Moments Inequalities for $NBRU_{rp}$ Distributions with Hypotheses Testing Applications

S. M. El-Arishy
A professor of Mathematical Statistics.
College of Science for (girls), Dept. of Mathematics,
Al-Azhar University, Nasr City, 11884, Egypt.

L. S. Diab
Assistant Professor of Mathematical Statistics.
College of Science for (girls), Dept. of Mathematics,
Al-Azhar University, Nasr City, 11884, Egypt.

E. S. El-Atfy
College of Science for (girls), Dept. of Mathematics,
Al-Azhar University, Nasr City, 11884, Egypt.

ABSTRACT
In this paper, we presented a new test statistic for testing exponentiality against new better than renewal used in the RP order $(NBRU_{rp})$ based on moment inequality. Pitman's asymptotic efficiency, The Pitman asymptotic relative efficiency $(PARE)$ are studied for other testes. Critical values are tabulated for sample size $n = 5(1)30(5)50$, the power of the test are calculate. Also we proposed a test for testing exponentiality versus $(NBRU_{rp})$ for right censored data and the power estimates of this test are also simulated for some commonly used distributions in reliability. Finally, real data are given to elucidate the use of the proposed test statistic in the reliability analysis.

Keywords
Life distributions, $(NBRU_{rp})$ aging class, moment inequalities, exponentiality U-statistic, asymptotic normality, efficiency, Monte Carlo method, power and censored data.

1. INTRODUCTION
In reliability, various aging classes of life distributions have been introduced to describe several types of improvement that accompany aging. The residual probability (RP) function is a well-known reliability measure which has applications in many disciplines such as reliability theory, survival analysis, and actuarial studies. The RP function uniquely determines the distribution function of F (and hence the distribution function of G), under the condition that the ratio of the hazard rates of X and Y is known. In addition, when the ratio of the hazard rates of X and Y is a monotone function of time, then RP function is also a monotone function of time. The study of the properties of RP function might be important for engineers and system designers to compare the lifetime of the products and, hence, to design better products. For example, consider a series system with two independent components. If X and Y denote the lifetime of the components, then clearly the lifetime of the system is $T = min\{X, Y\}$. It is easily seen that $R(t) = P(Y = T | T > t)$, that is, the probability that the component with lifetime Y causes the system failure given that the system has survived up to time t (cf. Zardasht and Asadi [27] for several reliability properties, Tan and Lü [26] for some biological background, and Lü and Chen [20] , Chen et al. [9] and Zhou et al. [28] for some real world applications). Formally, in view of the RP function, the lifetime random variable X is said to be smaller than Y in the RP order (denoted by $X \leq_{rp} Y$) if and only if

$$R(t) \leq 0.5, \qquad \forall t > 0.$$

The past decades witnessed some aging notions based on a stochastic comparison between a random life X and its equilibrium version X^* which are introduced and studied by Li and Xu [19] and Bhattacharjee et al. [6].

Kayid, M. et al. [16] defined the $NBRU_{rp}$ and investigated the probabilistic characteristics of this class of life distribution.

Definition 1. The random variable X is said to be smaller than Y in the residual probability order (denoted by $X \leq_{rp} Y$) if

$$\int_t^\infty [f(x)\bar{G}(x) - g(x)\bar{F}(x)]dx \geq 0, \quad \forall t \geq 0.$$

Definition 2. A random life X is said to be new better than renewal used in the RP order $(NBRU_{rp})$ if $X^* \leq_{rp} X$, or equivalently,

$$\int_t^\infty \left[\bar{F}^2(x) - f(x)\int_x^\infty \bar{F}(u)du\right] dx \geq 0, \quad \forall t \geq 0.$$

As the dual version, new worse than renewal used in the RP order $(NWRU_{rp})$ may be defined through $X^* \geq_{rp} X$.

On the other hand, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions. These categories are useful for modeling situations, maintenance, inventory theory, and biometry (cf. Barlow and Proschan [5] and Lai and Xie [17]). The random variable $\check{X}$ with distribution

$\check{F}(x) = \frac{1}{\mu}\int_0^x \bar{F}(u)du, \; for\ all \;\; x \geq 0$, Where μ is the mean of X, is known in the literature as the equilibrium distribution associated with X . The equilibrium distribution can be used to characterize some aging properties (cf. Mi [22] , Bon and Illayk ([7],[8]) , Mugdadi and Ahmad [23] , and Kayid et al. [15]). Ordinarily, when a stochastic order is proposed in the literature, its further properties in different forms of statistical analysis become important to study.

Based on moment inequalities, many statisticians derived the moment inequalities for the nonparametric families of aging distributions, among them Ahmad [4] for IFR, NBU, NBUE and HNBUE. Abu-Youssef [3] for DMRL. Mahmoud and Abdul Alim [21] introduced two tests statistics for testing exponentiality against NBUFR and NBAFR. El Arishy et al [12] derived moment inequalities for NRBU and similarly for the RNBU properties. In [10], Diab studied a U-statistic for testing exponentiality against the new better than used in the Laplace transform order aging class NBUL based on moment inequality. Finally, Diab [11] studied a new approach to moments inequalities for NRBU and RNBU classes of life distributions.

In this paper, we derive the moment inequalities for the $NBRU_{rp}$ class in section 2, we present attest statistic based on a U-statistic for testing H_0 : is exponential against H_1 : is $NBRU_{rp}$ and not exponential, the Pitman asymptotic efficiencies are calculated for some commonly used distributions in reliability, in Section 3.In Section 4 Monte Carlo null distribution critical points are simulated for sample sizes *n=5(1)30(5)50* , the power estimates of this test are calculated at the significant level $\alpha = 0.05$ for some common alternatives distribution and some application are given. In section 5, we dealing with right-censored data and selected critical values are tabulated; the power estimates for censor data of this test are tabulated. Finally, we discuss some applications to elucidate the usefulness of the proposed test in reliability analysis.

2. MOMENT INEQUALITY

The next result provides moments inequality for the $NBRU_{rp}$ distributions.

In this, as well as subsequent results all moments are assumed to exist and are finite.

Theorem 2.1 If F is $NBRU_{rp}$, then for all integer $r \geq 0$,

$$\int_0^\infty x^{r+2}\bar{F}(x)dF(x) \geq \frac{r+2}{r+4}\int_0^\infty x^{r+1}\left(\int_x^\infty tdF(t)\right)dF(x). \tag{2.1}$$

Proof: since F is $NBRU_{rp}$ then

$$\begin{aligned}\int_t^\infty \bar{F}^2(x)dx &\geq \int_t^\infty f(x)\left[\int_x^\infty \bar{F}(u)du\right]dx\\ &= \int_t^\infty \bar{F}(x)\left[\int_t^x f(u)du\right]dx\\ &= \int_t^\infty [\bar{F}(t)\bar{F}(x) - \bar{F}^2(x)]dx.\end{aligned}$$

This can be written in the form

$$2\int_t^\infty \bar{F}^2(x)dx \geq \bar{F}(t)\int_t^\infty \bar{F}(x)dx.$$

Multiplying both sides by t^r for $r \geq 0$, and integrating over $(0,\infty)$ w.r.t. t , we get,

$$2\int_0^\infty t^r\int_t^\infty \bar{F}^2(x)dxdt \geq \int_0^\infty t^r\bar{F}(t)\int_t^\infty \bar{F}(x)dxdt. \tag{2.2}$$

It is easy to show that,

L.H.S

$$\begin{aligned}&= 2\int_0^\infty t^r\int_t^\infty \bar{F}^2(x)dxdt,\\ &= 2\int_0^\infty \bar{F}^2(x)\left(\int_0^x t^r dt\right)dx,\\ &= \frac{4}{(r+1)(r+2)}\int_0^\infty x^{r+2}\bar{F}(x)dF(x).\end{aligned} \tag{2.3}$$

And,

R.H.S

$$\begin{aligned}&= \int_0^\infty t^r\bar{F}(t)\int_t^\infty \bar{F}(x)\,dxdt\,,\\ &= \int_0^\infty \bar{F}(x)\left(\int_0^x t^r\bar{F}(t)dt\right)dx,\\ &= \frac{1}{(r+1)}\left\{\frac{-r}{(r+2)}\int_0^\infty x^{r+2}\bar{F}(x)dF(x)\right.\\ &\quad\left.+\int_0^\infty x^{r+1}\left(\int_x^\infty tdF(t)\right)dF(x)\right\}.\end{aligned} \tag{2.4}$$

Making use of (2.3), (2.4) in (2.2) the result follows.

3. TESTING AGAINST $NBRU_{rp}$ CLASS FOR NON-CENSORED DATA

Here we present a test statistic based on the moment inequality, the test presented depends on a sample $X_1, X_2, \ldots, X_n$ from a population with distribution function F. We test the null hypothesis for testing $H_0 : F$ is exponential against an alternative that $H_1 : F$ is belongs to $NBRU_{rp}$ class and not exponential. We propose the following measure of departure using theorem (2.1) as

$$\delta_{rp} = \int_0^\infty x^{r+2}\bar{F}(x)dF(x) - \frac{r+2}{r+4}\int_0^\infty x^{r+1}\left(\int_0^\infty tI(t > x)dF(t)\right)dF(x). \tag{3.1}$$

Where,

$$I(t > x) = \begin{cases}1, & t > x\\ 0, & O.W\end{cases}$$

Note that under $H_0 : \delta_{rp} = 0$, while under $H_1 : \delta_{rp} > 0$

3.1 Empirical Test Statistic $NBRU_{rp}$ Alternative

To estimate δ_{rp}, let $X_1, X_2, \ldots, X_n$ be a random sample from F. Let $\bar{F}_n(x)$ denote the empirical distribution of the survival function $\bar{F}(x)$ where

$$\bar{F}_n(x) = \frac{1}{n}\sum_{j=1}^n I(X_j > x), \quad dF_n(x) = \frac{1}{n}, \tag{3.2}$$

and let $\hat{\delta}_{rp}$ be the empirical estimate of δ_{rp} where can be written as

$$\hat{\delta}_{rp} = \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\left\{X_i{}^{r+2} - \frac{r+2}{r+4}X_jX_i{}^{r+1}\right\}I(X_j > X_i) \tag{3.3}$$

To make the test $\hat{\delta}_{rp}$ scale invariant, we let

$$\hat{\Delta}_{rp} = \frac{\hat{\delta}_{rp}}{\bar{X}^{r+2}} \tag{3.4}$$

Set,

$$\phi(X_1, X_2) = \left\{X_1{}^{r+2} - \frac{r+2}{r+4}X_2X_1{}^{r+1}\right\}I(X_2 > X_1). \tag{3.5}$$

And define the symmetric kernel as

$$\psi(X_1,X_2)=\frac{1}{2!}\sum_R \phi(X_i,X_j).$$

Where the sum is over all arrangements of X_i and X_j , this leads to $\hat{\Delta}_{rp}$ is equivalent to U_n- statistic given by

$$U_n=\frac{1}{\binom{n}{2}}\sum_{i<j}\phi(X_i,X_j).$$

The following theorem summarizes the asymptotic normality of $\hat{\Delta}_{rp}$.

Theorem 3.1 As $n\to\infty$, $\sqrt{n}(\hat{\Delta}_{rp}-\delta_{rp})$ is asymptotically normal with mean 0 and variance,

$$\sigma^2=Var\left[\frac{1}{r+4}(2X^{r+2}-(r+2)X^{r+1})e^{-X}+\int_0^X x_1^{r+2}\,e^{-x_1}dx_1-\frac{r+2}{r+4}X\int_0^X x_1^{r+1}\,e^{-x_1}dx_1\right].$$

Under H_0 , and $r=0$. The variance σ^2 reduces to

$$\sigma_0^2=Var\left[-2Xe^{-X}-2e^{-X}-\frac{1}{2}X+2\right] \qquad (3.6)$$

Proof: Let

$$\eta_1(X_1)=E[\phi(X_1,X_2)|X_1]$$
$$=X_1^{r+2}\int_{X_1}^{\infty}e^{-x}dx-\frac{r+2}{r+4}X_1^{r+1}\int_{X_1}^{\infty}xe^{-x}\,dx$$
$$=\frac{1}{r+4}(2X_1^{r+2}-(r+2)X_1^{r+1})e^{-X_1}.$$

And,

$$\eta_2(X_1)=E[\phi(X_2,X_1)|X_1]$$
$$=\int_0^{X_1}x^{r+2}e^{-x}dx-\frac{r+2}{r+4}X_1\int_0^{X_1}x^{r+1}e^{-x}dx\,.$$

Set,

$$\zeta(X_1)=\eta_1(X_1)+\eta_2(X_1)$$
$$=\frac{1}{r+4}(2X_1^{r+2}-(r+2)X_1^{r+1})e^{-X_1}+\int_0^{X_1}x^{r+2}e^{-x}dx-\frac{r+2}{r+4}X_1\int_0^{X_1}x^{r+1}e^{-x}dx\,.$$

Then,

$$\sigma^2=Var[\zeta(X_1)].$$

Under H_0 the variance reduces to eq. (3.6), after calculation $\sigma_0^2=\frac{1}{54}$.

3.2 The Pitman Asymptotic Efficiency

To judge on the quality of this procedure, Pitman asymptotic efficiencies *(PAE)* are computed. We use the concept of Pitman's asymptotic efficiency *(PAE)* which is defined as

$$PAE\left(\delta_{rp}(\theta)\right)=\frac{1}{\sigma_0}\left|\left[\frac{d}{d\theta}\delta_{rp}(\theta)\right]_{\theta\to\theta_0}\right|.$$

And compared with some other tests for the following alternative distributions:

(i) Linear failure rate family *(LFR)*,

$$\bar{F}_1(x)=e^{-x-\frac{\theta}{2}x^2}\,,x\geq0,\theta\geq0,$$

(ii) Makeham family,

$$\bar{F}_2(x)=e^{-x-\theta(x+e^{-x}-1)}\,,x\geq0,\theta\geq0,$$

(iii) Gamma family,

$$\bar{F}_3(x)=\int_x^{\infty}e^{-u}\,u^{\theta-1}du/\Gamma(\theta)\,,x>0,\theta\geq0,$$

(iv) Weibull family,

$$\bar{F}_4(x)=e^{-x^{\theta}}\,,x>0,\theta\geq0.$$

Note that H_0 (the exponential distribution) is attained at $\theta_0=0$ in (i), (ii) and at $\theta_0=1$ in (iii), (iv)

Since,

$$\delta_{rp}(\theta)=\int_0^{\infty}x^{r+2}\bar{F}_\theta(x)dF_\theta(x)-\frac{r+2}{r+4}\int_0^{\infty}x^{r+1}\left(\int_x^{\infty}tdF_\theta(t)\right)dF_\theta(x).$$

The $PAE\left(\delta_{rp}(\theta)\right)$ can be written as,

$$PAE\left(\delta_{rp}(\theta)\right)=\frac{1}{\sigma_0}\left|\left\{\int_0^{\infty}x^{r+2}\bar{F}_\theta(x)d\acute{F}_\theta(x)+\int_0^{\infty}x^{r+2}\acute{\bar{F}}_\theta(x)dF_\theta(x)\right\}-\frac{r+2}{r+4}\left\{\int_0^{\infty}x^{r+1}\left(\int_x^{\infty}tdF_\theta(t)\right)d\acute{F}_\theta(x)+\int_0^{\infty}x^{r+1}\left(\int_x^{\infty}td\acute{F}_\theta(t)\right)dF_\theta(x)\right\}\right|.$$

Using *MATHEMATECA 9* program to calculate the Pitman asymptotic efficiency for $NBRU_{rp}$ test statistic in case of Weibull, Gamma family and direct calculations for linear failure rate family *(LFR)* and Makeham. In the above cases we get the following *PAE* values:

(i) Linear failure rate family $\bar{F}_1$:

$$PAE\left(\delta_{rp}(\theta)\right)=\frac{1}{\sigma_0}\left|\frac{r+2}{r+4}\,2^{-(r+2)}\Gamma(r+2)\right|,$$

(ii) Makeham family $\bar{F}_2$:

$$PAE\left(\delta_{rp}(\theta)\right)=\frac{1}{\sigma_0}\left|\frac{r+2}{2(r+4)}\,3^{-(r+2)}\Gamma(r+2)\right|,$$

(iv) Weibull family $\bar{F}_3$:

$$PAE\left(\delta_{rp}(\theta)\right)=0.709668,$$

(iii) Gamma family $\bar{F}_4$:

$$PAE\left(\delta_{rp}(\theta)\right)=0.291887.$$

Direct calculations of the asymptotic efficiencies of $NBRU_{rp}$ test are given in Table1. at $r=0$.

Table 1: Comparison between the *PAF* of our test and some other tests:

Test	$\bar{F}_1$	$\bar{F}_2$	$\bar{F}_3$	$\bar{F}_4$
Mugdadi et al δ_1[23]	0.408	0.039	0.170	-
Abdl-Aziz δ_2 [1]	0.535	0.184	0.223	-

Mahmoud et alδ_3 [21]	**0.**217	0.144	0.050	-
Kayid δ_4[16]	**0.**571	0.268	1.426	-
Our test δ_{rp}	**0.**919	0.204	0.710	0.292

It is clear from Table 1, that the new test statistic δ_{rp} for $NBRU_{rp}$ is more efficient than δ_1 , δ_2 , δ_3 and δ_4 . Also the Pittman asymptotic relative efficiency *(PARE)* of our test δ_{rp} comparing to δ_1, δ_2, δ_3, δ_4 is calculated where $PARE(T_1,T_2)=\frac{PAF\,(T_1)}{PAE\,(T_2)}$.

Table 2: show that the asymptotic relative efficiencies for our test:

Test	LFR	Makeham	Weibull
$PARE\left(\delta_{rp},\delta_1\right)$	2.252	5.231	4.176
$PARE\left(\delta_{rp},\delta_2\right)$	1.718	1.109	3.184
$PARE\left(\delta_{rp},\delta_3\right)$	4.235	1.417	14.20
$PARE\left(\delta_{rp},\delta_4\right)$	1.609	0.761	0.498

We can see from Table 2 that our test statistic δ_{rp} for $NBRU_{rp}$ is more efficiently than the other four cases.

4. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS

In this section the Monte Carlo null distribution critical points of $\hat{\Delta}_{rp}$ are simulated based on 5000 generated samples of size $n=5(1)30(5)50$. from the standard exponential distribution by using *Mathematica 9* program, simulated percentiles for small samples are commonly used by applied statisticians and reliability analyst. We have simulated the upper percentile values for 90%; 95%; 98% and 99%. Table 3, presented these percentile values of the statistics $\hat{\Delta}_{rp}$ 1n (3:4).

Table 3. Critical values of statistic $\hat{\Delta}_{rp}$

n	90%	95%	98%	99%
5	0.06131	0.07848	0.09722	0.10896
6	0.05981	0.07624	0.09403	0.10448
7	0.05612	0.07179	0.08898	0.09925
8	0.05206	0.06746	0.08243	0.09295
9	0.05046	0.06559	0.08152	0.09662
10	0.05067	0.06355	0.07557	0.08427
11	0.04704	0.06069	0.07652	0.08511
12	0.04479	0.05711	0.07185	0.08453
13	0.04436	0.05506	0.06692	0.07687
14	0.04246	0.05368	0.06761	0.07633
15	0.04085	0.05116	0.06260	0.06914
16	0.04045	0.05193	0.06533	0.07158
17	0.03932	0.04935	0.06146	0.06852
18	0.03784	0.04875	0.06083	0.06935
19	0.03766	0.04917	0.05873	0.06745
20	0.03619	0.04718	0.05759	0.06457
21	0.03581	0.04476	0.05676	0.06257
22	0.03426	0.04348	0.05467	0.06286
23	0.03346	0.04320	0.05413	0.06144
24	0.03325	0.04239	0.05465	0.06213
25	0.03279	0.04188	0.05183	0.05827
26	0.03251	0.04065	0.04995	0.05676
27	0.03094	0.03868	0.04950	0.05486
28	0.03099	0.03966	0.04886	0.05615
29	0.03115	0.03988	0.04853	0.05566
30	0.02957	0.03704	0.04616	0.05215
35	0.02857	0.03657	0.04483	0.05143
40	0.02680	0.03451	0.04228	0.04749
45	0.02526	0.03224	0.04002	0.04315
50	0.02291	0.02956	0.03733	0.04294

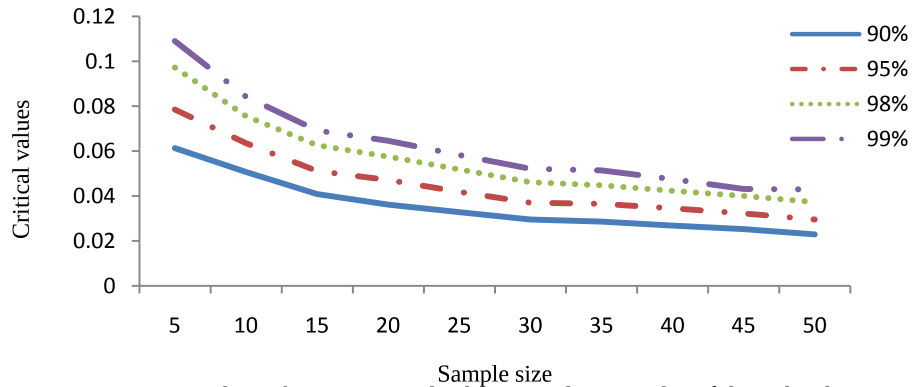

Fig. 1. Relation between critical values, sample size and confidence levels.

It is clear from Table 3, and Fig.1, that the critical values are increasing as the confidence level increasing and are almost decreasing as the sample size increasing.

4.1 Power Estimates of the Test $\hat{\Delta}_{rp}$

In this section, the power of our test $\hat{\Delta}_{rp}$ is estimated at $(1-\alpha)\%$ confidence level, $\alpha=0.05$ with suitable parameters values of at $n=10,20\ and\ 30$ with respect to three alternatives linear failure rate *(LFR)*, Weibull, and Gamma distributions based on 5000 simulated samples.

Table 4. Power estimates using $\alpha=0.05$.

Distribution	Parameter θ	Sample Size $n=10$	$n=20$	$n=30$
LFR	2	0.9982	0.9996	1.0000
	3	0.9990	1.0000	1.0000
	4	0.9986	0.9998	1.0000
	2	1.0000	1.0000	1.0000

Weibull	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Gamma	2	0.9944	0.9990	0.9996
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000

Table 4, shows that the power estimates of our test $\hat{\Delta}_{rp}$ increases as the value of the parameter θ and sample size n increases, and it is clear that our test has good powers.

4.2 Applications Using Complete (Uncensored) Data

In this section, we apply our test to some real data-sets at 95% confidence level.

Data-set #1.

Consider the data that given in Abouammoh et al. [2]. These data represent set of 40 patients suffering from blood cancer (leukemia) from one of ministry of health hospitals in Saudi Arabia. In this case, we get $\hat{\Delta}_{rp}= 0.107634$ and this value exceeds the tabulated critical value in Table 3. It is evident at the significant level %95, that the data set has $NBRU_{rp}$ property.

Data-set #2.

Consider the data set given in Grubbs [13], This data gives the times between arrivals of 25 customers at a facility. It is easily to show that $\hat{\Delta}_{rp}= 0.154287$ which is greater than the critical value of Table 3. Then we accept H_1 which shows that the data set have $NBRU_{rp}$ property but not exponential.

Data-set #3.

Consider the data, which represent failure times in hours, for a specific type of electrical insulation in an experiment where the insulation was subjected to a continuously increasing voltage stress (Lawless [18], p.138). The value of test statistic for the data set by formula (3.4) is given by $\hat{\Delta}_{rp}= 0.028851$ which is less than the critical value of table 3. Then we accept the null hypothesis of exponentially property. This means that this kind of data doesn't fit with $NBRU_{rp}$ property

5. TESTING AGAINST $NBRU_{rp}$ CLASS FOR CENSORED DATA

In this section, a test statistic is proposed to test H_0 versus H_1 with randomly right-censored data. Such a censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can formally be modeled as follows.

Suppose n objects are put on test, and $X_1, X_2, \ldots, X_n$ denote their true life time. We assume that $X_1, X_2, \ldots, X_n$ be independent, identically distributed (i.i.d.) according to a continuous life distribution F . Let $Y_1, Y_2, \ldots, Y_n$ be (i.i.d.) according to a continuous life distribution G . Also we assume that $X's$ and $Y's$ are independent. In the randomly right-censored model, we observe the pairs $(Z_j, \delta_j), j = 1, \ldots, n$ where $Z_j = min(X_j, Y_j)$ and

$$\delta_j = \begin{cases} 1 \;\; if\; Z_j = X_j (j-th\; observation\; is\; uncensored). \\ 0 \;\; if\; Z_j = Y_j (j-th\; observation\; is\; censored). \end{cases}$$

Let $Z(0) = 0 < Z(1) < Z(2) < \cdots < Z(n)$ denote the ordered $Z's$ and $\delta_{(j)}$ is the δ_j corresponding to $Z_{(j)}$ respectively.

Using the censored data $(Z_j, \delta_j), j = 1, \ldots, n$ Kaplan and Meier [14] proposed the product limit estimator.

$$\bar{F}_n(X) = 1 - F_n(X)$$

$$= \prod_{[j:Z_{(j)} \le X]} \left\{\frac{n-j}{n-j+1}\right\}^{\delta_{(j)}}, X \in [0, Z_n]$$

Now, for testing $H_0 : \hat{\Delta}_{rp}= 0$, against $H_1 : \hat{\Delta}_{rp}> 0$, using the randomly right censored data, we propose the following test statistic:

$$\hat{\Delta}^c_{rp}= \frac{1}{\mu^{r+2}}\left\{\int_0^\infty x^{r+2}\bar{F}_n(x)dF_n(x) - \frac{r+2}{r+4}\int_0^\infty x^{r+1}\left(\int_0^\infty tI(t > x)dF_n(t)\right)dF_n(x).\right\} \quad (5.1)$$

Since

$$\int_0^\infty x^{r+1}\,\bar{F}^2(x)dx = \frac{2}{r+2}\int_0^\infty x^{r+2}\bar{F}(x)dF(x)$$

For computational purposes and for $r = 0$, $\hat{\Delta}^c_{rp}$ can be rewritten as

$$\hat{\Delta}^c_{rp} = \frac{1}{\mu^2}(\eta - \beta). \quad (5.2)$$

Where,

$$\mu = \sum_{i=1}^n \prod_{m=1}^{i-1} C_m^{\delta_{(m)}}\left(Z_{(i)} - Z_{(i-1)}\right),$$

$$\eta = \sum_{i=1}^n Z_{(i)}\left(\prod_{m=1}^{i-1} C_m^{\delta_{(m)}}\right)^2 \left(Z_{(i)} - Z_{(i-1)}\right),$$

$$\beta = \frac{1}{2}\sum_{i=1}^n Z_{(i)}\left\{\sum_{j=i}^n Z_{(j)}\left[\prod_{q=1}^{j-2} C_q^{\delta_{(q)}} - \prod_{q=1}^{j-1} C_q^{\delta_{(q)}}\right]\right\} \left(\prod_{p=1}^{i-2} C_p^{\delta_{(p)}} - \prod_{p=1}^{i-1} C_p^{\delta_{(p)}}\right).$$

And

$$dF_n(Z_j) = \bar{F}(Z_{j-1}) - \bar{F}(Z_j), \qquad C_k = \frac{n-k}{n-k+1}.$$

Table 5. gives the critical values percentiles of $\hat{\Delta}^c_{rp}$ test for sample sizes $n = 5(5)30(10)70,81,86$, based on 5000 replications.

Table 5. Critical values for percentiles of $\hat{\Delta}^c_{rp}$ test

n	90%	95%	98%	99%
5	1.91182	2.26190	2.65184	2.97898
10	0.47058	0.53694	0.60793	0.67943
15	0.31263	0.36263	0.41703	0.45568
20	0.21942	0.26526	0.31678	0.35392

25	0.16632	0.21284	0.27035	0.32665
30	0.12593	0.16446	0.22834	0.27424
40	0.07460	0.10012	0.17921	0.22265
50	0.04491	0.07026	0.14553	0.19414
60	0.02625	0.04301	0.09216	0.15135
70	0.01507	0.02973	0.08784	0.14979
81	0.00306	0.01589	0.05811	0.11202
86	0.00012	0.00996	0.04267	0.11747

it is noticed from Table 5, and Fig.2, that the critical values are increasing as the confidence level increasing and decreasing as the sample size increasing.

5.1 Power Estimates of the Test $\hat{\Delta}^c_{rp}$

In this section, we present an estimation of the power for testing exponentiality versus $NBRU_{rp}$. Using significance level $\alpha = 0.05$ with suitable parameter values of θ at $n = 10{,}20 \; and \; 30$, and for commonly used distributions in reliability such as LFR family, Weibull family and Gamma family alternatives as shown in Table 6.

Table 6. Power estimates for $\hat{\Delta}^c_{rp}$ test

Distribution	Parameter θ	Sample Size $n = 10$	$n = 20$	$n = 30$
LFR	2	0.9294	0.9996	0.9992
	3	0.9230	1.0000	1.0000
	4	0.9042	0.9998	1.0000
Weibull	2	1.0000	1.0000	1.0000
	3	1.0000	1.0000	1.0000
	4	1.0000	1.0000	1.0000
Gamma	2	0.9412	0.9074	0.9918
	3	0.9256	0.8638	0.9972
	4	0.8936	0.8258	0.9960

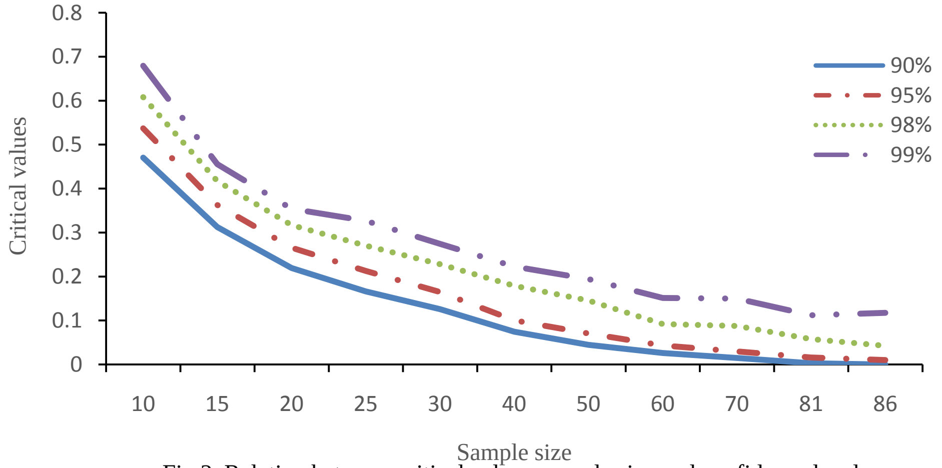

Fig.2. Relation between critical values, sample size and confidence levels.

We notice from Table 5. That our test has a good power, and the power increases as the sample size increases.

5.2 Applications for Censored Data

We present two good real examples to illustrate the use of our test statistics $\hat{\Delta}^c_{rp}$ in the case of censored data at 95% confidence level.

Data-set #4.

Consider the data from Susarla and Vanryzin [25], which represent 81 survival times (in months) of patients melanoma. Out of these 46 represents non-censored data. Now, taking into account the whole set of survival data (both censored and uncensored). It was found that the value of test statistic for the data set using formula (5.2) is given by $\hat{\Delta}^c_{rp} = 1.49772 * 10^{-95}$ and this value is less than the tabulated critical value in Table 5. This means that the data set have the exponential property.

Data-set #5.

On the basis of right censored data for lung cancer patients from Pena [24].These data consists of 86 survival times (in month) with 22 right censored. Now account the whole set of survival data (both censored and uncensored), and computing the test statistic given by formula (4.2). It was found that $\hat{\Delta}^c_{rp} = 1.42503 * 10^{-9}$. This is less than the tabulated value in Table 5.so we accept the null hypotheses.

6. CONCLUSION

Moments inequalities of $NBRU_{rp}$ class of life distributions are deduced. Based on these inequalities a new test for exponentiality versus $NBRU_{rp}$ class is constructed. The critical values of this test are calculated. Based on *PAEs* comparison between our test and tests of Mugdadi et al δ_1 [23], Abdl-Aziz δ_2 [1], Mahmoud et al δ_3 [21] and Kayid δ_4 [16] are given. Our study showed that our test performs higher PAE with respect to *r*. Based on right censored data a test for exponentiality versus $NBRU_{rp}$ class is also given. The power estimates of this test are simulated for uncensored and censored data. Finally sets of real data are used to elucidate the proposed test for practical problems.

7. REFERENCES

[1] Abdul Aziz, A. A., On testing exponentiality against RNBRUE alternatives, AppliedMathematical Science, 1, 1725-1736 (2007).

[2] Abouammoh, A.M, Abdulghani, S.A and Qamber, I.S (1994) On partial orderings and testing of new better than renewal used classes. Reliability Eng Syst Safety 43,pp. 37.41.

[3] Abu-Youssef, S. E. (2002). A moment inequality for decreasing (increasing) mean residual life distributions with hypothesis testing application, Statist. Probab. Lett., 57, 171-177.

[4] Ahmad, I. A. (2001). Moments inequalities of ageing families of distribution with hypothesis testing applications, J. Statist. Plan. Inf., 92,121-132.

[5] Barlow, R. E. and Proschan, F. (1981). Statistical Theory of Reliability and Life Testing, To BeginWith, Silver Spring,Md, USA.

[6] Bhattacharjee, M C., Abouammoh, A. M., Ahmed, A. N. and Barry, A. M. .Preservation results for life distributions based on comparisons with asymptotic remaining life under replacements,. Journal of Applied Probability, vol. 37, no. 4, pp. 999. 1009, 2000.

[7] Bon, J.-L. and Illayk, A. .A note on some new renewal ageing notions,.Statistics & Probability Letters, vol. 57, no. 2, pp. 151.155, 2002.

[8] Bon, J.-L. and Illayk, A. .Ageing properties and series systems,.Journal of Applied Probability, vol. 42, no. 1, pp. 279.286, 2005.

[9] Chen, Y., Lü, J., Yu, X. and Lin, Z. .Consensus of discrete-time second-order multiagent systems based on in.nite products of general stochastic matrices,.SIAMJournal on Control and Optimization, vol. 51, no. 4, pp. 3274.3301, 2013.

[10] Diab L.S. et al (2009). .Moments inequalities for NBUL distributions with hypotheses testing applications., Contem. Engin. Sci., 2, 319-332.

[11] Diab,L.S. A new approach to moments inequalities for NRBU and RNBU classes with hypothesis testing applications, International Journal of Basic & Applied Sciences IJBAS-IJENS 13(06) (2013), 7-13.

[12] EL-Arishy, S. M., Diab, L. S. and Mahmoud, M. A. W. (2003). .Moment inequalities for testing new renewal better than used and renewal new better than used classes, Int. J. Rel. Appl., 4, 97-123.

[13] Grubbs, F. E. (1971). Fiducial bounds on reliability for the two parameter negative exponential distribution. Technomet., 13, pp. 873-876.

[14] Kaplan, E. L. and Meier, P.(1958). .Nonparametric estimation from incomplete observation., J. Amer. Statist.Assoc., 53, pp.457-481.

[15] Kayid, M., Ahmad, I. A, Izadkhah, S. and Abouammoh, A. M. .Further results involving the mean time to failure order, and the decreasingmean time to failure class.,IEEE Transactions on Reliability, vol. 62, no. 3, pp. 670.678, 2013.

[16] Kayid, M., Izadkhah, S. and Alshami, 1, S. (2014)..Residual Probability Function, Associated Orderings, and Related Aging Classes. Mathematical Problems in Engineering http://dx.doi.org/10.1155/2014/490692.

[17] Lai, C-D . and Xie, M . (2006) Stochastic Ageing and Dependence for Reliability, Springer, New York, NY, USA.

[18] Lawless, J. F. (1982). Statistical Models & Methods for lifetime Data, John Wiley & sons, New York.

[19] Li, X. and Xu, M. .Reversed hazard rate order of equilibrium distributions and a related aging notion,. Statistical Papers, vol. 49, no. 4, pp. 749.767, 2008.

[20] Lü, J. and Chen, G..Atime-varying complex dynamical network model and its controlled synchronization criteria,.IEEE Transactions on Automatic Control, vol. 50, no. 6, pp. 841.846, 2005.

[21] Mahmoud, M. A.W. and Abdul Alim, A. N., A Goodness of .t approach to for testing NBUFR (NWUFR) and NBAFR (NWAFR) Properties, International Journal of Reliability Application, 9, 125-140 (2008).

[22] Mi, J..Some comparison results of system availability,.Naval Research Logistics, vol. 45, no. 2, pp. 205.218, 1998.

[23] Mugdadi, A. R. and Ahmed, I. A. Moment inequalities derived from comparing life with its equilibrium form, Journal of Statistical Planning and Inference, 134, 303-317 (2005).

[24] Pena, A. E. (2002).Goodness of fit tests with censored data. http://statmanStat.sc.edupenajtajkspresentedjtalk actronel

[25] Susarla, V. and Vanryzin, J (1978). Empirical bayes estimations of a survival function right censored observation. Ann. Statist., 6, pp. 710-755.

[26] Tan, S. and Lü, J..Characterizing the effect of population heterogeneity on evolutionary dynamics on complex networks,.Scienti. Reports, vol. 4, p. 5034, 2014.

[27] Zardasht, V. and Asadi, M. (2010)..Evaluation of $P\,(X_t > Y_t)$ when both X_t and Y_t are residual lifetimes of two systems. Statistics Neer landica, vol. 64, no. 4, pp. 460.481.

[28]Zhou, J. and Lü, J.(2006) .Adaptive synchronization of an uncertain complex dynamical network,.IEEE Transactions on Automatic Control, vol. 51, no. 4, pp. 652.656.