Fuzzy BG – Ideals in BG – Algebra

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Abstract

In this paper, we introduce the concept of fuzzy BG – ideals in BG – Algebra and we have discussed some of their properties.

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Keywords

BG-algebra, sub BG - algebra and BG-ideals, fuzzy BG – ideals, fuzzy BG – bi-ideal.

1.Introduction

Y. Imai and K. Iseki introduced two classes of abstract algebras: BCK – algebras and BCI – algebras. It is known that the class of BCK – algebras is a proper subclass of the class of BCI – algebras. J. Neggers and H.S.Kim introduced a new notion, called B – algebra. C.B.Kim and H.S.Kim introduced the notion of the BG – algebra which is a generalization of B – algebra. In this paper, we classify the fuzzy BG – ideals in BG – Algebra.

2. Preliminaries

In this section we site the fundamental definitions that will be used in the sequel.

Definition 2.1

A nonempty set X with a constant 0 and a binary operation ‘ $*$ ‘ is called a BG – Algebra if it satisfies the following axioms.

1. $x * x = 0$,
2. $x * 0 = x$,
3. $(x * y) * (0 * y) = x$, $\forall$ $x,y \in X$.

Example 2.1

Let X = { 0,1,2 } be the set with the following table.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X , * , 0)$ is a BG – Algebra.

Definition 2.2

Let S be a non empty subset of a BG -algebra X , then S is called a subalgebra of X if $x * y \in S$, for all $x ,y \in S$.

Definition 2.3

Let X be a BG-algebra and I be a subset of X ,then I is called a BG-ideal of X if it satisfies following conditions:

1. $0 \in I$,
2. $x * y \in I$ and $y \in I \Rightarrow x \in I$,
3. $x \in I$ and $y \in X \Rightarrow x * y \in I$, $I \times X \subseteq I$.

Definition 2.4

A mapping $f: X \to Y$ of a BG-algebra is called a homomorphism if $f(x*y) = f(x)*f(y)$ $\forall$ $x ,y \in X$

Remark:

If $f: X \to Y$ is a homomorphism of BG-algebra, then $f(0) = 0$.

Definition 2.5

Let X be a non-empty set .A fuzzy sub set μ of the set X is a mapping $\mu : X \to [0,1]$.

Definition 2.6

A fuzzy set μ in X is said to be a fuzzy BG – bi-ideal if $\mu(x * w * y) \geq \min \{ \mu (x) , \mu(y)\}$ $\forall$ $x,y,w \in X$.

3. FUZZY SUBALGEBRAS

Definition 3.1

Let μ be a fuzzy set in BG – Algebra. Then μ is called a fuzzy subalgebra of X if

$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}\ \forall\ x,y \in X.$

Example 3.1

Let X = { 0,1,2, 3 } be the set with the following table.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Then $(X, *, 0)$ is a BG – Algebra. Define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = \mu(1) = t_0$ and $\mu(2) = \mu(3) = t_1$ for $t_0, t_1 \in [0,1]$ with $t_0 > t_1$. Then μ is a fuzzy subalgebra of X .

Definition 3.2

Let μ be a fuzzy set in a set X. For $t \in [0,1]$, the set

$\mu_t = \{x \in X / \mu(x) \geq t\}$ is called a level subset of μ .

4. Fuzzy BG – Ideal

Definition 4.1

A fuzzy set μ in X is called fuzzy BG – Ideal of X if it satisfies the following inequalities.

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$,
3. $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}\ \forall\ x,y \in X.$

Definition 4.2

Let λ and μ be the fuzzy sets in a set X . The Cartesian product $\lambda \times \mu : X \times X \to [0,1]$ is defined by

$(\lambda \times \mu)(x,y) = \min\{\lambda(x), \mu(y)\}\ \forall\ x,y \in X.$

Theorem 4.1

If λ and μ are fuzzy BG – Ideals of a BG – algebra X, then $\lambda \times \mu$ is a fuzzy BG – Ideals of $X \times X$.

Proof

For any $(x,y) \in X \times X$, we have

$$(\lambda \times \mu)(0,0) = \min\{\lambda(0), \mu(0)\}$$
$$\geq \min\{\lambda(x), \mu(y)\}$$
$$= (\lambda \times \mu)(x,y)$$

That is, $(\lambda \times \mu)(0,0) = (\lambda \times \mu)(x,y)$.

Let (x_1,x_2) and $(y_1,y_2) \in X \times X$. Then,

$$(\lambda \times \mu)(x_1, x_2)$$
$$= \min\{\lambda(x_1), \mu(x_2)\}$$
$$\geq \min\{\min\{\lambda(x_1 * y_1), \lambda(y_1)\}, \min\{\mu(x_2 * y_2), \mu(y_2)\}\}$$
$$= \min\{\min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}, \min\{\lambda(y_1), \mu(y_2)\}\}$$
$$= \min\{(\lambda \times \mu)((x_1 * y_1, x_2 * y_2)), (\lambda \times \mu)(y_1, y_2)\}$$
$$= \min\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$$

That is,

$$(\lambda \times \mu)((x_1, x_2))$$
$$= \min\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$$

and $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))$

$$= (\lambda \times \mu)(x_1 * y_1, x_2 * y_2)$$
$$= \min\{\lambda(x_1 * y_1), \mu(x_2 * y_2)\}$$
$$\geq \min\{\min\{\lambda(x_1), \lambda(y_1)\}, \min\{\mu(x_2), \mu(y_2)\}\}$$
$$= \min\{\min\{\lambda(x_1), \mu(x_2)\}, \min\{\lambda(y_1), \mu(y_2)\}\}$$
$$= \min\{(\lambda \times \mu)((x_1,x_2)), (\lambda \times \mu)((y_1,y_2))\}$$

That is , $(\lambda \times \mu)((x_1, x_2) * (y_1, y_2))$

$$= \min\{(\lambda \times \mu)(x_1, x_2), (\lambda \times \mu)(y_1, y_2)\}$$

Hence $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$.

Theorem 4.2

Let λ and μ be fuzzy sets in a BG – algebra such that $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$. Then

i. Either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$ $\forall$ $x \in X$.

ii. If $\lambda(0) \geq \lambda(x)$ $\forall$ $x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$.

iii. If $\mu(0) \geq \mu(x)$ $\forall$ $x \in X$, then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$.

Proof

i. Assume $\lambda(x) > \lambda(0)$ and $\mu(y) > \mu(0)$ for some $x, y \in X$.

Then $(\lambda \times \mu)(x,y) = \min\{\lambda(x), \mu(y)\}$
$> \min\{\lambda(0), \mu(0)\}$
$= (\lambda \times \mu)(0,0)$

Therefore $(\lambda \times \mu)(x, y) > (\lambda \times \mu)(0, 0)$, $\forall x, y \in X$.

Which is a contradiction to $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$.

Therefore either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$ $\forall$ $x \in X$.

ii. Assume $\mu(0) < \lambda(x)$ and $\mu(0) < \mu(y)$ $\forall x, y \in X$.

Then $(\lambda \times \mu)(0,0) = \min\{\lambda(0), \mu(0)\}$
$= \mu(0)$.

And $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\} > \mu(0)$
$= (\lambda \times \mu)(0, 0)$.

This implies $(\lambda \times \mu)(x, y) > (\lambda \times \mu)(0, 0)$.

Which is a contradiction to $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$.

Hence if If $\lambda(0) \geq \lambda(x)$ $\forall$ $x \in X$, then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$

iii. This proof is quite similar to (ii).

Theorem 4.3

If $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$, then λ or μ is a fuzzy BG – ideal of X.

Proof

Firstly to prove that μ is a fuzzy BG – ideal of X.

Given $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$, then by Theorem4.2(i), either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$, $\forall$ $x \in X$.

Let $\mu(0) \geq \mu(x)$.

By theorem 4.2(iii) then either $\lambda(0) \geq \lambda(x)$ or $\lambda(0) \geq \mu(x)$.

Now $\mu(x).$ $= \min\{\lambda(0), \mu(x)\}$

$= (\lambda \times \mu)(0,x)$

$\geq \min\{(\lambda \times \mu)((0,x) * (0,y)), (\lambda \times \mu)(0,y)\}$

$= \min\{(\lambda \times \mu)((0 * 0, x * y), (\lambda \times \mu)(0,y)\}$

$= \min\{(\lambda \times \mu)((0, x * y), (\lambda \times \mu)(0,y)\}$

$= \min\{(\lambda \times \mu)((0 * 0, x * y), (\lambda \times \mu)(0,y)\}$

$= \min\{\mu(x * y), \mu(y)\}$.

That is, $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

$\mu(x * y) = \min\{\lambda(0), \mu(x * y)\}$

$= (\lambda \times \mu)(0, x * y)$

$= (\lambda \times \mu)(0 * 0, x * y)$

$= (\lambda \times \mu)((0,x) * (0,y))$

$\mu(x * y) \geq \min\{(\lambda \times \mu)(0,x), (\lambda \times \mu)(0,y)\}$

$= \min\{\mu(x), \mu(y)\}$.

That is, $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$.

This proves that μ is a fuzzy BG-ideal of X.

Secondly to prove that λ is a fuzzy BG – ideal of X.

Given $\lambda \times \mu$ is a fuzzy BG – ideal of $X \times X$, then by Theorem4.2(i), either $\lambda(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$, $\forall$ $x \in X$.

Let $\lambda(0) \geq \lambda(x)$

By theorem 4.2(ii) then either $\mu(0) \geq \lambda(x)$ or $\mu(0) \geq \mu(x)$.

Now ,

$\lambda(x) = \min\{\mu(0), \lambda(x)\}$

$= (\lambda \times \mu)(0, x)$

$\geq \min\{(\lambda \times \mu)((0,x) * (0,y)), (\lambda \times \mu)(0,y)\}$

$= \min\{(\lambda \times \mu)((0 * 0, x * y), (\lambda \times \mu)(0,y)\}$

$= \min\{(\lambda \times \mu)((0, x * y), (\lambda \times \mu)(0,y)\}$

$= \min \{(\lambda \times \mu)((0*0, x*y), (\lambda \times \mu)(0, y)\}$

$= \min \{ \lambda(x*y), \lambda(y)\}$

That is, $\lambda(x) \geq \min\{ \lambda(x*y), \lambda(y)\}$.

$\lambda(x*y) = \min \{\mu(0), \lambda(x*y)\}$

$= (\lambda \times \mu)(0, x*y)$

$= (\lambda \times \mu)(0*0, x*y)$

$= (\lambda \times \mu)((0, x)*(0, y))$

$\lambda(x*y) \geq \min \{(\lambda \times \mu)(0, x), (\lambda \times \mu)(0, y)\}$

$= \min \{ \lambda(x), \lambda(y)\}$

That is, $\lambda(x*y) \geq \min \{ \lambda(x), \lambda(y)\}$.

This proves that λ is a fuzzy BG-ideal of X.

Theorem 4.4

If μ is a fuzzy BG – ideal of X, then μ_t is a BG – ideal of X for all $t \in [0, 1]$.

Proof:

Let μ be a fuzzy BG – ideal of X. Then

1. $\mu(0) \geq \mu(x)$,
2. $\mu(x) \geq \min\{ \mu(x*y), \mu(y)\}$,
3. $\mu(x*y) \geq \min \{ \mu(x), \mu(y)\}$ $\forall$ $x, y \in X$.

To prove that μ_t is a BG – ideal of X

We know that $\mu_t = \{ x / \mu(x) \geq t\}$

Let $x, y \in \mu_t$ and μ is a fuzzy BG – ideal of X .

Since $\mu(0) \geq \mu(x) \geq t$ Implies $0 \in \mu_t$, $\forall$ $t \in [0, 1]$.

Let $x*y \in \mu_t$ and $y \in \mu_t$

Therefore, $\mu(x*y) \geq t$ and $\mu(y) \geq t$.

Now $\mu(x) \geq \min\{ \mu(x*y), \mu(y)\} \geq \min \{ t, t \} \geq t$.

Hence $\mu(x) \geq t$.

That is , $x \in \mu_t$.

Let $x \in \mu_t$, $y \in X$.

Choose y in X such that $\mu(y) \geq t$.

Since $x \in \mu_t$ implies $\mu(x) \geq t$.

We know that $\mu(x*y) \geq \min \{ \mu(x), \mu(y)\}$

$\geq \min \{ t, t \} \geq t$.

That is, $\mu(x*y) \geq t$ implies $x*y \in \mu_t$.

Hence μ_t is a BG – ideal of X.

Theorem 4.5

If X be a BG – algebra, $\forall$ $t \in [0, 1]$, and μ_t is a BG - ideal of X, then μ is a fuzzy BG – ideal of X.

Proof :

Since μ_t is a BG - ideal of X .

i. $0 \in \mu_t$,

ii. $x*y \in \mu_t$ and $y \in \mu_t$ implies $x \in \mu_t$,

iii. $x \in \mu_t$ $y \in X$ implies $x*y \in \mu_t$.

To prove that μ is a fuzzy BG – ideal of X.

i. Let $x, y \in \mu_t$ then $\mu(x) \geq t$ and $\mu(y) \geq t$.

Let $\mu(x) = t_1$ and $\mu(y) = t_2$, without loss of generality let $t_1 \leq t_2$

Then $x \in \mu_{t1}$.

Now $x \in \mu_{t1}$ and $y \in X$ implies $x*y \in \mu_{t1}$.

That is , $\mu(x*y) \geq t_1$

$= \min \{ t_1, t_2 \}$

$= \min \{ \mu(x), \mu(y)\}$.

That is , $\mu(x*y) \geq \min \{ \mu(x), \mu(y)\}$.

ii. Let $\mu(0) = \mu(x*x)$

$\geq \min \{ \mu(x), \mu(x)\}$ (by proof (i))

$\geq \mu(x)$.

That is , $\mu(0) \geq \mu(x)$ $\forall$ $x \in X$.

iii. Let $\mu(x) = \mu((x*y)*(0*y))$

$\geq \min \{ \mu(x*y), \mu(0*y) \}$ (by (i))

$\geq \min \{ \mu(x*y), \min \{ \mu(0), \mu(y) \}\}$

$\geq \min \{ \mu(x*y), \mu(y) \}$ (by (ii)).

$\mu(x) \geq \min \{ \mu(x*y), \mu(y) \}$.

Hence μ is a fuzzy BG – ideal of X.

Theorem 4.6

Every fuzzy BG – ideal is a fuzzy BG – bi-ideal.

Proof

It is trivial.

Remark:

Converse of the above theorem is not true. That is every fuzzy BG – bi –ideal is not fuzzy BG – ideal. Let us prove this by an example.

Example:

Let X = { 0,1,2 } be the set with the following table.

*	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Then $(X, *, 0)$ is a BG – Algebra.

We define a fuzzy set $\mu : X \to [0,1]$ by $\mu(0) = 0.8$ and

$\mu(x) = 0.2 \ \forall \ x \neq 0$.

Clearly μ is fuzzy BG – ideal of X . But μ is not a BG – bi-ideal of X .

For, Let x = 0, w = 1, y = 0. Then

$\mu(x * w * y) = \mu(0 * 1 * 0) = \mu(0 * 1) = \mu(1) = 0.02$.

$\min \{ \mu(x), \mu(y) \} = \min \{ \mu(0), \mu(0) \} = \mu(0) = 0.8$.

Hence $\mu(x * w * y) \leq \min \{ \mu(x), \mu(y) \}$.

Hence μ is not a fuzzy BG – bi-ideal of X.

Definition 4.3

Let f: $X \to Y$ be a mapping of BG –algebra and μ be a fuzzy set of Y then μ^f is the pre -image of μ under f if $\mu^f(x) = \mu(f(x)) \ \forall \ x \in X$.

Theorem 4.7

Let f: $X \to Y$ be a homomorphism of BG – algebra if μ is a fuzzy BG – ideal of Y then μ^f is a fuzzy BG – ideal of X.

Proof

For any $x \in X$, we have

$$\mu^f(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = \mu^f(0)$$

Let $x, y \in X$, then

$$\begin{aligned}
\min \{ \mu^f(x * y), \mu^f(y) \} &= \min \{ \mu(f(x * y)), \mu(f(y)) \} \\
&= \min \{ \mu(f(x) * f(y)), \mu(f(y)) \} \\
&\leq \mu(f(x)) \\
&= \mu^f(x)
\end{aligned}$$

That is , $\mu^f(x) \geq \min \{ \mu^f(x * y), \mu^f(y) \}$

$$\begin{aligned}
\min \{ \mu^f(x), \mu^f(y) \} &= \min \{ \mu(f(x)), \mu(f(y)) \} \\
&\leq \mu(f(x) * f(y)) \\
&= \mu(f(x * y)) \\
&= \mu^f(x * y)
\end{aligned}$$

That is , $\mu^f(x * y) \geq \min \{ \mu^f(x), \mu^f(y) \}$

Hence μ^f is a fuzzy BG – ideal of X.

Theorem 4.8

Let f: $X \to Y$ be an epimorphism of BG – algebra. If μ^f is a fuzzy BG – ideal of X, then μ is a fuzzy BG – ideal of Y.

Proof

Let $y \in Y$, $\exists \ x \in X$ such that $f(x) = y$,

Then
$$\begin{aligned}
\mu(y) &= \mu(f(x)) \\
&= \mu^f(x) \\
&\leq \mu^f(0) \\
&= \mu(f(0)) = \mu(0).
\end{aligned}$$

Again let $x, y \in Y$ then $\exists \ a, b \in X$ such that $f(a) = x$ and $f(b) = y$.

It follows that
$$\begin{aligned}
\mu(x) &= \mu(f(a)) \\
&= \mu^f(a) \\
&\geq \min \{ \mu^f(a * b), \mu^f(b) \} \\
&= \min \{ \mu(f(a * b)), \mu(f(b)) \} \\
&= \min \{ \mu(f(a) * f(b)), \mu(f(b)) \} \\
&= \min \{ \mu(x * y), \mu(y) \}.
\end{aligned}$$

That is, $\mu(x) \geq \min \{ \mu(x * y), \mu(y) \}$.

and
$$\begin{aligned}
\mu(x * y) &= \mu(f(a) * f(b)) \\
&= \mu(f(a * b)) \\
&= \mu^f(a * b) \\
&\geq \min \{ \mu^f(a), \mu^f(b) \} \\
&= \min \{ \mu(f(a)), \mu(f(b)) \} \\
&= \min \{ \mu(x), \mu(y) \}
\end{aligned}$$

Hence $\mu(x * y) \geq \min \{ \mu(x), \mu(y) \}$.

Hence μ is fuzzy BG – ideal of Y.

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