Fixed Points Theorems in G-metric Spaces

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ABSTRACT
In this paper, we prove common fixed point theorems for a pair of mappings satisfying contractive condition of integral type in G- metric spaces.

Keywords
G-metric spaces, fixed point, integral type contractive condition.

1. INTRODUCTION
In 2004, Mustafa and Sims [4] introduced the concept of G-metric spaces as follows:

Let X be a nonempty set. Let G: $X \times X \times X \longrightarrow R^+$ be a function satisfying the following properties:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables),
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality).

The function G is called a generalized metric or, more specifically, a G – metric on X, and the pair (X, G) is called a G – metric space.

Definition 1.1. Let (X,G) be a G-metric space then for $x_0 \in X$, $r > 0$,the G-ball with center x_0 and radius r is, $B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}$.

Proposition 1.1. Let (X,G) be a G-metric space, then for any $x_0 \in X$ and $r > 0$,we have,

(i) if $G(x_0,x,y) < r$, then $x, y \in B_G(x_0,r)$.

(ii) if $y \in B_G(x_0,r)$,then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B(x_0,r)$.

Definition 1.2. Let (X,G) and (X´,G´) be G-metric spaces .A function f: $X \longrightarrow X'$ is G-continuous at a point $x_0 \in X$ if $f^{-1}(B_{G'}(f(x_0),r)) \in X$ for all $r > 0$.We say f is G-continuous if it is G-continuous at all points of X.

Proposition 1.2. Let (X,G) and (X´,G´) be G-metric spaces. A function f: $X \longrightarrow X'$ is G-continuous at a point $x \in X$ iff it is G-sequentially continuous at x ;that is, whenever (x_n) is G-convergent to x we have (fx_n) is G-convergent to f(x).

Proposition 1.3. Let (X,G) be a G-metric space, then the function G(x,y,z) is jointly continuous in all three of its variables .

Definition 1.3. Let (X, G) be a G – metric space. A sequence $\{x_n\}$ in X is G – convergent to x if $\lim_{m,n \to \infty} G(x, x_n, x_m) = 0$; i.e., for each $\in > 0$ there exists an N such that $G(x, x_n, x_m) < \in$ for all $m, n \geq N$.

Proposition 1.4. Let (X, G) be a G – metric space. Then the following are equivalent:

(i) $\{x_n\}$ is G convergent to x,
(ii) $G(x_n, x_n, x) \longrightarrow 0$ as $n \longrightarrow \infty$,
(iii) $G(x_n, x, x) \longrightarrow 0$ as $n \longrightarrow \infty$,
(iv) $G(x_m, x_n, x) \longrightarrow 0$ as $m, n \longrightarrow \infty$.

Definition 1.4. Let (X, G) be a G – metric space. A sequence $\{x_n\}$ is called G – Cauchy if, for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$.

Proposition 1.6. In a G – metric space (X, G) the following are equivalent:

(i) the sequence $\{x_n\}$ is G – Cauchy,
(ii) for each $\in > 0$ there exists an N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$.

Proposition 1.7. A G – metric space (X, G) is G – complete if and only if (X, d_G) is a complete metric space.

2. MAIN RESULTS.
In 2002, Branciari [1] and Rhoades [7] proved fixed point theorems for integral type contractive condition which further strengthens the Banach Fixed Point Theorem.

Now, we prove a common fixed point theorem using E.A. property for a pair of weakly compatible mappings satisfying a contractive condition of integral type in G-metric spaces.

Theorem 2.1. Let f and g be weakly compatible self maps of a G- metric space (X, G) satisfying the following conditions :

(2.1) f and g satisfy E.A. property,

(2.2) $$\int_0^{G(fx,fy,fz)} \varphi(t)\,dt \leq c \int_0^{G(gx,gy,gz)} \varphi(t)\,dt,$$

for each $x,y \in X$, $c \in [0,1)$,

where φ: $R^+ \rightarrow R^+$ is a Lebesgue-integral mapping which is summable (with finite integral) on each compact subset of R^+, non-negative, and such that for each $\varepsilon > 0$,

(2.3) $$\int_0^{\varepsilon} \varphi(t)\,dt > 0,$$

(2.4) g(X) is a closed subspace of X.

Then f and g have a unique common fixed point.

Proof. Since f and g satisfy the E.A. property, therefore, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = u \in X$. Since g(X) is a closed subspace of X, every convergent sequence of point of g(X) has a limit point in g(X) .

Therefore, $\lim_{n\to\infty} fx_n = u = ga = \lim_{n\to\infty} gx_n$ for some $a \in X$. This implies $u = ga \in g(X)$.

Now From (2.2), we have

$$\int_0^{G(fa, fx_n, fx_n)} \varphi(t)\,dt \le c \int_0^{G(ga, gx_n, gx_n)} \varphi(t)\,dt .$$

Letting $n\to\infty$ and using Lebesgue dominatd convergence theorem and $c \in [0, 1)$ it follows in view of (2.3) that u= fa. This implies u= ga =fa .Thus a is the coincidence point of f and g.

Since f and g are weakly compatible, therefore, fu = fga= gfa = gu.

From (2.2), we have

$$\int_0^{G(fu, fa, fa)} \varphi(t)\,dt \le c \int_0^{G(gu, ga, ga)} \varphi(t)\,dt,$$ since $c \in [0,1)$,which in turns implies that fu=u. Hence u is the unique common fixed point of f and g.

For uniqueness: Suppose that $w\ (\neq z)$ is also another common fixed point of f and g. From (2.2), we have

$$\int_0^{G(z,w,w)} \varphi(t)\,dt = \int_0^{G(fz, fw, fw)} \varphi(t)\,dt \le c \int_0^{G(gz, gw, gw)} \varphi(t)\,dt = c \int_0^{G(z,w,w)} \varphi(t)\,dt,$$ since $c \in [0,1)$, therefore, z = w and uniqueness follows .

Example 2.1. Consider $X = [0,2]$ with usual G- metric space G .i.e., $G(x,y,z) = (|x-y| + |y-z| + |z-x|)$. Define the self maps f and g on X as follows:

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 2 \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 2, & \text{if } 1 \le x \le 2. \end{cases}$$

Consider the sequence $x_n = \frac{1}{n}$. Clearly $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$.Then f and g satisfy E. A . property. Also $f(X) = \{0,1\}$ and $g(X) = \{0,2\}$. Here we note that neither f(X) is contained in g(X) nor g(X) is contained in f(X).

Moreover φ defined by φ(t) =t is a Lebesgue-integral mapping which is summable (with finite integral) on each compact subset of R^+, non-negative, and such that for each $\varepsilon > 0$,

$\int_0^{\varepsilon} \varphi(t)dt > 0$. Theorem 2.1 holds for $\frac{1}{2} \le q < 1$.

Theorem 2.2. Let f and g be weakly compatible self maps of a G- metric space (X, G) satisfying (2.1),(2.2) ,(2.3) and the following :

(2.5) $f(X) \subset g(X)$.

If the range of either of f or g is a closed subspace of X, then f and g have a unique common fixed point.

Proof follows easily from Theorem 2.1.

3. REFERENCES

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