e_1), A_2(e_2)\}$, is the smallest and we have the chain :

$$\{e_1, e_2\} = \bar{A}t_0 \subset \bar{A}t_1 \subset \bar{A}t_2 \subset \dots \subset \bar{A}t_n = G, \text{ where } t_0 < t_1 < t_2 < \dots < t_n.$$

2. Now we proceed on to define the anti fuzzy bigroup of a group.

2.1 Definition

A fuzzy subset A of a group G is said to be an anti fuzzy sub-bigroup of the group G if there exist two anti fuzzy subgroups A_1 and A_2 of A ($A_1 \neq A$ and $A_2 \neq A$) such that $A = A_1 \cup A_2$.

2.1 Theorem

Every anti fuzzy sub-bigroup of a group G is an anti fuzzy subgroup of a group G but not conversely.

Proof

It is clear. Since the union of any two anti fuzzy subgroup is an anti fuzzy subgroup.

Converse part is not true in general and is proved by the following example.

2.1 Example

Consider the group $G = \{1, -1, i, -i\}$ under usual multiplication. Define $A : G \rightarrow [0, 1]$ by

$$A(x) = \begin{cases} 0 & \text{for } x = \{1, -1\} \\ 1 & \text{for } x = \{i, -i\} \end{cases}$$

Clearly A is an anti fuzzy subgroup as all of its lower level subsets are subgroups of G . Further $\alpha(\text{Im}(A)) = 2$.

If A_k is an anti fuzzy subgroup of A such that A_k is a subset of A . ($A_k \neq A$). Then A_k takes following form.

$$A_k(x) = \begin{cases} 0 & \text{for } x = \{1, -1\} \\ \alpha_k & \text{for } x = \{i, -i\} \end{cases}$$

with $0 \leq \alpha_k < 1$ for every $i \in I$.

It is easy to verify that $A_j \cup A_k \neq A$ for any $j, k \in I$. Thus there does not exist two anti fuzzy subgroups A_1 and A_2 of A ($A_1 \neq A$ and $A_2 \neq A$) such that $A = A_1 \cup A_2$.

2.2 Theorem

Let A be an anti fuzzy subgroup of a group G with $3 \leq \alpha(\text{Im}(A)) < \infty$ then there exist two anti fuzzy subgroups A_1 and A_2 of A ($A_1 \neq A$ and $A_2 \neq A$) such that $A = A_1 \cup A_2$.

Proof

Let A be an anti fuzzy subgroup of a group G . Suppose $\text{Im}(A) = \{a_1, a_2, \dots, a_n\}$ with $a_1 < a_2 < \dots < a_n$ and $3 \leq n < \infty$.

Choose $b_1, b_2 \in [0, 1]$ be such that $b_1 \leq a_1 < b_2 < a_2 < \dots < a_n$ and define $A_1, A_2 : G \rightarrow [0, 1]$ by,

$$A_1(x) = \begin{cases} a_1 & \text{for } x \in \bar{A}_{a_1} \\ b_2 & \text{for } x \in \bar{A}_{a_2} - \bar{A}_{a_1} \\ A(x) & \text{otherwise} \end{cases}$$

and

$$A_2(x) = \begin{cases} b_1 & \text{for } x \in \bar{A}_{a_1} \\ a_2 & \text{for } x \in \bar{A}_{a_2} - \bar{A}_{a_1} \\ A(x) & \text{otherwise} \end{cases}$$

Clearly A_1 and A_2 are anti fuzzy subgroups.

2.3 Theorem

Let A be a fuzzy subset of a group G with $3 \leq \alpha(\text{Im}(A)) < \infty$. Then A is an anti fuzzy subgroup of the group G iff A is an anti fuzzy sub-bigroup of G .

Proof

It is clear from **Theorem 2.1** and **Theorem 2.2**

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