Fixed Point Theorems in Multiplicative Metric Spaces

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ABSTRACT
In this paper, we prove fixed point theorems in multiplicative metric spaces.

General Terms
The set of positive real numbers ℝ+ is not complete according to the usual metric IR, but it is complete in the sense of multiplicative metric spaces.

Keywords
Multiplicative metric spaces, fixed point.

1. INTRODUCTION
The set of positive real numbers ℝ+ is not complete according to the usual metric, IR . To overcome this problem, in 2008, Bashirov et. al. [2] introduced the concept of multiplicative metric spaces as follows:

Definition 1.1.[2] Let X be a non-empty set. A multiplicative metric is a mapping d: X×X → ℝ+ satisfying the following conditions:

i. d(x, y) ≥ 1 for all x, y ∈ X and d(x, y) = 1 if and only if x=y;

ii. d(x, y) = d(y, x) for all x, y ∈ X;

iii. d(x, y) ≤ d(x, z). d(z, y) for all x, y, z ∈ X (multiplicative triangle inequality). (iii) d(x, y) ≤ d(x, z). d(z, y) for all x, y, z ∈ X (multiplicative triangle inequality).

Example 1.1. [4] Let d: ℝ × ℝ→ [1, ∞) be defined as

d(x, y) = $a^{|x-y|}$,where x, y ∈ ℝ and a > 1. Then d(x, y) is a multiplicative metric and (X, d) is called a multiplicative metric space. We call it usual multiplicative metric spaces.

Example1.2.[4] Let (X, d) be a metric space .Define a mapping d_a on X by $d_a(x, y) = a^{d(x,y)}$ where a > 1 is a real number and $d_a(x, y) = a^{d(x,y)} = \begin{cases} 1 \ if \ x = y \\ a \ if \ x \neq y. \end{cases}$

The metric $d_a(x, y)$ is called discrete multiplicative metric and X together with metric d_a i.e., (X, d_a) is known as a discrete multiplicative metric space.

For more detail on multiplicative metric topology one can refer to ([3]).

Definition 1.2.([3]) Let (X, d) be a multiplicative metric space. A sequence $\{x_n\}$ in X said to be a

(i) multiplicative convergent sequence to x, if for every multiplicative open ball

(ii) $B_\epsilon(x)$ = { y | d(x, y) < ∈} , ∈ > 1, there exists a natural number N such that $x_n \in B_\epsilon(x)$ for all n ≥ N, i. e, $d(x_n, x) \to 1$ as n → ∞.(ii) multiplicative Cauchy sequence if for all ∈> 1, there exists N ∈ ℕ such that $d(x_n, x_m)$ < ∈ for all m, n > N i. e , $d(x_n, x_m) \to 1$ as n → ∞.

A multiplicative metric space is called complete if every multiplicative Cauchy sequence in X is multiplicative converging to x ∈ X.

In 2012, Özavşar and Çevikel [3] proved Banach-contraction principle mappings in the setting of multiplicative metric spaces akin to Banach-contraction principle mappings in metric spaces.

Let (X, d) be a complete multiplicative metric space and let f: X → X be a multiplicative contraction if there exists a real constant λ ∈ [0, 1) such that

d(f(x), f(y)) ≤ $d(x,y)^\lambda$ for all x, y ∈ X. Then f has a unique fixed point.

2. MAIN RESULTS
Now we prove a result for a map that satisfy the contractive type condition.

Theorem 2.1. Let (X, d) be a complete multiplicative metric space. Suppose the mapping

f : X → X be a continuous self- mapping satisfies the condition

(2.1)d(fx,fy)≤ $[d(x,y)]^{a_1}.[d(x,fy)]^{a_2}.[d(fx,y)]^{a_3}.[d(fy,y)]^{a_4}$

for all x, y ∈ X, where a_1, a_2, a_3, a_4 ≥ 0 and $a_1 + 2a_2 + 2a_3 + a_4 < 1$.

Then f has a unique fixed point in X.

Proof. Let $\{x_n\}$be a sequence in X defined as follows.

Let $x_0 \in X$. For this x_0 there exists x_1 such that $f(x_0) = x_1$. Again, for this x_1 there exists x_2 such that $f(x_1) = x_2$. Continue like this we get $f(x_n) = x_{n+1}$.

Consider (2.1), we have

$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},fx_n)]^{a_2}.[d(fx_{n-1},x_n)]^{a_3}.[d(fx_n,x_n)]^{a_4}$

$\leq [d(x_{n-1},x_n)]^{a_1}.[d(x_{n-1},x_{n+1})]^{a_2}.$

$[d(x_n,x_n)]^{a_3}.[d(x_{n+1},x_n)]^{a_4}.$

On simplification ,we have

$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n)$

$\leq [d(x_{n-1},x_n)]^{a_1+a_2+a_3}.$

$[d(x_{n+1},x_n)]^{a_2+a_3+a_4}.$

This implies that

$d(x_n, x_{n+1}) \leq [d(x_{n-1},x_n)]^{h}$,where

$$h = \frac{a_1+a_2+a_3}{1-(a_2+a_3+a_4)} < 1.$$

Similarly, $d(x_{n-1}, x_n) \leq [d(x_{n-2},x_{n-1})]^{h}$.

$d(x_n, x_{n+1}) \leq [d(x_{n-2},x_{n-1})]^{h^2}$.

Continue in a similar fashion, we get

$d(x_n, x_{n+1}) \leq [d(x_0,x_1)]^{h^n}$

For n > m,

$d(x_n, x_m) \leq d(x_n, x_{n-1}) \cdot d(x_{n-1}, x_{n-2}) \cdots$

$d(x_m, x_{m+1})$

$\leq d(x_0,x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$

$\leq d(x_0,x_1)^{\frac{h^m}{1-h}}$.

This implies $d(x_n, x_m) \to 1$ as $n, m \to \infty$.

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now we show that z is fixed point of f by assuming that f is continuous or f is not continuous.

(i) f is continuous, since $x_n \to z$ $(n \to \infty)$ and f is continuous so, $\lim_{n\to\infty} f\,x_n = fz = \lim_{n\to\infty} x_{n+1} = z$, i.e., z is a fixed point of f.

(ii) f is not continuous then

$d(fz, z) \leq d(fx_n, fz).\, d(fx_n z)$

$\leq [d(z,x_n)]^{a_1}.[d(x_n,fz)]^{a_2}.[d(fx_n,z)]^{a_3}.[d(fz,z)]^{a_4}.$

$d(fz, z) \leq [d(z,fz)]^{a_2+a_4}$ gives fz = z, i.e., z is a fixed point of f.

Uniqueness. Suppose z, w $(z \neq w)$ be two fixed point of f, then

$d(z, w) = d(fz, fw) \leq [d(z,w)]^{a_1}.[d(z,fw)]^{a_2}.[d(fz,w)]^{a_3}.[d(fw,w)]^{a_4}$

$d(z, w) \leq [d(z,w)]^{a_1+a_2+a_3}$ this implies that $d(z, w) = 1$ i.e., z = w.

Hence f has a unique fixed point.

Cor. 2.1. On Putting $a_2 = a_3 = a_4 = 0$ in (2.1) , we get Banach-contraction [3] in the sense of multiplicative metric spaces.

Now we prove a result for a map that satisfy the rational type contractive condition .

Theorem 2.2. Let f be a continuous self- mapping defined on a complete multiplicative metric space X, further f satisfies the following conditions

(2.2) d(fx, fy)

$$\leq \max\left\{d(x,fx), d(y,fy), d(x,fy), d(y,fx), \frac{d(x,fx).d(y,fy).d(y,fx)}{d(x,y)}\right\}^{a_1}$$

for all $x, y \in X$ and $a_1 < 1$. Then f has a unique fixed point.

Proof. Let $\{x_n\}$ be a sequence in X, defined as follows:

Let $x_0 \in X$, $f(x_0) = x_1, f(x_1) = x_2, \cdots, f(x_n) = x_{n+1}$.

If $x_n = x_{n+1}$ for some $n \in N$ then x_n is a fixed point of f.

Taking $x_n \neq x_{n+1}$ for all $n \in N$

Consider (2.2), we have

$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1})$

$$\leq \left\{\max\left\{d(x_n,fx_n), d(x_{n-1},fx_{n-1}), d(x_n,fx_{n-1}), d(x_{n-1},fx_n), \frac{d(x_n,fx_n).d(x_{n-1},fx_{n-1}).d(x_{n-1},fx_n)}{d(x_n,x_{n-1})}\right\}\right\}^{a_1}$$

$$\leq \max\left\{d(x_n,x_{n+1}), d(x_{n-1},x_n), d(x_n,x_n), d(x_{n-1},x_{n+1}), \frac{d(x_n,x_{n+1}).d(x_{n-1},x_n).d(x_{n-1},x_{n+1})}{d(x_n,x_{n-1})}\right\}^{a_1}$$

$$= [d(x_n,x_{n+1})^2.\, d(x_{n-1},x_n)]^{a_1}.$$

d(x_{n+1}, x_n) $\leq [d(x_n, x_{n+1})]^{2a_1}. [d(x_n, x_{n-1})]^{a_1}$,

d(x_n, x_{n+1}) $\leq [d(x_{n-1}, x_n)]^h$,where h = $\frac{a_1}{1-2a_1}$ < 1.

Similarly, d(x_{n-1}, x_n) $\leq [d(x_{n-2}, x_{n-1})]^h$,

d(x_n, x_{n+1}) $\leq [d(x_{n-2}, x_{n-1})]^{h^2}$.

Continue like this we get,

d(x_n, x_{n+1}) $\leq [d(x_0, x_1)]^{h^n}$

For n > m, d(x_n, x_m) ≤ d(x_n, x_{n-1}) · d(x_{n-1}, x_{n-2}) · · · d(x_m, x_{m+1})

$$\leq d(x_0, x_1)^{h^{n-1}+h^{n-2}+\cdots h^m}$$

$$\leq d(x_0, x_1)^{\frac{h^m}{1-h}}$$

. This implies d(x_n, x_m) →1(n, m → ∞).

Hence (x_n) is a Cauchy sequence. By the multiplicative completeness of X, there is z ∈ X such that x_n → z as n →∞.

Now we show that z is fixed point of f.

Since f is continuous and x_n → z (n →∞) so, $\lim_{n\to\infty} f\, x_n$= fz = $\lim_{n\to\infty} x_{n+1}$= z,

i.e., z is a fixed point of f.

Uniqueness follows easily.

3. REFERENCES

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