Hybrid Fuzzy-Linear Programming with Shadowed Fuzzy Numbers

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ABSTRACT
Linear programming problem in an environment that includes different types of uncertainties represents real-world situations. In such situations, different forms of uncertain data parameters are commonly found in that problem. Fuzzy sets and their extensions are important tools of representing vague information. For decades, a lot of approaches are developed to solve fuzzy-linear programming problems. The existence of hybrid types of uncertainties in the fuzzy-linear programming problem imposes a real challenge to solve it. There is a need for introducing an efficient methodology to transform different types of uncertainties into a unified form. This paper introduces a new approach to solve hybrid fuzzy-linear programming using an improved version of shadowed fuzzy numbers (SFNs). SFNs are useful transformation tool for different types of uncertainties. They have the advantage of preserving the characteristics of uncertainty for different types of fuzzy sets used in the problem.

Keywords
Shadowed sets , Fuzzy numbers , Intuitionistic fuzzy numbers , Non-specificity measure , Entropy measure , Fuzzy linear programming

1. INTRODUCTION
Vague information are represented and processed by using fuzzy sets and their extensions. The fuzzy membership function contains two types of uncertainties, namely: fuzziness and nonspecificity [1]. Fuzziness and nonspecificity are related to the imprecise boundaries and the cardinality of fuzzy sets respectively [1, 2]. Intuitionistic fuzzy sets (IFSs) are characterized by two functions (membership and non-membership). It's more flexible than fuzzy sets. Entropy and nonspecificity represent basic types of uncertainty for IFSs [3, 4]. The shadowed sets are firstly proposed by Pedrycz to approximate fuzzy sets by using a three-valued constructs, namely (0, [0,1], 1) [5]. Pedrycz's method was motivated by the principle of uncertainty localization in which an optimization problem is formulated to redistribute the uncertainty content of a fuzzy set into the above three-valued construct [6]. The important properties of shadowed sets are preserving characteristics of uncertainty of fuzzy set and simplifying computations complexity.

Three areas are formed the shadowed set. Excluded area for the elements almost not included in set, core area for the elements that almost certainly belonging to it and shadow area or uncertain area for the elements approximately belong to it. The authors developed an improved shadowed fuzzy numbers (SFNs) that induced form fuzzy numbers and intuitionistic fuzzy numbers [7, 8]. A new approach is considered to be more accurate as it preserves more than one type of uncertainty. Also, it is appropriate for inducing shadowed sets from higher types of fuzzy sets. Over the past several decades linear programming models have been primarily developed in a crisp environment [1]. After fuzzy decision making is proposed by Bellman and Zadeh [9], many researchers developed this concept for solving fuzzy linear programming problems. In the literature [1, 10], the most of the fuzzy linear programming models can be classified to three categories. First, fuzziness related to the objective function coefficients and precise constraints. Second, a precise objective function coefficients and fuzzy constraints coefficients and right hand side values of the constraints. Third, a fuzzy objective function coefficients, fuzzy constraints coefficients and fuzzy right hand side of constraints. Intuitionistic fuzzy- linear programming is proposed by Angelov[11]. Different approaches are developed to solve Intuitionistic fuzzy- linear programming [11], [12], [13], [14]. In all previous mentioned approaches, some or all parameters of fuzzy-linear programming problem is considered from one type of fuzzy numbers and no explicit approach is proposed to solve the problem in the case of hybrid types of them. In this paper, the authors develop a new approach to solve hybrid fuzzy-linear programming which includes different uncertain types of parameters. The hybrid data include exact numbers, type-1 fuzzy numbers and intuitionistic fuzzy numbers. The new approach is based on the use of the shadowed fuzzy numbers. The reset of this paper is organized as follows: Section 2 presents an introduction about fuzzy numbers (FNs). It reviews intuitionistic fuzzy sets (IFSs) and an intuitionistic fuzzy numbers (IFNs). Also, this section introduces shadowed sets. Section 3 defines shadowed fuzzy number (SFN) and the new approaches to induce it from FN and IFN and define basic operations for SFNs. Section 4 introduces to proposed steps for solving hybrid fuzzy-linear programming problems. Section 5 provides illustrative examples to explain a new approach. Finally, conclusions are discussed in section 6.

2. PRELIMINARIES
In this section, some basic definitions and properties relevant to the present work are mentioned for fuzzy numbers (FN), intuitionistic fuzzy numbers (IFNs) and shadowed sets.

2.1 Fuzzy Numbers (FNs)
Some types of fuzzy sets $\widetilde{F}$ are defined on set $\mathbb{R}$ of real numbers and used to approximate uncertainty concept about numbers or intervals. Such types are called fuzzy numbers or fuzzy intervals and have the following conditions [1, 15]

- $\widetilde{F}$ is normal, i.e. at least one element x_i such that $\mu(x_i) = 1$.

- $\widetilde{F}$ is convex such that $\widetilde{F}(\lambda x + (1 - \lambda)y) \geq min\left(\widetilde{F}(x), \widetilde{F}(y)\right) \forall x, y \in U$ and $\lambda \in [0, 1]$ where **U** is a universe of discourse.

- The support of $\widetilde{F}$ is bounded.

The membership function of the real fuzzy number $\widetilde{F}$ is defined by [16]

$$\mu_{\widetilde{F}}(x) = \begin{cases} l_{\widetilde{F}}(x) & if \, a \leq x \leq b, \\ 1 & if \, b \leq x \leq c, \\ r_{\widetilde{F}}(x) & if \, c \leq x \leq d, \\ 0 & otherwise \end{cases} \quad (1)$$

where $l_{\widetilde{F}}$ and $r_{\widetilde{F}}$ are two continuous increasing and decreasing functions for left and right side of fuzzy number and a, b, c, d are real numbers.

2.2 Intuitionistic fuzzy numbers (IFNs)

An intuitionistic fuzzy number is an intuitionistic fuzzy set that defines on real numbers scale as in Fig. 1 and is defined as [17, 18]

$$A = \{< x, \mu_A(x), v_A(x) >|x \in \mathbb{R}\}, \quad (2)$$

where $\mu_A(x) : X \to [0, 1]$ is membership function, $v_A(x) : X \to [0, 1]$ is non-membership function, such that [3, 4]:

$$0 \leq \mu_A(x) + v_A(x) \leq 1. \quad (3)$$

Let

$$\pi_A = 1 - \mu_A(x) - v_A(x), \quad (4)$$

where π_A is called the intuitionistic index or hesitancy degree of x in A . The IFN A satisfy the following conditions:

1. A is normal, i.e. at least two points x_0, x_1 belong to A such that $\mu_A(x_0) = 1, v_A(x_1) = 1$.

2. A is convex, i.e. μ_A is fuzzy convex and v_A is fuzzy concave.

3. μ_A is upper semicontinuous and v_A is lower semicontinuous.

4. support$(A) = \{x \in X \,|v_A(x) < 1\}$ is bounded.

Some type like triangular intuitionistic fuzzy set as in Fig.1 and identified as [17]

$$TIFS = < (a, b, c), (\bar{a}, b, \bar{c}); M_A, V_A >, \quad (5)$$

where $(a, b, c), (\bar{a}, b, \bar{c})$ are basic values of TIFS , M_A is largest membership value and V_A is lowest non-memberaship value.
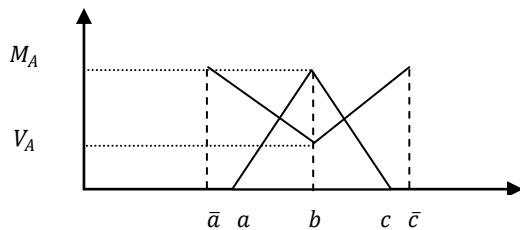

Fig. 1. Triangular intuitionistic fuzzy number

2.3 Shadowed sets

Shadowed sets are induced from fuzzy sets which preserving the essence of fuzzy sets and reducing the computational costs. Shadowed set S is defined as [5, 19]

$$S : X \to \{0, 1, [0, 1]\}, \quad (6)$$

The membership function of fuzzy set is transformed to the form is similar to three-valued logic. For preserving the uncertainty modeled by fuzzy sets, the process of creating shadowed set start by selecting the threshold $\alpha \in (0, 0.5)$. Selecting α depend on achieve a balance of uncertainty between three regions. The shadow region r_3, the region r_2 which represent the uncertainty lost due to elevating membership values to 1 and the region r_1 where the membership values is reduced to 0. This balance is defined as the following equation [5].

$$u(r_1) + u(r_2) = u(r_3), \quad (7)$$

where **u** is uncertainty of regions r_1, r_2 and r_3. Pedrycz proposed minimization of performance index V_α for the threshold α as

$$V_\alpha = |u(r_1) + u(r_2) - u(r_3)|. \quad (8)$$

After getting an optimal α, three regions can be induced from fuzzy set to represent the core, shadow and exclusion areas as illustrated in Fig. 2. [19].
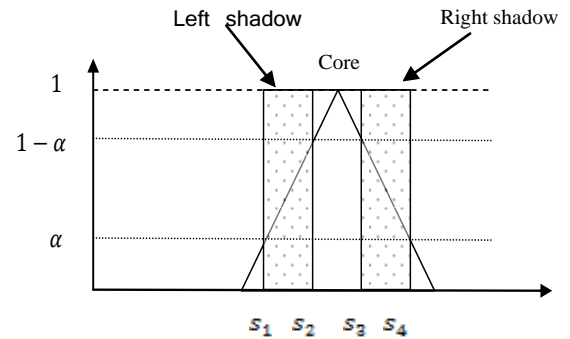

Fig. 2. Shadowed sets induced from triangular fuzzy number

3. SHADOWED FUZZY NUMBERS

Shadowed fuzzy number (SFN) is defined as a shadowed set induced from fuzzy number [20]. The authors proposed a new method to induce shadowed fuzzy numbers from fuzzy numbers by constructing core interval and fuzziness intervals [7]. The core interval induces using Hartley non-specificity measure [1] as the following equation

$$A_R(\alpha) - A_L(\alpha) + 1 = 2^{H_A}, $$

where H_A is Hartley non-specificity value of fuzzy $\quad (9)$ $A_L(\alpha), A_R(\alpha)$ are left and right α-cut of fuzzy set A, the core interval is deduced by using the α-core at level 1. The fuzziness intervals are based on the use of cardinality of left and right fuzziness sets as.

$$w_L = \sum_{x_L} f_A(x_L) \quad (10)$$

$$w_R = \sum_{x_R} f_A(x_R) \quad (11)$$

where w_L, w_R are width of left and right of fuzziness intervals. x_L and x_R are the left support and right support of fuzzy numbers from core value. $f_A(x_L), f_A(x_R)$ are fuzziness sets of fuzzy number which proposed by Tahayori [21] that defined as the following.

$$f_A = (x, fuzz(x)),$$

$$fuzz(x) = 1 - |2\mu_A(x) - 1|. \qquad (13)$$

Then shadowed fuzzy number (SFN) is constructed by using core interval and fuzziness intervals as in Fig. 3.
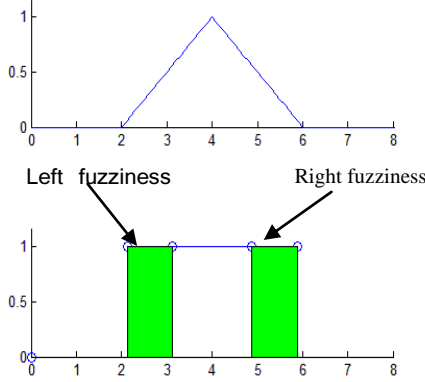

Fig. 3. SFN for triangular fuzzy number

3.1 Shadowed intuitionistic fuzzy numbers (SIFN)

Shadowed intuitionistic fuzzy numbers (SIFN) is extended version of SFNs in the case of normal intuitionistic fuzzy numbers IFNs [8]. The proposed approach is developed to induce the core interval of IFN as the following steps. The IFN is transformed to interval fuzzy number as [8]

$$A(x) = [\mu(x), 1 - \nu(x)] \qquad (14)$$

where $\mu(x)$ is a lower membership function and $1 - \nu(x)$ is an upper membership function. Hartley non-specificity value of IFN is obtained as the following equation

$$H_{avg}(A) = \frac{H(\mu(x)) + H(1 - \nu(x))}{2} \qquad (15)$$

where H_{avg} is the average of non-specificity measure, $H(\mu(x))$ and $H(1 - \nu(x))$ are the Hartley non-specificity measure of membership function and the complement of non-membership functions. Then H_{avg} is used to induce the α-cores for lower and upper membership functions of intuitionistic fuzzy number (IFN) as the following equations.

$$A_r^L(\alpha) - A_l^L(\alpha) + 1 = 2^{H_{avg}^L} \qquad (16)$$

$$A_r^U(\alpha) - A_l^U(\alpha) + 1 = 2^{H_{avg}^U} \qquad (17)$$

where $A_l^L(\alpha), A_r^L(\alpha), A_l^U(\alpha)$ and $A_r^U(\alpha)$ are left and right α-cut equations for lower and upper membership functions respectively for IFN A. Then α_L *and* α_U are induced and using them in previous left and right α-cut equations to get the core intervals $[c_l^L, c_r^L]$ **and** $[c_l^U, c_r^U]$.

In the case of symmetric membership and non-membership functions, two core intervals are equal. In the case of non-symmetric, the average of two intervals is defined as the following.

$$[c_l, c_r] = [\frac{(c_l^L + c_l^U)}{2}, \frac{(c_r^L + c_r^U)}{2}] \qquad (18)$$

The shadow areas are induced by using entropy intervals. The entropy is used to estimate the fuzziness of IFNs. The entropy sets is proposed to measures the fuzziness of IFNs based on Xia Liang method [22]. It is defined as the following.

$$E(A) = (x, ent(x)) \qquad (19)$$

$$ent(x) = \frac{1 - |\mu_A(x) - \nu_A(x)| + \pi_A(x)}{1 + \pi_A(x)} \qquad (20)$$

where $ent(x)$ is entropy measure based on Xia Liang method [22] and μ_A, ν_A and π_A are membership, non-membership and hesitancy degree functions for every element belong to IFN. The cardinality of left and right entropy sets to get the wide of entropy intervals as the following

$$w_L = \sum_{x_L} E(A) \qquad (21)$$

$$w_R = \sum_{x_R} E(A) \qquad (22)$$

where w_L **and** w_R are the wide of left and right entropy intervals and $E(A)$ is entropy set for IFN. These intervals represent shadow areas of shadowed intuitionistic fuzzy numbers (SIFN) and start from left and right end points of core interval. The core interval and entropy intervals are used to composition the shadowed intuitionistic fuzzy numbers (SIFN).

3.2 Shadowed fuzzy numbers operations

For applying shadowed numbers in decision making applications, the authors defined the basic arithmetic operations between SFNs as the following [23]

Let $SH_1 = (S_1^1, S_2^1, S_3^1, S_4^1)$, $SH_2 = (S_1^2, S_2^2, S_3^2, S_4^2)$ are two shadowed fuzzy numbers (SFNs).

- Addition

The addition operation of the two SFNs is defined as

$$SH_1 + SH_2 = (S_1^1 + S_1^2, S_2^1 + S_2^2, S_3^1 + S_3^2, S_4^1 + S_4^2) \qquad (23)$$

- Subtraction

The subtraction operation of the two SFNs where $SH_1 \geq SH_2$ is defined as

$$SH_1 - SH_2 = (S_1^1 - S_4^2, S_2^1 - S_3^2, S_3^1 - S_2^2, S_4^1 - S_1^2)$$

- Multiplication $\qquad (24)$

The multiplication operation of the two SFNs where SH_1 **and** SH_2 are induced from two positive fuzzy numbers is defined as

$$(25)$$
$$SH_1 \times SH_2 = (S_1^1 \times S_1^2, S_2^1 \times S_2^2, S_3^1 \times S_3^2, S_4^1 \times S_4^2)$$

- Division

The division operation between the two SFNs (SH_1, SH_2), where $S_1^2 \neq 0$, $S_2^2 \neq 0$, $S_3^2 \neq 0$ and $S_4^2 \neq 0$ and they induced from two positive fuzzy numbers is defined as

$$SH_1/SH_2 = (S_1^1/S_4^2, S_2^1/S_3^2, S_3^1/S_2^2, S_4^1/S_1^2) \qquad (26)$$

4. HYBRID FUZZY-LINEAR PROGRAMMING PROBLEM

The general fuzzy-linear programming is formulated as [1]

$$\max \quad \sum_{j=1}^{n} \tilde{c}_j x_j \qquad (27)$$

subject to

$$\sum_{j=1}^{n} \tilde{A}_{ij} x_j \leq \tilde{B}_i$$

$$x_j \geq 0$$

where

$$i \in \mathbb{N}, i = 1, \ldots \ldots, m,$$

$$j \in \mathbb{N}, j = 1, \ldots \ldots, n,$$

where $\tilde{A}_{ij}, \tilde{B}_i$ **and** $\tilde{c}_j$ are fuzzy numbers and x_j is a fuzzy variable. The $\leq$ is an inequality relation between fuzzy numbers. The arithmetic operations used in FLP model is fuzzy arithmetic operations. Three important cases are distinguished in the literature for modeling FLP. The first case is a precise objective function coefficients and fuzzy constraints coefficients and right hand side values of the constraints. The second case is a fuzzy objective function coefficients, fuzzy constraints coefficients and fuzzy right hand side of constraints. The Third case is a fuzzy objective function coefficients and precise constraints. Many approaches are developed to solve FLPPs [10].

In some situations, the FLP in the form of a hybrid fuzzy-linear programming problem (HFLPP) that has hybrid uncertain types about parameters. In such a situation the parameters of linear programming problems may be represented as different types of fuzzy numbers.

In all of the works in [1, 10 - 14], the FLP problems have been studied in which some or all parts of the problem were assumed to be one type of fuzzy numbers (not hybrid types).

In the following subsections, the authors will solve HFLPP in three different cases as mentioned in this section.

4.1 Precise objective function coefficients and fuzzy constraints

Let us consider the following fuzzy linear programming problem with m constraints and n variables may be formulated as follows:

$$\max \quad \sum_{j=1}^{n} c_j x_j \qquad (28)$$

subject to

$$\sum_{j=1}^{n} \tilde{k}_{ij} x_j \leq \tilde{t}_i$$

$$\sum_{j=1}^{n} \tilde{a}_{ij} x_j \leq \tilde{b}_i$$

$$x_j \geq 0$$

where

$$i \in \mathbb{N}, i = 1, \ldots \ldots, m$$

$$j \in \mathbb{N}, j = 1, \ldots \ldots, n$$

such that $\tilde{k}_{ij} = \left(k_{ij}^1, k_{ij}^2, k_{ij}^3\right), \mu_k; \left(\overline{k}_{ij}^1, \overline{k}_{ij}^2, \overline{k}_{ij}^3\right), v_k$ and $\tilde{t}_i = \left(t_i^1, t_i^2, t_i^3\right), \mu_t; \left(\overline{t}_i^1, \overline{t}_i^2, \overline{t}_i^3\right), v_t$ are triangular

intuitionistic fuzzy numbers TIFNs and μ, v are membership and and non-membership functions respectively. $\tilde{a}_{ij} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3\right), \tilde{b}_i = \left(b_i^1, b_i^2, b_i^3\right)$ are triangular fuzzy numbers TFNs. We use the following steps to solve this HFLPP problem.

Step 1 : The TIFNs $\tilde{k}_{ij}$ and $\tilde{t}_i$ transform to SFNs using equations $(15 - 23)$ where

$$sh(\tilde{k}_{ij}) = (s_{k_{ij}}^1, s_{k_{ij}}^2, s_{k_{ij}}^3, s_{k_{ij}}^4), \, sh(\tilde{t}_i) = (s_{t_i}^1, s_{t_i}^2, s_{t_i}^3, s_{t_i}^4).$$

Step 2 : The TFNs $\tilde{a}_{ij}$ and $\tilde{b}_i$ transform to SFNs using equations (10 - 14) where

$$sh(\tilde{a}_{ij}) = (s_{a_{ij}}^1, s_{a_{ij}}^2, s_{a_{ij}}^3, s_{a_{ij}}^4), sh(\tilde{b}_i) = (s_{b_i}^1, s_{b_i}^2, s_{b_i}^3, s_{b_i}^4).$$

Step 3 : The problem is reformulated as

$$\max \quad \sum_{j=1}^{n} c_j x_j \qquad (29)$$

subject to

$$\sum_{j=1}^{n} s_{k_{ij}}^1 x_j \leq s_{t_i}^1$$

$$\sum_{j=1}^{n} s_{k_{ij}}^2 x_j \leq s_{t_i}^2$$

$$\sum_{j=1}^{n} s_{k_{ij}}^3 x_j \leq s_{t_i}^3$$

$$\sum_{j=1}^{n} s_{k_{ij}}^4 x_j \leq s_{t_i}^4$$

$$\sum_{j=1}^{n} s_{a_{ij}}^1 x_j \leq s_{b_i}^1$$

$$\sum_{j=1}^{n} s_{a_{ij}}^2 x_j \leq s_{b_i}^2$$

$$\sum_{j=1}^{n} s_{a_{ij}}^3 x_j \leq s_{b_i}^3$$

$$\sum_{j=1}^{n} s_{a_{ij}}^4 x_j \leq s_{b_i}^4$$

$$x_j \geq 0$$

where

$$i \in \mathbb{N}, i = 1, \ldots \ldots, m$$

$$j \in \mathbb{N}, j = 1, \ldots \ldots, n$$

Step 4 : The problem can be solved as a classical linear programming problem.

4.2 Fuzzy objective function coefficients and fuzzy constraints

Let us consider the following fuzzy linear programming problem with m constraints and n variables may formulate as follows:

$$\text{max} \qquad \sum_{j=1}^{n} \tilde{c}_j x_j \qquad (30)$$

subject to

$$\sum_{j=1}^{n} \widetilde{k}_{ij} x_j \leq \tilde{t}_i$$

$$\sum_{j=1}^{n} \widetilde{a}_{ij} x_j \leq \widetilde{b}_i$$

$$x_j \geq 0$$

where

$$i \in \mathbb{N}, i = 1, \ldots \ldots, m$$

$$j \in \mathbb{N}, j = 1, \ldots \ldots, n$$

such that $\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3\right), \mu_c; \left(\overline{c}_{ij}^1, \overline{c}_{ij}^2, \overline{c}_{ij}^3\right), v_c, \widetilde{k}_{ij} = \left(k_{ij}^1, k_{ij}^2, k_{ij}^3\right), \mu_k; \left(\overline{k}_{ij}^1, \overline{k}_{ij}^2, \overline{k}_{ij}^3\right), v_k$ and
$\tilde{t}_i = \left(t_i^1, t_i^2, t_i^3\right), \mu_t; \left(\overline{t}_i^1, \overline{t}_i^2, \overline{t}_i^3\right), v_t$ are triangular intuitionistic fuzzy numbers TIFNs and μ, v are membership and and non-membership functions respectively. $\widetilde{a}_{ij} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3\right), \widetilde{b}_i = \left(b_i^1, b_i^2, b_i^3\right)$ are triangular fuzzy numbers TFNs. We use the following steps to solve this HFLPP.

Step 1 : The TIFNs $\tilde{c}_{ij}$, $\widetilde{k}_{ij}$ and $\tilde{t}_i$ transform to SFNs using equations $(14 - 22)$ where $sh(\tilde{c}_{ij}) = \left(s_{c_{ij}}^1, s_{c_{ij}}^2, s_{c_{ij}}^3, s_{c_{ij}}^4\right)$, $sh(\widetilde{k}_{ij}) = \left(s_{k_{ij}}^1, s_{k_{ij}}^2, s_{k_{ij}}^3, s_{k_{ij}}^4\right)$, and $sh(\tilde{t}_i) = \left(s_{t_i}^1, s_{t_i}^2, s_{t_i}^3, s_{t_i}^4\right)$.

Step 2 : The TFNs $\widetilde{a}_{ij}$ and $\widetilde{b}_i$ transform to SFNs using equations (10 - 14) where $sh(\widetilde{a}_{ij}) = \left(s_{a_{ij}}^1, s_{a_{ij}}^2, s_{a_{ij}}^3, s_{a_{ij}}^4\right)$ and $sh(\widetilde{b}_i) = \left(s_{b_i}^1, s_{b_i}^2, s_{b_i}^3, s_{b_i}^4\right)$.

Step 3 : The problem is reformulated as

$$\text{Max} \qquad \sum_{j=1}^{n} s_{c_{ij}}^t x_j \qquad (31)$$

subject to

$$\sum_{j=1}^{n} s_{k_{ij}}^1 x_j \leq s_{t_i}^1$$

$$\sum_{j=1}^{n} s_{k_{ij}}^2 x_j \leq s_{t_i}^2$$

$$\sum_{j=1}^{n} s_{k_{ij}}^3 x_j \leq s_{t_i}^3$$

$$\sum_{j=1}^{n} s_{k_{ij}}^4 x_j \leq s_{t_i}^4$$

$$\sum_{j=1}^{n} s_{a_{ij}}^1 x_j \leq s_{b_i}^1$$

$$\sum_{j=1}^{n} s_{a_{ij}}^2 x_j \leq s_{b_i}^2$$

$$\sum_{j=1}^{n} s_{a_{ij}}^3 x_j \leq s_{b_i}^3$$

$$\sum_{j=1}^{n} s_{a_{ij}}^4 x_j \leq s_{b_i}^4$$

$$x_j \geq 0$$

where

$$i \in \mathbb{N}, i = 1, \ldots \ldots, m$$

$$j \in \mathbb{N}, j = 1, \ldots \ldots, n$$

$$t \in \mathbb{N}, t = 1, 2, 3, 4$$

Step 4 : The problem can be solved four times with four parameters of a SFN for the objective function coefficients using a classical linear programming problem.

Step 5 : The resulting maximum optimal solution of HFLPP is a SFN and we need to transform it to one crisp number. We proposed the following steps to obtain the maximum optimal solution:

Let maximum optimal solution $H_{op} = \left(S_1^H, S_2^H, S_3^H, S_4^H\right)$ is a SFN that consists of three intervals $L_{H_{op}}, C_{H_{op}}, R_{H_{op}}$ where $L_{H_{op}} = \left[S_1^H, S_2^H\right], C_{H_{op}} = \left[S_2^H, S_3^H\right], R_{H_{op}} = \left[S_3^H, S_4^H\right]$ as in Fig. 4
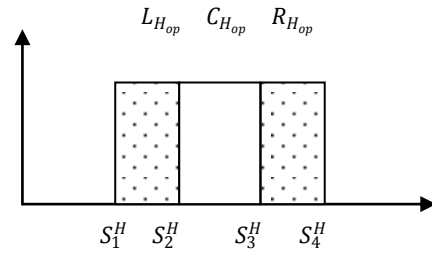
Fig. 4: Shadowed fuzzy number

We obtain the weights of the three intervals as follows:

$$w_L = \frac{S_2^H - S_1^H}{S_4^H - S_1^H} \qquad (32)$$

$$w_C = \frac{S_3^H - S_2^H}{S_4^H - S_1^H} \qquad (33)$$

$$w_R = \frac{S_4^H - S_3^H}{S_4^H - S_1^H} \qquad (34)$$

- The center values for the three intervals calculate as follows:

$$C_L = \frac{S_1^H + S_2^H}{2} \qquad (35)$$

$$C_C = \frac{S_2^H + S_3^H}{2} \qquad (36)$$

$$C_R = \frac{S_3^H + S_4^H}{2} \qquad (37)$$

- The value of the optimal solution M_{op} obtains as follows:

$$M_{op} = C_L \times w_L + C_C \times w_C + C_R \times w_R \qquad (38)$$

4.3 Fuzzy objective function coefficients and Precise constraints

Let us consider the following fuzzy linear programming problem with *m* constraints and *n* variables may formulate as follows:

$$\max \qquad \sum_{j=1}^{n} \tilde{c}_j x_j \qquad (39)$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$x_j \geq 0$$

$$\text{where} \qquad i \in \mathbb{N}, i = 1, \dots \dots, m$$

$$j \in \mathbb{N}, j = 1, \dots \dots, n$$

such that $\tilde{c}_{ij} = (c_{ij}^1, c_{ij}^2, c_{ij}^3), \mu_c; (\bar{c}_{ij}^1, \bar{c}_{ij}^2, \bar{c}_{ij}^3), v_c,$ is a triangular intuitionistic fuzzy numbers TIFNs and μ, v are membership and non-membership functions respectively. We can also use type-1 fuzzy number but the TIFN is used to describe steps of solving problem. a_{ij} and b_i are crisp numbers. The authors use the following steps to solve this HFLPP :

Step 1 : The TIFNs $\tilde{c}_{ij}$ transform to a SFNs using equations $(14 - 22)$ where $sh(\tilde{c}_{ij}) = (s_{c_{ij}}^1, s_{c_{ij}}^2, s_{c_{ij}}^3, s_{c_{ij}}^4)$.

Step 2 : The problem is reformulated as:

$$\max \qquad \sum_{j=1}^{n} s_{c_{ij}}^t x_j \qquad (40)$$

$$\text{subject to} \qquad \sum_{j=1}^{n} a_{ij} x_j \leq b_i$$

$$x_j \geq 0$$

$$\text{where} \qquad i \in \mathbb{N}, i = 1, \dots \dots, m$$

$$j \in \mathbb{N}, j = 1, \dots \dots, n$$

Step 3 : The problem can be solved four times with four parameters of a SFN of for the objective function coefficients using a classical linear programming problem.

Step 4 : The resulting maximum optimal solution of HFLPP is a SFN and transforms to one crisp number using equations $(32 - 38)$.

5. ILLUSTRATIVE EXAMPLES
Example 1

We use the adaptive example as mentioned in [1] where the parameters of the constraints are only type-1 fuzzy numbers. In these examples, TIFNs and TFNs are used to represent the constraints coefficients and right hand side values of the constraints. The problem is solved using steps in section 4.1. Consider the following HFLPP:

$$\max \qquad z = 5x_1 + 4x_2 \qquad (41)$$

$$\text{subject to} \qquad \bar{k}_{11} x_1 + \bar{k}_{12} x_2 \leq \tilde{t}_1$$

$$\tilde{a}_{11} x_1 + \tilde{a}_{12} x_2 \leq \bar{b}_1$$

$$x_1, x_2 \geq 0$$

where $\tilde{k}_{11}$= (3, 4, 5), (2, 4, 6); 1, 0 , $\tilde{k}_{12}$= (4, 5, 6), (2, 5, 7); 1, 0 and $\tilde{t}_1$= (20, 24, 30), (18, 24, 32); 1, 0. Also, $\tilde{a}_{11}$= (3, 4, 6) , $\tilde{a}_{12}$ = (0.5, 1, 2) and $\tilde{b}_1$= (6, 12, 15) are three TFNs. We transform the TIFNs to SFNs and the results are as the following:

$sh(\tilde{k}_{11})$ = (2.76, 3.35, 4.65, 5.24),

$sh(\tilde{k}_{12})$ = (4.12, 4.2, 5.66, 6.25),

$sh(\tilde{t}_{12})$ = (19.43, 21.98, 26.85, 30.54).

Also, we transform the TFNs to SFNs as the following

$sh(\tilde{a}_{11})$ = (3.06, 3.56, 4.89, 5.89),

$sh(\tilde{a}_{12})$ = (0.52, 0.77, 1.46, 1.96),

$sh(\tilde{b}_1)$ = (6.5, 9.5, 13.25, 14.75).

The problem can be reformulated to be as the following:

$$\max \qquad z = 5x_1 + 4x_2 \qquad (42)$$

$$\text{subject to} \qquad 2.76\,x_1 + 4.12\,x_2 \leq 19.43$$

$$3.35\,x_1 + 4.2\,x_2 \leq 21.98$$

$$4.65\,x_1 + 5.66\,x_2 \leq 26.85$$

$$5.24\,x_1 + 6.25\,x_2 \leq 30.54$$

$$3.06\,x_1 + 0.52\,x_2 \leq 6.5$$

$$3.56\,x_1 + 0.77\,x_2 \leq 9.5$$

$$4.89\,x_1 + 1.46\,x_2 \leq 13.25$$

$$5.89\,x_1 + 1.96\,x_2 \leq 14.75$$

$$x_1, x_2 \geq 0$$

Then we apply the classical linear programming steps using Matlab tool to obtain $x_1 = 1.27$, $x_2 = 3.7$ and $z = 21.16$. While the original problem used triangular fuzzy numbers only and its results were $x_1 = 1.5$, $x_2 = 3$ and $z = 19.5$

[1]. We find that, the results of the new approach are approximately equivalent the results of the original problem.

Example 2

This example is a modified version from [13] where TIFNs only are used to represent objective function coefficients, constraints coefficients and right hand side values. We use TIFNs, TFNs and Gaussian fuzzy numbers GFNs to represent objective function parameters and constraints parameters. The problem is solved using the steps in section 4.2. Consider the following HFLPP:

$$\max \quad z = \breve{5}x_1 + \breve{3}x_2 \qquad (43)$$

$$\text{subject to} \quad \widetilde{4}x_1 + \breve{3}x_2 \leq \widetilde{12}$$

$$\bar{1}x_1 + \breve{3}x_2 \leq \breve{6}$$

$$x_1, x_2 \geq 0$$

where $\breve{5}$ = (4, 5, 6), (4, 5, 6.1); 1, 0 and $\breve{3}$ = (2.5,3, 3.2), (2, 3, 3.5); 1, 0 are two TIFNs. $\widetilde{4}$ = (3, 4, 5), $\breve{3}$ = (2.3, 3, 4), are two TFNs. $\widetilde{1} = (m = 1; \sigma = 0.4)$ and $\breve{3} = (m = 3; \sigma = 0.2)$ are two GFNs. Also, $\widetilde{12}$ = (11, 12, 13) , (11, 12, 14) ; 1, 0 and $\breve{6}$ = (5.5, 6, 7.5), (5, 6, 8.1) ; 1, 0 are two TIFNs. We transform the TIFNs to SFNs and the results are the following:

$sh(\breve{5})$ = (4.06, 4.55, 5.48, 5.98),

$sh(\breve{3})$ = (2.18, 2.65, 3.16, 3.6),

$sh(\widetilde{12})$ = (11.05, 11.54, 12.65, 13.12),

$sh(\breve{6})$ = (5.21, 5.68, 6.81, 7.3).

Also, we transform TFNs to SFNs as the following:

$sh(\widetilde{4})$ = (3.04, 3.54, 4.46, 4.96),

$sh(\breve{3})$ = (2.33, 2.68, 3.46, 3.96),

$sh(\widetilde{1})$ = (0.36, 0.66, 1.34, 1.64),

$sh(\breve{3})$ = (2.68, 2.83, 3.17, 3.32).

The problem can be reformulated as the following:

$$\max \quad z = sh(\breve{5})x_1 + sh(\breve{3})x_2 \qquad (44)$$

$$\text{subject to} \quad 3.04\,x_1 + 2.33\,x_2 \leq 11.05$$

$$3.54\,x_1 + 2.68\,x_2 \leq 11.54$$

$$4.46\,x_1 + 3.46\,x_2 \leq 12.65$$

$$4.96\,x_1 + 3.96\,x_2 \leq 13.12$$

$$0.36\,x_1 + 2.68\,x_2 \leq 5.21$$

$$0.66\,x_1 + 2.83\,x_2 \leq 5.68$$

$$1.34\,x_1 + 3.17\,x_2 \leq 6.81$$

$$1.64x_1 + 3.32x_2 \leq 7.3$$

$$x_1, x_2 \geq 0$$

The resulting optimal value is SFN = (10.74, 12.04 , 14.5 , 15.82) , x_1= 2.65 and x_2 = 0 . We calculate the crisp optimal value where M_{op} = 13.28.

Example 3

This example is a modified version from [13] and [24] where TIFNs in [13] and interval-valued fuzzy numbers in [24] are used to represent the objective function coefficients. The TIFNs are used to represent parameters of the objective function. The problem is solved using steps in section 4.3. Let a factory produces automobiles and truck where each product need three processes. The production conditions are given in Table 1.

Table 1: Production conditions

Product type	Process 1 hours	Process 2 hours	Process 3 hours	Profit hundred $
Automobile	15	24	21	25
Truck	30	6	14	48
Total hours	45000	24000	28000	

Consider the following HFLPP:

$$\max \quad z = \widetilde{25}x_1 + \widetilde{48}x_2 \qquad (45)$$

$$\text{subject to} \quad 15x_1 + 30x_2 \leq 45000$$

$$24x_1 + 6x_2 \leq 24000$$

$$21x_1 + 14x_2 \leq 28000$$

$$x_1, x_2 \geq 0$$

where $\widetilde{25}$ = (19, 25, 33), (18, 25, 34); 1, 0 and $\widetilde{48}$ = (44, 48, 54), (43, 48, 56); 1, 0 are two TIFNs. The constraints parameters are crisp numbers. We transform the TIFNs to SFNs and the results are as the following:

$sh(\widetilde{25})$ = (21.87, 22.37, 28.44, 28.94),

$sh(\widetilde{48})$ = (45.66, 46.16, 50.86, 51.36).

The problem can be reformulated as the following:

$$\max \quad z = sh(\widetilde{25})x_1 + sh(\widetilde{48})x_2 \qquad (46)$$

$$\text{subject to} \quad 15x_1 + 30x_2 \leq 45000$$

$$24x_1 + 6x_2 \leq 24000$$

$$21x_1 + 14x_2 \leq 28000$$

$$x_1, x_2 \geq 0$$

The resulting optimal value is the SFN = (68490, 69240, 77795, 78670) , x_1= 500 , x_2 = 1250 . We calculated the crisp optimal value where M_{op} = 73587. In the original problem [24], x_1= 500 , x_2 = 1250 and the optimal solution = 73104.7.

Remark 1

When solving the problem in example 3 using a new approach, the values of objective function variables are (x_1= 0, x_2 = 1500) with SFN parameters (68490, 69240) and (x_1= 500 , x_2 = 1250) with SFN parameters (77795, 78670). We select solution that more near to optimal crisp solution where M_{op} = 73587.

In [13], the optimal solution for example 3 is (x_1= 0, x_2 = 1500) with optimal objective value 70500 and (x_1= 0, x_2 = 1500) with optimal objective value 62000.

5.1 Discussions

Based on the previous results, the following remarks are found:

1. The new approach can handle type-1 fuzzy numbers and higher type of fuzzy numbers i.e. intuitionistic fuzzy numbers. This feature provides more flexibility for representing linear programming problems.

2. The new approach can use different types of membership functions in one problem as in example 2 which TFNs and GFNs are used.

3. The new approach satisfies all the constraints including non-negative constraint.

4. Solving the same illustrative examples as in [1] and [24], using the newly proposed approach results in approximately the same optimal solution. This proves the validity of the proposed approach.

5.2 Advantage of the proposed method

The basic improvement in the new approach against previous approaches, it can handle different types of fuzzy numbers and also higher type like IFNs in the same problem with different cases of FLPP. This advantage provides more flexibility for solving FLPPs.

6. CONCLUSION

The hybrid fuzzy-linear programming problem is characterized by addressing different types of uncertain numbers that represent parameters values of problem. In this paper, we propose a new approach to solve this kind of problems using an improved form of shadowed fuzzy numbers. The proposed approach includes getting SFN from type-1 fuzzy numbers and intuitionistic fuzzy numbers which preserving the characteristics of uncertainty for them. The SFNs used in hybrid fuzzy-linear programming example to unify the form of parameters and in the same time simplify calculation operations. The newly proposed approach results are the same as the optimization results of the original example which prove the validity of the new approach. This new technique provides the ability to treat with more hybrid uncertain numbers for the future works in solving FLPPs.

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