Fixed Point Theorems for Iterated mappings via Caristi-Type Results

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ABSTRACT

In this paper we apply our former result [S. Lazaiz, K. Chaira, M. Aamri, and El M. Marhrani. Some remarks on Caristi type fixed point theorem. International Journal of Pure and Applied Mathematics, 104 (4): 585–597, 2015] to give a new results of iterated contraction mapping in complete metric space. As application we investigate the existence and uniqueness of solution for the nonlinear integral equation.

General Terms

Fixed point, Caristi's theorem

Keywords

Fixed point, Caristi's theorem, Remarks, Nonlinear Integral equations

1. INTRODUCTION

Fixed point theory plays an important role in nonlinear functional analysis and provides one of the best and important techniques for the existence of fixed point, coincidence point, coupled fixed point, and common fixed point for self-map under different situations. It is useful for the solution of fractional differential equations, functional equations, integral equations, matrix equations, linear inequalities, or integrodifferential equations and control theory (see [1, 2, 3, 4, 5, 6] and references therein).
Banach [7] has sorted out the successful and well-known result which later called the Banach contraction principle. Banach contraction principle [7] is the most versatile results in fixed point theory. This theorem have been studied by many authors (e.g. [8, 9, 10, 11, 12, 13]) and generalized in various ways.
In Hilbert space, Alber and Guerre-Delabriere [14] presented weak contraction by generalizing contraction and showed the presence of fixed points for a self map. Rhoades [15] proved this results in metric space under ϕ-weak contraction. Dutta and Choudhury [16] generalized ϕ-weak contraction to the concept of weak contraction and examined results for fixed point. Sehgal [17], in 1969, proves the following result for iterate mappings :

THEOREM 1. *Let (X, d) be a complete metric space and let $T : X \longrightarrow X$ be a continuous mapping. Assume that for some $k < 1$ and each $y \in B$ there is an integer $p \geq 1$ such that*

$$d\left(T^{p}x, T^{p}y\right) \leq kd\left(x, y\right)$$

for all $x \in X$.
Then there is a unique $u \in X$ such that $Tu = u$ and $T^{n}(y_0) \to u$ for each $y_0 \in X$.

Many results of iterate mappings has been studied and generalized [18, 19]. It is well known that Caristi's fixed point theorem (see [20]) generalizes also the Banach principle and leads to another forms of inequalities. Recently, Lazaiz et al [21] improve the Caristi fixed point theorem in the setting of iterated mapping and prove the following result :

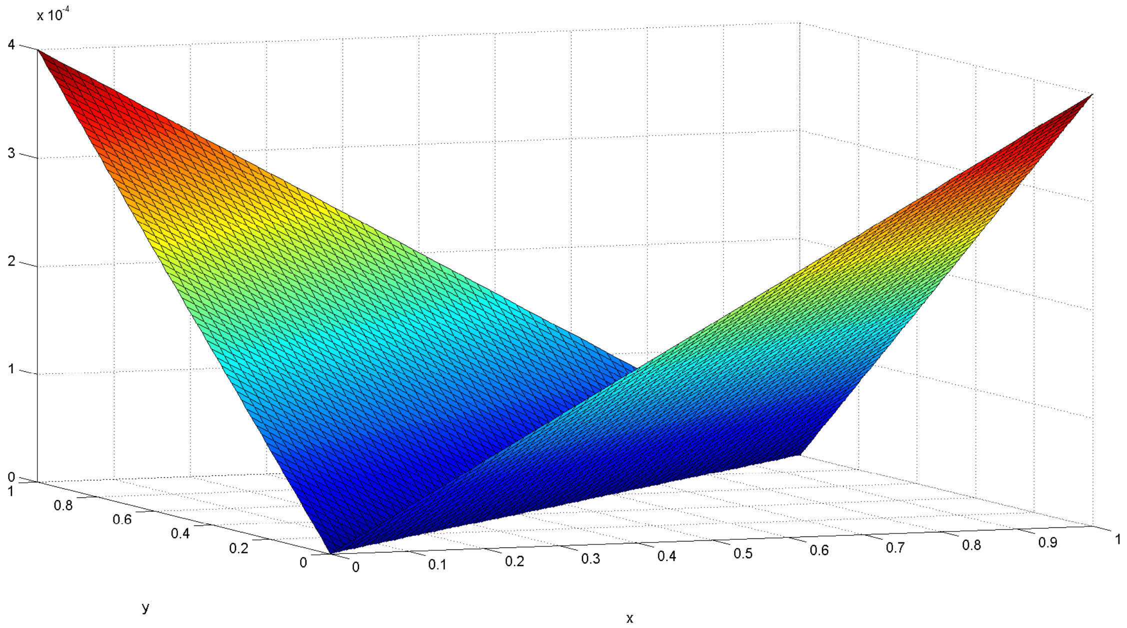
THEOREM 2. *Let (X, d) be a complete metric space, φ a mapping from X into a non-negative numbers and two co-prime positive integers p and q. T a self map of X, suppose that for all $x \in X$:*

$$\max\left\{d\left(x, T^{p}x\right), d\left(x, T^{q}x\right)\right\} \leq \varphi\left(x\right) - \varphi\left(T^{pq}x\right) \qquad (1)$$

and the mappings $x \mapsto d\left(x, T^{p}x\right)$ and $x \mapsto d\left(x, T^{q}x\right)$ are lower semi continuous. Then T has fixed point in X.

In view of the fact that the continuity of the mapping T was essential to prove Seghal's theorem, in the present paper, this assumption is dropped and replaced by a weak assumption, that is the iterate T^p is just lower semi-continuous for some positive integer $p \geq 1$. For that, we shall apply our former result (see [21]) to prove some contraction fixed point theorems for lower semi-continuous mappings and, extend results of Geraghty [22], Sehgal [17], Bryant [18] and Banach [7] to iterate mappings. The most results obtained follows easily from Caristi's inequality. As application, we investigate the existence and uniqueness of solution for a nonlinear integral equation.
The rest of the manuscript is organized as follows: In Section II, some standard assumptions are introduced, with the main theoreti-

Fig. 1. Geometric representation of function $\Pi(x,y) = \left|T^2(x) - T^2(y)\right|$ over $[0,1]^2$



cal results. In Section III, an application to prove the the existence and uniqueness of the solution for a nonlinear integral equation.

2. MAIN RESULTS

Let α be a function from $[0,\infty[$ into $[0,1[$ such that $\limsup_{s\to t^+} \alpha(s) < 1$ for all $t \in [0,\infty[$ and $\alpha(.)$ is non-decreasing. The following result is of Geraghty-type fixed point result.

THEOREM 3. *Let (X,d) be complete metric space, p and q two co-prime integers. Let $T : X \longrightarrow X$ be a mapping such that T^p and T^q are lower semi-continuous and for each $x, y \in X$*

$$d(T^{pq}x, T^{qp}y) \le \alpha(d(x,y))\, d(x,y) \tag{2}$$

Then T has a unique fixed point.

PROOF. Since T is a self mapping we get for all $x \in X$ and $y = T^p x$,

$$\begin{aligned} d\left(T^{pq}x, T^{qp+p}x\right) &\le \alpha(d(x,T^px))\, d(x,T^px) \\ &\le d(x,T^px) - (1-\alpha(d(x,T^px)))\, d(x,T^px) \end{aligned}$$

then

$$(1-\alpha(d(x,T^px)))\, d(x,T^px) \le d(x,T^px) - d\left(T^{pq}x, T^{qp+p}x\right) \tag{3}$$

it means that

$$d(x,T^px) \le \frac{d(x,T^px)}{1-\alpha(d(x,T^px))} - \frac{d(T^{pq}x, T^{qp+p}x)}{1-\alpha(d(x,T^px))}$$

and Since $\alpha(t)$ is non-decreasing and $d(T^{pq}x, T^{qp}y) < d(x,y)$ we get

$$\alpha(d(T^{pq}x, T^{qp}y)) < \alpha(d(x,y)) \Rightarrow \frac{1}{1-\alpha(d(T^{pq}x, T^{qp}y))} < \frac{1}{1-\alpha(d(x,y))}$$

Define $\varphi : X \longrightarrow \mathbb{R}_+$ by

$$\varphi(x) = \frac{d(x,T^px)}{1-\alpha(d(x,T^px))}$$

hence by (3) it follows that :

$$d(x,T^px) \le \varphi(x) - \varphi(T^{pq}x)$$

and since $x \mapsto d(x,T^px)$ is lower semi-continuous, all assumptions of theorem 2 hold, which implies that T has a fixed point u in X.
Uniqueness : Suppose that there exists $v \in X$ such that $Tv = v$ with $v \ne u$, then

$$d(u,v) = d(T^{pq}u, T^{pq}v) \le \alpha(d(u,v))\, d(u,v) < d(u,v)$$

contradiction, then $u = v$.

$\square$

REMARK 1. *In view of the fact that the continuity of T was essential in the proof of [[22], Theorem 1.3], it is remarkable that this result remains true without such assumption.*

EXAMPLE 1. *In this example we use the following data: $X = [0,1]$, $d(x,y) = |x-y|$, $\alpha(x) = \frac{2\arctan(x)}{\pi+1}$ and*

$$T(x) = -\frac{x}{50} + 1 \tag{4}$$

thus, for $p = 1$ and $q = 2$, we have

$$T^2(x) = \frac{x}{2500} + \frac{49}{50}$$

Fig. 2. Geometric representation of functions Π and Φ: Φ is the higher surface and Π is the lower one.

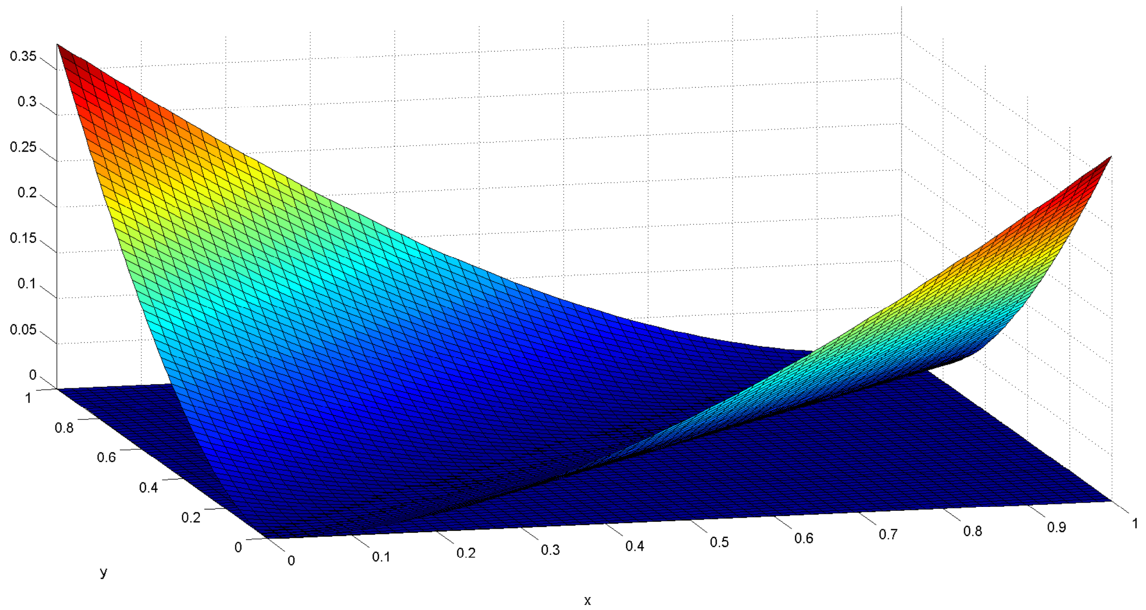
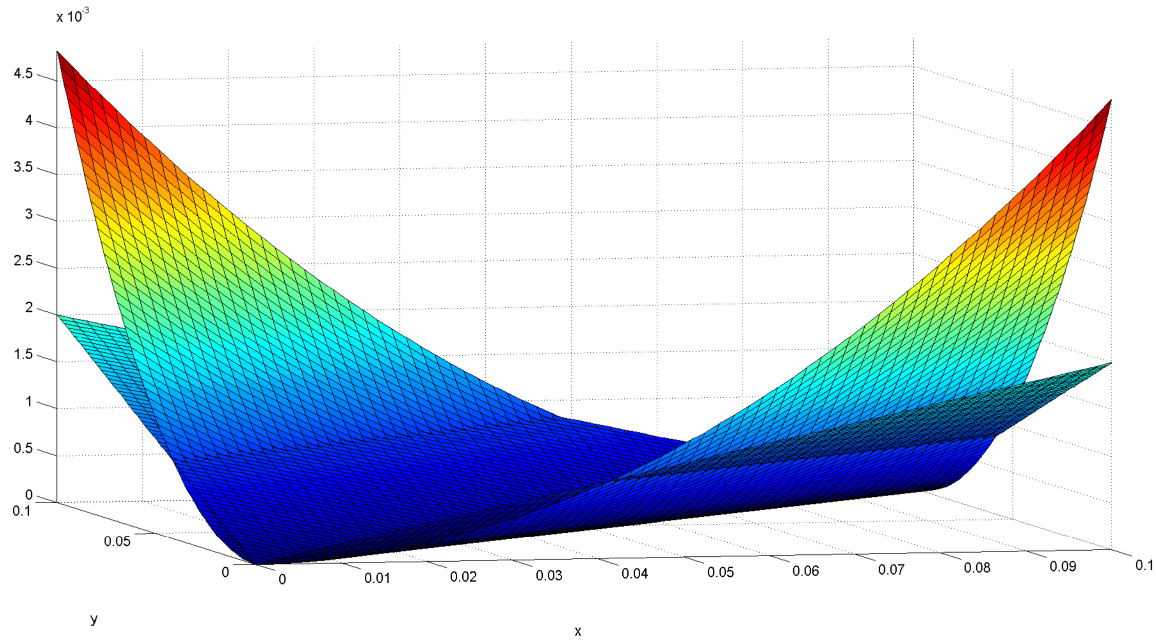


Fig. 3. Geometric representation of functions Γ and Φ over $[0, 0.1]^2$



In order to verify our results, we use the following geometric representations. For that we need some notations

$$\begin{aligned}\Gamma(x,y) &= |T(x) - T(y)| \\ \Pi(x,y) &= \left|T^2(x) - T^2(y)\right| \\ \Phi(x,y) &= \frac{2\arctan(|x-y|)}{\pi+1}|x-y|\end{aligned}$$

Figure 1 shows the geometric representation of function Π *over* $[0,1]^2$*, we should note that the vertical axis, of this Figure, have a maximum value equal to* 4×10^{-4}*, while in Figure 2, when the both functions* Π *and* Φ *are plotted, the maximal value of the vertical axis is* 0.4*, for that, it seems that the lower surface takes a zero* *values. It is clear from Figure 2 that*

$$\Pi(x,y) \leq \Phi(x,y), \textit{ for all } (x,y) \in [0,1]^2$$

which means that

$$\left|T^2(x) - T^2(y)\right| \leq \frac{2\arctan(|x-y|)}{\pi+1}|x-y|, \textit{ for all } (x,y) \in [0,1]^2 \quad (5)$$

Then all conditions of Theorem 3 are satisfied and hence T has a unique fixed point in $[0,1]$, that is $\frac{50}{51}$.
Figure 3 proves that the chosen function T in (4) doesn't verify the condition (5), which means that for $p = q = 1$, the condition (5) is not satisfied for all $(x,y) \in [0,1]^2$. To show that, we chose to plot the functions Γ and Φ over $[0,0.1]^2$.
If $q = 1$ and $\alpha \equiv k < 1$ Sehgla's theorem follows as a corollary.

COROLLARY 1. *Let (X,d) be complete metric space and $p \geq 1$ an integer. Let $T : X \longrightarrow X$ be a continuous mapping such that for each $x, y \in X$*

$$d(T^p x, T^p y) \leq k d(x,y) \quad (6)$$

for some $k \in [0,1[$, then T has a unique fixed point $u \in X$ and $T^n(y_0) \to u$ for each $y_0 \in X$.

PROOF. By the above theorem, T has a unique fixed point $u \in X$. We must only show that $T^n(y_0) \to u$ for each $y_0 \in X$, let $y_0 \in X$ then by inequality (2) we get for n large enough, write $n = rpq + s$ with $m > 0$ and $0 \leq r < n$ then

$$\begin{aligned} d(T^n y_0, u) &= d\left(T^{mpq+r} y_0, T^{pq} u\right) \\ &\leq kd\left(T^{(m-1)pq+r} y_0, u\right) \leq \cdots \\ &\leq k^m d(T^r y_0, u) \leq k^m M \end{aligned}$$

and it follows that $T^n y_0 \to u$.
If $p = q = 1$ and $\alpha \equiv k < 1$, we get the famous Banach principle.

□

COROLLARY 2. *Let (X,d) be complete metric space and $T : X \longrightarrow X$ a continuous mapping such that for each $x, y \in X$*

$$d(Tx, Ty) \leq k d(x,y) \quad (7)$$

for some $k \in [0,1[$, then T has a unique fixed point and $T^n(y_0) \to u$ for each $y_0 \in X$.

THEOREM 4. *Let (X,d) be complete metric space, p and q two co-prime integers. $T : X \longrightarrow X$ a mapping, such that T^p and T^q are lower semi-continuous and for each $x, y \in X$*

$$\eta(d(T^{pq}x, T^{qp}y)) \leq \eta(d(x,y)) - d(x,y) \quad (8)$$

where $\eta : [0,\infty) \to [0,\infty)$ and $\eta(t) = 0$ if $t = 0$.
Then T has a unique fixed point.

PROOF. It suffices to choose

$$\varphi(x) = \eta(d(x, T^p x))$$

for all $x \in X$, $y = T^p x$, then the inequality (8) shows that

$$d(x, T^p x) \leq \varphi(x) - \varphi(T^{pq} x)$$

Using theorem 2, T has a fixed point $u \in X$. For the uniqueness, suppose there exists $v \in X$ with $v \neq u$ such that $Tv = v$ then by (8) we get

$$\begin{aligned} \eta(d(u,v)) &= \eta(d(T^{pq}u, T^{qp}v)) \\ &\leq \eta(d(u,v)) - d(u,v) \\ &< \eta(d(u,v)) \end{aligned}$$

Contradiction, hence $u = v$.

□

THEOREM 5. *Let (X,d) be complete metric space, $p \geq 1$ an integer. Let $T : X \longrightarrow X$ a mapping such that T^p is lower semi-continuous and for all $x, y \in X$*

$$d(x,y) \leq \psi(x,y) - \psi(T^p x, T^p y) \quad (9)$$

Then T has a unique fixed point.

PROOF. For each $x \in X$, let $y = T^p x$ and $\varphi(x) = \psi(x, T^p x)$. Then for each $x \in X$

$$d(x, T^p) \leq \varphi(x) - \varphi(T^p x)$$

and T^p is a lower semi-continuous mapping. Thus, applying Theorem 2 for $q = 1$ implies that T has a fixed point $u \in X$ such that $Tu = u$.
Uniqueness : Suppose that there exists $v \in X$ such that $Tv = v$, then

$$d(u,v) \leq \psi(u,v) - \psi(T^p u, T^p v) = \psi(u,v) - \psi(u,v) = 0$$

thus, $u = v$.

□

THEOREM 6. *Let (X,d) be complete metric space and $p \geq 1$ an integer. Let $T : X \longrightarrow X$ a mapping such that T^p is lower semi-continuous and for all $x, y \in X$*

$$\max\{d(T^p x, T^p y), d(T^q x, T^q y)\} \leq \theta(d(x,y)) \quad (10)$$

where $\theta : [0,\infty) \to [0,\infty)$ is such that $\theta(t) < t$ for each $t > 0$ and $\theta(.)$ is a non-decreasing map.
Then T has a unique fixed point.

PROOF. Define

$$\psi(x,y) = \begin{cases} \frac{d(x,y)}{1 - \frac{\theta(d(x,y))}{d(x,y)}} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

then 10 shows that

$$d(T^p x, T^p y) \leq d(x,y) - \left(1 - \frac{\theta(d(x,y))}{d(x,y)}\right) d(x,y)$$

hence

$$d(x,y) \leq \frac{d(x,y)}{1 - \frac{\theta(d(x,y))}{d(x,y)}} - \frac{d(T^p x, T^p y)}{1 - \frac{\theta(d(x,y))}{d(x,y)}}$$

since $\frac{\theta(t)}{t}$ is non-decreasing and $d(T^p x, T^p y) < d(x,y)$, we get

$$d(x,y) \leq \frac{d(x,y)}{1 - \frac{\theta(d(x,y))}{d(x,y)}} - \frac{d(T^p x, T^p y)}{1 - \frac{\theta(d(T^p x, T^p y))}{d(T^p x, T^p y)}}$$

then

$$d(x,y) \leq \psi(x,y) - \psi(T^p x, T^p y)$$

and so by applying Theorem 6, one can conclude that T has a unique fixed point.

□

3. APPLICATION

The main aim of this section is to investigate the existence and uniqueness of solution for the nonlinear integral equation:

$$x(t) = \phi(t) + \int_0^t K(s, x(s))\, ds \tag{11}$$

where $T > 0$, $x \in \mathcal{C}[0,T]$ space of all continuous functions form $[0,T]$ into $\mathbb{R}$, $\phi : [0,T] \to \mathbb{R}$ and $K : [0,T] \times \mathbb{R} \to \mathbb{R}$ are two given function.
Let $X = \mathcal{C}[0,T]$ endowed by the metric $d : X \times X \to \mathbb{R}_+$such that for each $x, y \in X$

$$d(x,y) = \sup_{t \in [O,T]} |x(t) - y(t)|$$

clearly (X, d) is complete metric space.
Consider the mapping $F : X \to X$ defined by

$$F(x)(t) = \phi(t) + \int_0^t K(s, x(s))\, ds$$

for all $x \in X$.
In the nonlinear integral equation (11), suppose that the following condition hold

THEOREM 7. *(1) $K : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous mapping;*
(2) For all $x, y \in X$ and $s \in [0,T]$ there exists some positive integer $p \geq 1$ such that

$$|K(s, F^p(x)(s)) - K(s, F^p(y)(s))| \leq \frac{1}{T} \alpha(|x(s) - y(s)|)\, |x(s) - y(s)|$$

where $\alpha(.)$ is non-decreasing function from $[0, \infty[$ into $[0, 1[$ such that $\limsup_{s \to t^+} \alpha(s) < 1$ for all $t \in [0, \infty[$.
Then the nonlinear integral equation (11) has a unique solution.

PROOF. We will show that F is contraction mapping in the sense of theorem 3. Assume that $x, y \in X$ and $t \in [0,T]$. Then we get

$$\begin{aligned}
|F^p(x)(t) - F^p(y)(t)| &= \left| \int_0^t K(s, F^p(x)(s))\, ds - \int_0^t K(s, F^p(y)(s))\, ds \right| \\
&= \left| \int_0^t [K(s, F^p(x)(s))\, ds - K(s, F^p(y)(s))]\, ds \right| \\
&\leq \int_0^t |K(s, F^p(x)(s))\, ds - K(s, F^p(y)(s))|\, ds \\
&\leq \frac{1}{T} \int_0^t \alpha(|x(s) - y(s)|)\, |x(s) - y(s)|\, ds \\
&\leq \frac{T}{T} \alpha(|x(s) - y(s)|)\, d(x,y) \\
&\leq \alpha(d(x,y))\, d(x,y)
\end{aligned}$$

which implies that

$$\sup_{t \in [O,T]} |F^p(x)(t) - F^p(y)(t)| \leq \alpha(d(x,y))\, d(x,y)$$

and hence

$$d(F^p x, F^p y) \leq \alpha(d(x,y))\, d(x,y)$$

for all $x, y \in X$. It follows that where $q = 1$ that all the conditions of Theorem 3 are satisfied and hence F has a unique fixed point in X. This implies that there exists a unique solution of the nonlinear equation (11).

□

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