Edge Domination Number of Corona Product Graph of a Cycle with a Complete Graph

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ABSTRACT
Graph theory is one of the most flourishing branches of modern mathematics and computer applications. Domination in graphs has been studied extensively in recent years and it is an important branch of graph theory. An introduction and an extensive overview on domination in graphs and related topics is surveyed and detailed in the two books by Haynes et al. [7, 8]. In this paper we present some results on minimal edge dominating sets of corona product graph of cycle with a complete graph.

Keywords
Corona Product, edge dominating set, edge domination number.

Subject Classification: 68R10

1. INTRODUCTION
Domination Theory has a wide range of applications to many fields like Engineering, Communication Networks, Social sciences, linguistics, physical sciences and many others. Allan, R.B. and Laskar, R.[1], Cockayne, E.J. and Hedetniemi, S.T. [4] have studied various domination parameters of graphs.

Frucht and Harary [6] introduced a new product on two graphs G1 and G2, called corona product denoted by G1⊙G2. The object is to construct a new and simple operation on two graphs G1 and G2 called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of G1 and of G2 .

The concept of edge domination was introduced by Mitchell and Hedetniemi [11] and it is explored by many researchers. Arumugam and Velammal [3] have discussed the edge domination in graphs while the fractional edge domination in graphs is discussed in Arumugam and Jerry [2]. The complementary edge domination in graphs is studied by Kulli and Soner [10] while Jayaram [9] has studied the line dominating sets and obtained bounds for the line domination number. The bipartite graphs with equal edge domination number and maximum matching cardinality are characterized by Dutton and Klostermeyer [5] while Yannakakis and Gavril [12] have shown that edge dominating set problem is NP-complete even when restricted to planar or bipartite graphs of maximum degree. The edge domination in graphs of cubes is studied by Zelinka [13].

2. CORONA PRODUCT OF C_n AND K_m AND ITS PROPERTIES
The corona product of a cycle C_n with a complete graph K_m is a graph obtained by taking one copy of a n – vertex graph C_n and n copies of K_m and then joining the i^{th} vertex of C_n to every vertex of i^{th}copy of K_m.This graph is denoted by $C_n \odot K_m$.

The vertices of C_n are denoted by$v_1, v_2, \ldots, v_n$. The edges in C_n are denoted by$e_1, e_2, \ldots, e_n$where $e_i$is the edge joining the vertices v_i andv_{i+1},$i \neq n$. For $i = n, e_n$ is the edge joining the vertices$v_n$andv_1.

The vertices in the i^{th} copy of K_m are denoted by$w_{i1}, w_{i2}, \ldots, w_{im}$.The edges in the i^{th} copy of K_m are denoted byl_{ij} , $j = 1,2, \ldots, \frac{m(m-1)}{2}$.

The edges in the outer Hamilton cycle of thei^{th} copy of K_m are labelled byl_{ij} where l_{ij} is the edge joining the vertices w_{ij} and$w_{i(j+1)}$, $j = 1,2, \ldots, (m-1)$. For $j = m$, l_{im} is the edge joining the vertices w_{im} andw_{i1} .

There are another type of edges in G denoted byh_{ij} , $i = 1,2, \ldots, n$where$j = 1,2, \ldots, m$ is the edge joining thevertices v_i of C_n to w_{ij} of i^{th} copy ofK_m. These edges which are in G and related to the i^{th} copy of $K_m$are denoted by$h_{i1}, h_{i2}, \ldots, h_{im}$ and these are adjacent to each other and incident with the vertex $v_i$of C_n.

Theorem 2.1: The graph $G = C_n \odot K_m$ is a connected graph.

Proof: Consider the graph $G = C_n \odot K_m$.

By the definition of the corona product, we know that the i^{th} vertex of C_n is adjacent to each vertex of i^{th} copy of $K_m$in G. That is the vertices in C_n are connected to the vertices of K_m. Since C_n and K_m are connected, it follows that $G = C_n \odot K_m$ is also connected. ■

Theorem 2.2: The degree of a vertex v in

G = $C_n \odot K_m$ is given by

$$d(v) = \begin{cases} m+2, & if v \in C_n, \\ m, & if v \in K_m. \end{cases}$$

Proof: By the definition of the graph G, the i^{th} vertex v of C_n is joined to m vertices of i^{th}copy of K_m in G. The vertex $v \in C_n$ is adjacent to two of vertices of C_n . Therefore the degree of a vertex $v \in C_n$ in G is $m + 2$.

There are m vertices in each copy of K_m , and each vertex $v \in K_m$ is of degree $m - 1$. Since this vertex is adjacent to a corresponding vertex of C_n in G, it follows that the degree of a vertex $v \in K_m$ in G is m.■

Theorem 2.3: The number of vertices and edges in $G = C_n \odot K_m$ is given respectively by

1. $|V(G)| = n(m+1)$,
2. $|E(G)| = \frac{n}{2}(m^2 + m + 2)$.

Proof: Consider the graph $G = C_n \odot K_m$ with vertex set V.

In G, we know that n, m denote the number of vertices of C_n and the complete graph K_m respectively.

By the definition, the vertex set of G contains the vertices of C_n and the vertices of K_m in$n -$ copies.

Hence, it follows that

$$|V(G)| = n + nm = n(m+1) .$$

By Theorem 2.2, the degree of a vertex v is given by

$$d(v) = \begin{cases} m+2 , & if v \in C_n, \\ m , & if v \in K_m. \end{cases}$$

Hence $|E(G)|$

$$= \frac{1}{2}\left(\sum_{v \in C_n} \deg(v) + n \sum_{v \in K_m} \deg(v) \right)$$

$$= \frac{1}{2}\left(n(m+2) + n(\underbrace{m+m+.....+m}_{m-times}) \right)$$

$$= \frac{n}{2}(m + 2 + m^2)$$

$$= \frac{n}{2}(m^2 + m + 2).$$ ■

Theorem 2.4: The graph $G = C_n \odot K_m$ is non - hamiltonian.

Proof: Let V denote the vertex set of the graph$G = C_n \odot K_m$.

Let us denote the vertices of C_n by $v_1, v_2, \dots\dots., v_n$ and denote the vertices of i^{th} copy of K_m by $u_{i1,} u_{i2}, \dots\dots\dots, u_{im}$.

In G, the i^{th} vertex of C_n is adjacent to each vertex of i^{th} copy of K_m .

Then $v_i, u_{i1,} u_{i2}, \dots\dots\dots, u_{im}, v_i$ form a cycle in G.

In this way, we have n disjoint cycles in G, and these cycles are formed at each vertex of C_n. Since the vertices of C_n are in a cycle, we can not find a closed path in G that contains all the vertices of G.

Therefore$G = C_n \odot K_m$ is non - hamiltonian.■

Theorem 2.5: The graph $G = C_n \odot K_m$ is eulerian, if m is even.

Proof: We know that a n– vertex cycle is eulerian and a complete graph with m – vertices is eulerian, if m is odd.

By Theorem 2.2, there are n- vertices of C_n which are of degree $m+2$ in G and there are m – vertices in each copy of K_m whose degree is m.

Then the following cases arise:

Case 1: Suppose m is even.

Then each vertex of G is of even degree. Hence, it follows that G is eulerian.

Case 2: Suppose m is odd.

Then each vertex of G is of odd degree. Hence, it follows that G is not eulerian. ■

Theorem 2.6: The graph $G = C_n \odot K_m$ is not bipartite.

Proof: Suppose m is odd. Then K_m contains an odd cycle,so that G has an odd cycle. Hence G is not bipartite.

Suppose m is even. Then K_m contains an even cycle. Since the vertices of K_m in each copy are joined to a corresponding vertex in C_n, it follows that these $m+1$ vertices form an odd cycle in G. Hence G is not bipartite. ■

Theorem 2.7: The adjacency of an edge e in $G = C_n \odot K_m$ is given by

$$adj(e) = \begin{cases} 2m+2, \text{if } e = e_i \in C_n, \\ 2m-2, \text{if } e = l_{ij} \in i^{th} \text{ copy of} K_m, \\ 2m, \text{if } e = h_{ij} \in G = C_n \odot K_m. \end{cases}$$

Proof: Consider the graph $G = C_n \odot K_m$.We know that if $e = uv$ is any edge in a graph, joining the vertices u and v then $adj(e) = d(u) + d(v) - 2$.

Case 1: Let $e = e_i \in C_n$.The edge $e = e_i \in C_n$ is incident to the vertices v_i and $v_{i+1}$of C_n. The vertex v_i is incident to two edges $viz., e_i , e_{i+1}$, if $i \neq n$ of $C_n$and incidentto two edges e_n , e_1 of C_n if

$i = n$ and incident to m edges drawn from vertex$v_i$to the vertices$w_{i1}, w_{i2}, \dots, w_{im}$of i^{th} copy of K_m which are $h_{i1}, h_{i2}, \dots, h_{im}$ respectively.

Similarly the vertex $v_{i+1}$is incident to two edges $viz., e_i , e_{i+1}$of C_n and m edges $h_{(i+1)1}, h_{(i+1)2}, \dots, h_{(i+1)m}$ drawn from the vertex $v_{i+1}$to the m vertices $w_{(i+1)1}, w_{(i+1)2}, \dots, w_{(i+1)m}$ of $(i+1)^{th}$ copy of K_m.

So $deg(v_i) = m+2$ and $deg(v_{i+1}) = m+2$.

Therefore $adj(e) = (m+2) + (m+2) - 2 = 2m+2$.

Case 2: Let $e = l_{ij} \in i^{th}$ copy of K_m. Then

l_{ij} is the edge joining the vertex w_{ij} to vertex$w_{i(j+1)}$ in the i^{th} copy ofK_m.Since K_m is a complete graph, every vertex w_{ij} in$K_m$has degree$(m-1)$. But every vertex w_{ij} in K_m is joined to a vertex v_i of $C_n$and this edge is denoted byh_{ij}.

Therefore $deg(w_{ij}) = (m-1) + 1 = m$ and

$deg(w_{i(j+1)}) = (m-1) + 1 = m$.

Hence $adj(l_{ij}) = m + m - 2 = 2m - 2$.

Case 3: Let $e = h_{ij} \in C_n \odot K_m$. We know that h_{ij} is the edge joining the vertices v_i of C_n to the vertex w_{ij} in the i^{th} copy of K_m. As every vertexw_{ij} in $K_m$has degree$(m-1)$ we have

$deg(w_{ij}) = (m-1) + 1 = m$.

Since every vertex v_i in $C_n$has degree 2 and incident to m edges$h_{i1}, h_{i2}, \dots, h_{im}$ we have $deg(v_i) = m+2$.

Therefore$adj(h_{ij}) = m + m + 2 - 2 = 2m$. Thus in the graph $G = C_n \odot K_m$ we have

$$adj(e) = \begin{cases} 2m+2, \text{if } e = e_i \in C_n, \\ 2m-2, \text{if } e = l_{ij} \in i^{th} \text{ copy of} K_m, \\ 2m, \text{if } e = h_{ij} \in G = C_n \odot K_m. \end{cases}$$

3. EDGE DOMINATING SETS OF $G = C_n \odot K_m$

In this section we discuss minimal edge dominating sets of corona product graph $G = C_n \odot K_m$ and the edge domination numbers of this graph are obtained. First let us recall some definitions.

Definition: Let $G(V, E)$ be a graph. A subset D of E is said to be an **edge dominating set** (EDS) of G if every edge in (E– D) is adjacent to some edge in D.

An edge dominating set D is called a **minimal edge dominating set** (MEDS) if no proper subset of D is an edge dominating set of G.

Definition: The **edge domination number** of G is the minimum cardinality taken over all minimal dominating sets in G and is denoted by $\gamma_e^{'}(G)$.

Theorem 3.1: The edge domination number of $G = C_n \odot K_m$, $m = 3$ is given by

$$\begin{cases} \dfrac{3n}{2}, \text{if } n \text{ is even,} \\ \dfrac{3n+1}{2}, \text{if } n \text{ is odd.} \end{cases}$$

Proof: Let $m = 3$.

Consider the graph $G = C_n \odot K_3$ with vertex set V and edge set E given by

$V = \{v_1, v_2, \dots, v_n; w_{i1}, w_{i2}, w_{i3}, i = 1,2,3, \dots, n\}$. Here $v_1, v_2, \dots, v_n$ are vertices of C_n and w_{i1}, w_{i2}, w_{i3} are vertices in the i^{th} copy of K_3.

$$E = \begin{Bmatrix} e_1, e_2, \dots, e_n; h_{i1}, h_{i2}, h_{i3}, i = 1,2,3, \dots, n;\ l_{i1}, l_{i2}, l_{i3}, \\ i = 1,2,3, \dots, n \end{Bmatrix}$$

Here for $i = 1,2,3, \dots, n$; $h_{ij}, j = 1,2,3$ are the edges drawn from the vertex v_i to the vertices w_{i1}, w_{i2}, w_{i3} of i^{th} copy of K_3 respectively.And l_{ij}, $j = 1,2,3$ are the edges in the i^{th} copy of K_m.

Case 1: Suppose *n* is even.

Let
$D = \{l_{ij}$, for any $j = 1,2,3;\ i = 1,2,3, \dots, n; e_1, e_3, \dots, e_{n-1}\}$.

Since n is even, $|D| = n + \frac{n}{2} = \frac{3n}{2}$. Suppose $j = 1$. Then the edge l_{i1} is adjacent to the remaining two edges in the i^{th} copy of K_3 and is adjacent to the edges drawn from v_i to the end vertices w_{i1}, w_{i2} of l_{i1} in the i^{th} copy of K_3 $viz.$, h_{i1} , h_{i2}. The edge h_{i3} is dominated by the edge e_i. Similar is the case when

$j = 2$ or 3 .Therefore an edge l_{ij} , for any $j = 1,2,3$ and for $i = 1,2,3, \dots, n$ dominates all the edges in the *n*-copies of K_3 and any two edges h_{ij}. The remaining one edge h_{ij} is dominated by the edge e_i respectively.

The edges e_1 , $e_3, \dots$, e_{n-1} in C_n will dominate the rest of the edges in C_n respectively.

Therefore D dominates all the edges in $G = C_n \odot K_m$.Hence D becomes an edge dominating set (EDS) of $G = C_n \odot K_m$.

We show that D is minimal.

Delete the edge e_i from D for some *i*. Then the edge h_{ik}, for some $k = 1,2,3$ is not dominated by any edge in D $-\{e_i\}$.

Therefore D is minimal.

The edge domination number of $G = C_n \odot K_m$ is given by

$\gamma_e^{'}(G) = \frac{3n}{2}$, if n is even**.**

Case2: Suppose *n* is odd**.**

Let
$D = \{l_{ij}$, for any $j = 1,2,3$ and $i = 1,2,3, \dots, n; e_1, e_3, \dots, e_n\}$

Then $|D| = n + \frac{n+1}{2} = \frac{3n+1}{2}$. As in Case1 we can show that D is a minimal edge dominating set of $G = C_n \odot K_3$.

Therefore $\gamma_e^{'}(C_n \odot K_3) = \begin{cases} \frac{3n}{2}, \text{if } n \text{ is even,} \\ \frac{3n+1}{2}, \text{if } n \text{ is odd.} \end{cases}$

Theorem 3.2: The edge domination number of $G = C_n \odot K_m$, where $m > 3$ and n is even is given by

$$\begin{cases} \left(\dfrac{n}{2}\right) m\,, \quad \text{if } m \text{ is odd .} \\ \left(\dfrac{n}{2}\right)(m+1), \text{if } m \text{ is even .} \end{cases}$$

Proof: Let $G = C_n \odot K_m$ be the given graph and

$m > 3$. Suppose *n* is even.

Case 1: Let m be odd.
Let $D = \begin{Bmatrix} e_2, e_4, \dots, e_n; l_{ij}\ , \\ i = 1,2,3, \dots, n, j = 1,3,5, \dots, m-2 \end{Bmatrix}$.

We can see that $|l_{ij}| = \frac{m-1}{2}$ for $j = 1,3,5, \dots, m-2$.

We show that D is an edge dominating set of $G = C_n \odot K_m$**.**

Clearly the edges $e_2, e_4, \dots, e_n$ dominate the edges $e_1, e_3, \dots, e_{n-1}$ of C_n respectively.

Consider the edges h_{ij} and $h_{(i+1)j}$, $j = 1,2, \dots, m$. The edge h_{ij} is incident with the vertex v_i and the edge $h_{(i+1)j}$ is incident with the vertex $v_{(i+1)}$. Since the edges in D are alternately the edges in C_n, it follows that all the edges h_{ij} ,

$i = 1,2,3, \dots, n, j = 1,2, \dots, m$ are dominated by the edges

$e_i \in C_n, i = 2,4, \dots, n$.

Now consider an edge, say

$l_{ij} \in i^{th}$ copy of K_m. There are $\frac{m(m-1)}{2}$ edges in each copy of K_m. If we select one edge say l_{ik} then it dominates

$(2m-3)$ edges $l_{ij}, j \neq k$ in K_m.The remaining undominated edges are $\frac{(m-2)(m-3)}{2}$. Then we require at least $\frac{(m-3)}{2}$ edges to dominate these edges.

Thus $1 + \frac{(m-3)}{2} = \frac{(m-1)}{2}$ edges will dominate the edges l_{ij} in each copy of K_m.

Thus every edge in E(G) is dominated by some edge in D**.**

Therefore D becomes an EDS of $G = C_n \odot K_m$

We show that D is minimal.

Delete an edge, say e_k for some $k = 2,4, \dots, n$ from D.

Then the edge e_k is not dominated by any edge in $D - \{e_k\}$**.**

Again if we delete some edge $l_{ij} \in i^{th}$ copy of K_m, then also we can see that some edges in the i^{th} copy of K_m are not dominated by any edge in $D - \{l_{ij}\}$ as the edges l_{ij} are taken in minimum number. Thus D becomes a minimal edge dominating set of $G = C_n \odot K_m$.

We have chosen $(\frac{n}{2})$ edges from C_n into the dominating set of D. From each i^{th} copy of K_m, we have chosen $\frac{(m-1)}{2}$

edges into the dominating set of D. As there n such copies of K_m, we have chosen a total of $n\left(\frac{m-1}{2}\right)$ edges from all copies of K_m into D.

Therefore$|D| = \frac{n}{2} + n\left(\frac{m-1}{2}\right) = \left(\frac{n}{2}\right)m$.

Thus$\gamma_e^{'}(C_n \odot K_m) = |D| = \left(\frac{n}{2}\right)m$.

Case 2: Let m be even**.** Let $D = \left\{ \begin{matrix} e_2, e_4, \dots, e_n; l_{ij},\ i = 1,2,3,\dots,n, \\ j = 1,3,5,\dots,m-1 \end{matrix} \right\}$.

We can see that $|l_{ij}| = \frac{m}{2}$ for $j = 1,3,5,\dots,m-1$.

As in Case1 we can show that D is a minimal edge dominating set of$G = C_n \odot K_m$.

Further as in Case 1 we can see that $1 + \left(\frac{m-2}{2}\right) = \frac{m}{2}$ edges l_{ij} are requiredin each copy of $K_m$and there are $\left(\frac{n}{2}\right)$edges fromC_n into D.

Therefore$|D| = \frac{n}{2} + n\left(\frac{m}{2}\right) = \frac{n}{2}(m+1)$. Thus

$$\gamma_e^{'}(C_n \odot K_m) = \begin{cases} \left(\frac{n}{2}\right)m, & \text{if } m \text{ is odd}, \\ \left(\frac{n}{2}\right)(m+1), & \text{if } m \text{ is even.} \end{cases}$$

Theorem 3.3: The edge domination number of $G = C_n \odot K_m$ where $m > 3$ and$n$ is odd is given by

$$\begin{cases} \left(\frac{mn+1}{2}\right), & \text{if } m \text{ is odd}, \\ \left(\frac{n(m+1)+1}{2}\right), & \text{if } m \text{ is even}. \end{cases}$$

Proof: Let $G = C_n \odot K_m$ be the given graph**.** Suppose n is odd**.**

Case 1: Let m be odd**.** Let $D = \left\{ \begin{matrix} e_1, e_3, \dots, e_n; l_{ij}, \\ i = 1,2,3,\dots,n, j = 1,3,5,\dots,m-2 \end{matrix} \right\}$. In similar lines to the proof of Case 1 of Theorem 3.2. we can show that D is a MEDS of G.We have chosen$\left(\frac{n+1}{2}\right)$ edges from$C_n$into the dominating set D and$\left(\frac{m-1}{2}\right)$ edges l_{ij} from each copy of K_m to D, as there are n such copies of K_m we have chosen a total of $n\left(\frac{m-1}{2}\right)$ edges into D**.**

Therefore$|D| = \frac{n+1}{2} + n\left(\frac{m-1}{2}\right) = \frac{nm+1}{2}$.

Thus$\gamma_e^{'}(C_n \odot K_m) = |D| = \frac{nm+1}{2}$.

Case 2: Let m be even**.** Let $D = \left\{ \begin{matrix} e_1, e_3, \dots, e_n; l_{ij}, \\ i = 1,2,3,\dots,n, j = 1,3,5,\dots,m-1 \end{matrix} \right\}$. Then in similar lines to Case 2of Theorem 3.2 we can show that D is a minimal edge dominating set of$G = C_n \odot K_m$. Again there are $\left(\frac{m}{2}\right)$edges l_{ij} fromeach copy of K_m and $\left(\frac{n+1}{2}\right)$edges e_i from C_n are chosen into D. Therefore$|D| = \left(\frac{n+1}{2}\right) + \left(\frac{nm}{2}\right) = \frac{n(m+1)+1}{2}$.

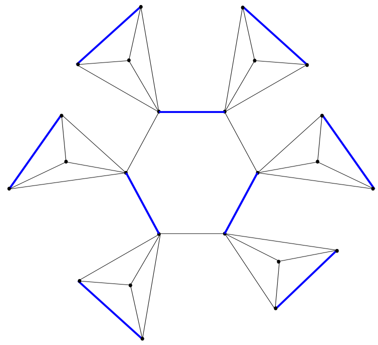
Hence$\gamma_e^{'}(C_n \odot K_m) =$

$$\begin{cases} \left(\frac{mn+1}{2}\right), & \text{if } m \text{ is odd}, \\ \left(\frac{n(m+1)+1}{2}\right), & \text{if } m \text{ is even}. \end{cases}$$

3.1 Illustrations

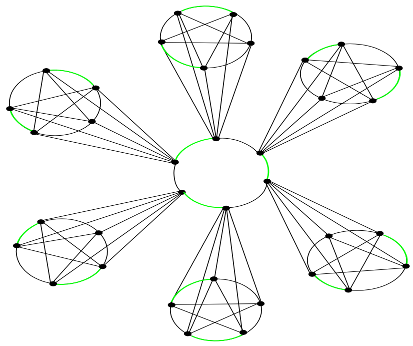
Minimal Edge Dominating Set

Theorem 3.1



$C_6 \odot K_3$

Theorem 3.2



$C_6 \odot K_5$

4. CONCLUSION

It is interesting to study the edge domination number and edge dominating sets of corona product graph of a cycle with a complete graph. This work gives the scope for an extensive study of edge dominating sets in general of these graphs.

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