Solution of Initial Value Prpblem of Bratu – Type Equation using Modifications of Homotopy Perturbation Method

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ABSTRACT
Homotopy perturbation method (HPM) is an effective method for solving nonlinear differential equations. In this paper, some modifications of this method has been proposed to initial value problem of Bratu - Type model. The combination of Laplace transform and homotopy perturbation (LHPM), the new homotopy perturbation method (NHPM) and Laplace new homotopy perturbation method (LNHPM) are applied, and the solutions are considered as an infinite series that converge rapidly to the exact solutions.

Keywords
Bratu – Type equation, Homotopy perturbation method, Laplace homotopy perturbation method, new homotopy perturbation method, Laplace new homotopy perturbation method

1. INTRODUCTION
The Bratu-type equation arise from a simplification of the solid fuel ignition model in thermal combustion theory. This type has many other physical applications, such as chemical reactor theory, radiative heat transfer, electrostatics, fluid mechanics physical applications ranging from chemical reaction theory and nanotechnology to the expansion of universe [1]. Several numerical techniques had been applied to obtain the approximate solution of the Bratu-type problem. For example spline method [2-3], wavelet method [4-6], spectral method [7-10] and Adomain decomposition method [11-14].

In 1992, the homotopy analysis method (HAM) was proposed by Liou [15], some authors used that method and its modifications to solve Bratue – type equation [16-17].

In 1999, J. He presented perturbation technique coupled with the homotopy technique, it was known as HPM [18]. Recently, The homotopy perturbation method and its modification can effectively, easily, and accurately solve a large class of linear and nonlinear, ordinary or partial, deterministic or stochastic differential equations. The approximate solutions converge rapidly to accurate solutions. The method is well-suited to physical problems since it does not require the linearization and other restrictive methods, which may change the problem. The homotopy perturbation method and variational homotopy perturbation method had been applied for Bratue – type problem in [19-20].

In this paper, some modifications of homotopy perturbation method are presented to solve the initial value problem of second order differential equation of Bratu – Type. In section 2, HPM is reviewed. In section 3, the combination of Laplace transform and homotopy perturbation method is presented. NHPM is reviewed in section 4. In section 5, the combination of Laplace transform and new homotopy perturbation method is presented. Finally, in section 6, the previous methods are applied to second order initial value problems of Bratu – Type.

2. THE HOMOTOPY PERTURBATION METHO
HPM is combining the classical perturbation technique with the homotopy concept in topology [18].

Consider the nonlinear problem

$$A(u)-f(x)=0, \qquad x\in\Omega \tag{0.1}$$

with boundary condition

$$B(u,\frac{du}{dx})=0, \qquad x\in\Gamma \tag{0.2}$$

where A is a general differential operator, B is a boundary operator, $f(x)$ is a known analytic function, Γ is the boundary of the domain Ω .

The operator A can be divided into two operators' L and N, where L is linear and N is nonlinear, Eq. (0.1) can be rewritten as

$$L(u)+N(u)-f(x)=0 \tag{0.3}$$

Construct a homotopy $v(x,p):\Omega\times[0,1]\to\mathbf{R}$ which satisfies

$$\begin{aligned}\phi(v,p)&=(1-p)[L(v)-L(u_0)]\\&\quad+p[A(v)-f(x)]=0, \quad p\in[0,1],\ x\in\Omega\end{aligned} \tag{0.4}$$

or

$$\begin{aligned}\phi(v,p)&=L(v)-L(u_0)+pL(u_0)\\&\quad+p[N(v)-f(x)]=0\end{aligned} \tag{0.5}$$

where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (0.1), which satisfies the boundary conditions.

Assume that the solution of Eq. (0.4) can be written as a power series in p:

$$v = \sum_{n=0}^{\infty} p^n v_n \tag{0.6}$$

Substituting Eq. (0.6) into Eq. (0.5) and collecting terms of the same power of p gives

$$p^0: \qquad v_0 = u_0 \tag{0.7}$$

$$p^1: \qquad \begin{cases} L(v_1) = -L(u_0) - N(v_0) + f(x) \\ B(v_1, \dfrac{dv_1}{dx}) = 0 \end{cases} \tag{0.8}$$

$$p^2: \qquad \begin{cases} L(v_2) = -N(v_0, v_1) \\ B(v_2, \dfrac{dv_2}{dx}) = 0 \end{cases} \tag{0.9}$$

$$\vdots$$

$$p^i: \qquad \begin{cases} L(v_i) = -N(v_0, v_1, \ldots, v_{i-1}) \\ B(v_i, \dfrac{dv_i}{dx}) = 0 \end{cases} \tag{0.10}$$

Setting $p = 1$, the approximate solution of Eq. (0.1) will be

$$u = \lim_{p \to 1} v = \sum_{i=0}^{\infty} v_i \tag{0.11}$$

The series in Eq. (0.11) converges for most cases and it depends upon the nonlinear operator $A(v)$.

3. LAPLACE HOMOTOPY PERTURBATION METHOD

LHPM is combining Laplace transform and HPM, it has good stability properties, this method is started by applying Laplace transform on both sides of (0.5), we get

$$\begin{aligned} &\mathcal{L}\{L(v) - L(u_0) + pL(u_0) \\ &\qquad + p[N(v) - f(x)]\} = 0 \end{aligned} \tag{0.12}$$

By using the differential property of Laplace transform

$$\begin{aligned} \mathcal{L}\{v\} = &\frac{1}{s^n}\{s^{n-1}v(0) + s^{n-2}v'(0) + \cdots + v^{(n-1)}(0) \\ &+ \mathcal{L}\{L(u_0) - pL(u_0) - p[N(v) - f(x)]\}\} \end{aligned} \tag{0.13}$$

By applying inverse Laplace transform on both side of Eq. (0.13)

$$\begin{aligned} v = &\mathcal{L}^{-1}\{\frac{1}{s^n}\{s^{n-1}v(0) + s^{n-2}v'(0) + \cdots + v^{(n-1)}(0) \\ &+ \mathcal{L}\{L(u_0) - pL(u_0) - p[N(v) - f(x)]\}\} \end{aligned} \tag{0.14}$$

Rewrite Eq. (0.14) is in the form

$$\begin{aligned} \sum_{n=0}^{\infty} p^n v_n = &\mathcal{L}^{-1}\{\frac{1}{s^n}\{s^{n-1}v(0) + s^{n-2}v'(0) + \cdots \\ &+ v^{(n-1)}(0) + \mathcal{L}\{L(u_0) - pL(u_0) \\ &- p[N(\sum_{n=0}^{\infty} p^n v_n) - f(x)]\}\}\} \end{aligned}$$

(0.15)

By equating the terms with identical powers of p

$$\begin{aligned} p^0: \qquad v_0 = &\mathcal{L}^{-1}\{\frac{1}{s^n}\{s^{n-1}v(0) + s^{n-2}v'(0) \\ &+ \cdots + v^{(n-1)}(0) + \mathcal{L}\{L(u_0)\}\}\} \end{aligned} \tag{0.16}$$

$$\begin{aligned} p^1: \qquad v_1 = &\mathcal{L}^{-1}\{\frac{1}{s^n}\mathcal{L}\{-L(u_0) - \\ &[N(v_0) - f(x)]\}\} \end{aligned} \tag{0.17}$$

$$p^2: \qquad v_2 = \mathcal{L}^{-1}\{\frac{1}{s^n}\mathcal{L}\{-[N(v_0, v_1)]\}\} \tag{0.18}$$

$$\vdots$$

$$p^i: \qquad v_i = \mathcal{L}^{-1}\{\frac{1}{s^n}\mathcal{L}\{-[N(v_0, v_1, \ldots, v_{i-1})]\}\}$$

(0.19)

Assuming that the initial approximation has the form

$v(0) = u_0 = \alpha_0, v'(0) = \alpha_1, \ldots, v^{(n-1)}(0) = \alpha_{n-1}$

4. NEW HOMOTOPY PERTURBATION METHOD

For solving Eq. (0.1) by NHPM [21], construct the following homotopy

$$\begin{aligned} \phi(v, p) = &(1 - p)[L(v) - u_0] \\ &+ p[A(v) - f(x)] = 0, \\ &p \in [0,1], \ x \in \Omega \end{aligned} \tag{0.20}$$

or equivalently

$$\begin{aligned} \phi(v, p) = &L(v) - u_0 + pu_0 \\ &+ p[N(v) - f(x)] = 0 \end{aligned} \tag{0.21}$$

Applying the inverse operator L^{-1} to both sides of Eq. (0.21) gives

$$\begin{aligned} v = &L^{-1}(u_0) - p\,L^{-1}(u_0) \\ &- p\,L^{-1}[N(v) - f(x)] = 0 \end{aligned} \tag{0.22}$$

Assume that the initial approximation of Eq. (0.1) has the form

$$u_0 = \sum_{n=0}^{\infty} a_n P_n(x) \tag{0.23}$$

Where $a_0, a_1, a_2, \dots$ are unknown coefficients which must be computed and $P_0, P_1, P_2 \dots$ are specific functions depending on the problem. Substituting Eq. (0.23) into Eq. (4.3) leads to

$$\sum_{n=0}^{\infty} p^n v_n = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) - p\, L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) - p\, L^{-1}\left[N\left(\sum_{n=0}^{\infty} p^n v_n\right) - f(x)\right] = 0 \tag{0.24}$$

By equating the terms with identical powers of p

$$p^0: \quad v_0 = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) \tag{0.25}$$

$$p^1: \quad v_1 = -L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) - L^{-1}[N(v_0) - f(x)] \tag{0.26}$$

$$p^2: \quad v_2 = -L^{-1}[N(v_0, v_1)] \tag{0.27}$$

$$\vdots$$

$$p^i: \quad v_i = -L^{-1}[N(v_0, v_1, \dots, v_{i-1})] \tag{0.28}$$

If these equations are solved by considering $v_1(x) = 0$, then $v_2(x) = v_3(x) = \cdots = 0$. Hence the exact solution may be written as

$$u(x) = v_0(x) = L^{-1}\left(\sum_{n=0}^{\infty} a_n P_n(x)\right) \tag{0.29}$$

5. LAPLACE NEW HOMOTOPY PERTURBATION METHOD

In LNHPM Laplace transform and NHPM are combined [22], taking Laplace transform on both sides of Eq. (0.21) leads to

$$\mathcal{L}\{L(v)\} = \mathcal{L}\{u_0 - p\, u_0 - p[N(v) - f(x)]\} \tag{0.30}$$

Using the differential property of Laplace transform gives

$$\mathcal{L}\{v\} = \frac{1}{s^n}\{s^{n-1} v(0) + s^{n-2} v'(0) + \cdots + v^{(n-1)}(0) + \mathcal{L}\{u_0 - p\, u_0 - p[N(v) - f(x)]\}\} \tag{0.31}$$

Applying inverse Laplace transform to both side of Eq. (0.31) leads to

$$v = \mathcal{L}^{-1}\{\frac{1}{s^n}\{s^{n-1} v(0) + s^{n-2} v'(0) + \cdots + v^{(n-1)}(0) + \mathcal{L}\{u_0 - p\, u_0 - p[N(v) - f(x)]\}\}\} \tag{0.32}$$

By substituting the series of v, u_0 once can rewrite Eq. (0.32) as

$$\sum_{n=0}^{\infty} p^n v_n = \mathcal{L}^{-1}\{\frac{1}{s^n}\{s^{n-1} v(0) + s^{n-2} v'(0) + \cdots + v^{(n-1)}(0) + \mathcal{L}\{\sum_{n=0}^{\infty} a_n P_n(x) - p \sum_{n=0}^{\infty} a_n P_n(x) - p[N(\sum_{n=0}^{\infty} p^n v_n) - f(x)]\}\}\} \tag{0.33}$$

By equating the terms with identical powers of p

$$p^0: \quad v_0 = \mathcal{L}^{-1}\{\frac{1}{s^n}\{s^{n-1} v(0) + s^{n-2} v'(0) + \cdots + v^{(n-1)}(0) + \mathcal{L}\{\sum_{n=0}^{\infty} a_n P_n(x)\}\}\} \tag{0.34}$$

$$p^1: \quad v_1 = \mathcal{L}^{-1}\{\frac{1}{s^n} \mathcal{L}\{-\sum_{n=0}^{\infty} a_n P_n(x) - [N(v_0) - f(x)]\}\}\} \tag{0.35}$$

$$p^2: \quad v_2 = \mathcal{L}^{-1}\{-\frac{1}{s^n} \mathcal{L}\{N(v_0, v_1)\}\} \tag{0.36}$$

$$\vdots$$

$$p^i: \quad v_i = \mathcal{L}^{-1}\{-\frac{1}{s^n} \mathcal{L}\{N(v_0, v_1, \dots, v_{i-1})\}\} \tag{0.37}$$

As in NHPM $a_0, a_1, a_2, \dots$ are computed and solved these equations by considering $v_1(x) = 0$.

6. ILLUSTRATIVE EXAMPLE

Consider the initial value problem of Bratu-Type

$$u'' - 2e^u = 0 \qquad 0 < x < 1$$
$$u(0) = u'(0) = 0 \tag{0.38}$$

The exact solution of above equation is

$$u(x) = -2\ln(\cos x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \frac{62x^{10}}{14175} + \frac{691x^{12}}{467775} + \cdots \tag{0.39}$$

To solve Eq. (0.38) by HPM, construct the following homotopy

$$(1-p)[v'' - u_0''] + p[v'' - 2e^v] = 0, \quad p \in [0,1]$$

(0.40)

By applying Eq. (0.7) - (0.10)

$$p^0: \quad v_0 = u_0 = x^2$$

$$p^1: \quad \begin{cases} v_1'' = -2 + 2\sum_{i=0}^{\infty} \frac{v_0^{\,i}}{i\,!} \\ v_1(0) = v_1'(0) = 0 \end{cases} \tag{0.41}$$

$$p^2: \quad \begin{cases} v_2'' = 2\sum_{i=0}^{\infty} \frac{v_0^{\,i} v_1}{i\,!} \\ v_2(0) = v_2'(0) = 0 \end{cases}$$

$\vdots$

By solving the differential equations above $v_0, v_1, v_2, \ldots$ are found

$$v_0(x) = x^2$$

$$v_1(x) = \frac{x^4}{6} + \frac{x^6}{30} + \frac{x^8}{168} + \frac{x^{10}}{1080} + \frac{x^{12}}{7920} + \cdots$$

$$v_2(x) = \frac{x^6}{90} + \frac{x^8}{140} + \frac{103x^{10}}{37800} + \frac{97t^{12}}{124740} + \cdots$$

$$v_3(x) = \frac{x^8}{2520} + \frac{x^{10}}{1400} + \frac{659x^{12}}{1247400} + \cdots$$

$\vdots$

(0.42)

And once can get

$$v(x) = \sum_{i=0}^{\infty} v_i(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \frac{62x^{10}}{14175} + \frac{691x^{12}}{467775} + \cdots$$

(0.43)

Which converge to the exact solution in Eq. (0.39).

To solve Eq. (0.38) by LHPM, apply Laplace transform to both sides of Eq. (0.40)

$$\mathcal{L}\{v'' - u_0'' + p\,u_0'' - 2p\,e^v\} = 0 \tag{0.44}$$

By applying inverse Laplace transform

$$v = \mathcal{L}^{-1}\{\frac{1}{s^n}\{\mathcal{L}\{u_0'' - p\,u_0'' - p\,e^v\}\} \tag{0.45}$$

Equating the terms with identical powers of *p* as Eq. (0.16) - (0.37) gives

$$p^0: \quad v_0 = \mathcal{L}^{-1}\{\frac{1}{s^n}\mathcal{L}\{2\}\} = x^2$$

$$p^1: \quad v_1 = \mathcal{L}^{-1}\{\frac{1}{s^n}\mathcal{L}\{-2 + 2\sum_{i=0}^{\infty}\frac{v_0^{\,i}}{i\,!}\}\} = \frac{x^4}{6} + \frac{x^6}{30} + \frac{x^8}{168} + \frac{x^{10}}{1080} + \cdots$$

$$p^2: \quad v_2 = \mathcal{L}^{-1}\{\frac{1}{s^n}\mathcal{L}\{2\sum_{i=0}^{\infty}\frac{v_0^{\,i} v_1}{i\,!}\}\} = \frac{x^6}{90} + \frac{x^8}{140} + \frac{103x^{10}}{37800} + \cdots \tag{0.46}$$

$\vdots$

Thus

$$v(x) = \sum_{i=0}^{\infty} v_i(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \frac{62x^{10}}{14175} + \frac{691x^{12}}{467775} + \cdots \tag{0.47}$$

Which converge to the exact solution in Eq. (0.39).

For solving Eq. (0.38) by NHPM, consider the homotopy

$$(1-p)[v'' - u_0] + p[v'' - 2e^v] = 0 \tag{0.48}$$

By applying the inverse operator $L^{-1} = \int_0^x \int_0^t (.)\,dr\,dt$ to both sides of Eq. (0.48)

$$v(x) = v(0) + v'(0) + \int_0^x \int_0^t u_0(r)\,dr\,dt - p\int_0^x \int_0^t u_0(r)\,dr\,dt + 2p\int_0^x \int_0^t e^{v(r)}\,dr\,dt \tag{0.49}$$

By substituting the series of v, u_0 once can rewrite Eq. (0.49) as

$$\sum_{n=0}^{\infty} p^n v_n = \int_0^x \int_0^t \sum_{n=0}^{\infty} a_n r^n\,dr\,dt - p\int_0^x \int_0^t \sum_{n=0}^{\infty} a_n r^n\,dr\,dt + 2p\int_0^x \int_0^t \sum_{n=0}^{\infty} \frac{v^i(r)}{i\,!}\,dr\,dt \tag{0.50}$$

Equating the terms with identical powers of *p* as Eq. (0.25) - (0.28) gives

$$p^0: \quad v_0 = \int_0^x\int_0^t \sum_{n=0}^{\infty} a_n r^n \, dr\, dt = \frac{x^2 a_0}{2} + \frac{x^3 a_1}{6} + \frac{x^4 a_2}{12} + \frac{x^5 a_3}{20} + \frac{x^6 a_4}{30} + \frac{x^7 a_5}{42} + \frac{x^8 a_6}{56} + \cdots$$

$$p^1: \quad v_1 = -\int_0^x\int_0^t \sum_{n=0}^{\infty} a_n r^n \, dr\, dt + 2\int_0^x\int_0^t \sum_{n=0}^{\infty} \frac{v_0^i(r)}{i\,!} \, dr\, dt$$

$$= x^2\left(1-\frac{a_0}{2}\right) - \frac{x^3 a_1}{6} + x^4\left(\frac{a_0}{12}-\frac{a_2}{12}\right) + x^5\left(\frac{a_1}{60}-\frac{a_3}{20}\right) + x^6\left(\frac{a_0^2}{120}+\frac{a_2}{180}-\frac{a_4}{30}\right) + x^7\left(\frac{a_0 a_1}{252}+\frac{a_3}{420}-\frac{a_5}{42}\right) + x^9\left(\frac{a_0^2 a_1}{1728}+\frac{a_1 a_2}{2592}+\frac{a_0 a_3}{1440}+\frac{a_5}{1512}\right) + x^{11}\left(\frac{a_0^3 a_1}{15840}+\frac{a_1^3}{71280}+\frac{a_0 a_1 a_2}{7920}+\frac{a_0^2 a_3}{8800}+\frac{a_2 a_3}{13200}+\frac{a_1 a_4}{9900}+\frac{a_0 a_5}{4620}\right) + x^8\left(\frac{a_0^3}{1344}+\frac{a_1^2}{2016}+\frac{a_0 a_2}{672}+\frac{a_4}{840}-\frac{a_6}{56}\right) + \cdots \quad (0.51)$$

Vanishing $v_1(x)$ lets the coefficients $a_n \;(n = 0,1,2,\ldots)$ to take the following values

$$a_0 = 2, a_2 = 2, a_4 = \frac{4}{3}, a_6 = \frac{34}{45}, \ldots \qquad a_1 = a_3 = a_5 = \cdots = 0 \quad (0.52)$$

Hence the solution of Eq. (0.38) is written as

$$u(x) = v_0(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \cdots \quad (0.53)$$

And this in the limit of infinity many terms, yields the exact solution in Eq. (0.39).

For solving Eq. (0.38) by LNHPM, apply Laplace transform to both sides of Eq. (0.48)

$$\mathcal{L}\{v\} = \frac{1}{s^2}\{s\,v(0) + v'(0) + \mathcal{L}\{u_0 - p\,u_0 + 2p\,e^v\}\} \quad (0.54)$$

By applying inverse Laplace transform

$$v = \mathcal{L}^{-1}\{\frac{1}{s^2}\mathcal{L}\{u_0 - p\,u_0 + 2p\,e^v\}\} \quad (0.55)$$

By substituting the series of v, u_0 once can rewrite Eq. (0.55) as

$$\sum_{n=0}^{\infty} p^n v_n = \mathcal{L}^{-1}\{\frac{1}{s^2}\mathcal{L}\{\sum_{n=0}^{\infty} a_n r^n - p\sum_{n=0}^{\infty} a_n r^n + 2p\sum_{n=0}^{\infty}\frac{v^i(r)}{i\,!}\}\} \quad (0.56)$$

Equating the terms with identical powers of p leads to

$$p^0: \quad v_0 = \mathcal{L}^{-1}\{\frac{1}{s^2}\{s^2 v(0) + v'(0) + \mathcal{L}\{\sum_{n=0}^{\infty} a_n x^n\}\} = \mathcal{L}^{-1}\{\frac{1}{s^2}\mathcal{L}\{\sum_{n=0}^{\infty} a_n x^n\}\} = \frac{x^2 a_0}{2} + \frac{x^3 a_1}{6} + \frac{x^4 a_2}{12} + \frac{x^5 a_3}{20} + \frac{x^6 a_4}{30} + \frac{x^7 a_5}{42} + \frac{x^8 a_6}{56} + \cdots$$

$$p^1: \quad v_1 = \mathcal{L}^{-1}\{\frac{1}{s^2}\mathcal{L}\{-\sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty}\frac{v_0^i}{i\,!}\}\} = \frac{1}{2}x^2(2-a_0) - \frac{x^3 a_1}{6} + \frac{1}{12}x^4(a_0 - a_2) + \frac{1}{60}x^5(a_1 - 3a_3) + \frac{1}{360}x^6(3a_0^2 + 2a_2 - 12a_4) + x^7\frac{(5a_0 a_1 + 3a_3 - 30a_5)}{1260} + x^8\frac{(15a_0^3 + 10a_1^2 + 30a_0 a_2 + 24a_4 - 360a_6)}{20160} + x^9\frac{(15a_0^2 a_1 + 10a_1 a_2 + 18a_0 a_3)}{25920}\cdots \quad (0.57)$$

Vanishing $v_1(x)$ lets the coefficients $a_n \;(n = 0,1,2,\ldots)$ to take the following values

$$a_0 = 2, a_2 = 2, a_4 = \frac{4}{3}, a_6 = \frac{34}{45}, \dots \qquad (0.58)$$
$$a_1 = a_3 = a_5 = \cdots = 0$$

Therefore, the solution of Eq. (0.38) can be written as

$$u(x) = v_0(x) = x^2 + \frac{x^4}{6} + \frac{2x^6}{45} + \frac{17x^8}{1260} + \cdots \qquad (0.59)$$

Which converge to the exact solution in Eq. (0.39).

7. CONCLUSION

In this paper, some modifications of homotopy perturbation method have used for solving second order initial value problem of Bratu type. It should be mentioned the results of the methods exhibit excellent agreement with the exact solution. The computations corresponding to the examples have been performed using Mathematica.

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