Common Fixed Point Theorem for Compatible Maps of Type (β) and Type (α) using Integral Type Mapping

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ABSTRACT
In this paper we prove a common fixed point theorem for compatible map of type (β) in Fuzzy 2- metric space.

Keywords
Fuzzy 2- metric space, G- Cauchy sequence, Weakly compatible, point of coincidence, complete metric space

1. INTRODUCTION
Integral type contraction principle is one of the most popular contraction principle in fixed point theory. The first known result in this direction was given by Branciari [1] in general setting of lebgesgue integrable function and proved following fixed point theorems in metric spaces. In 1988, Grabiec [3] defined contraction and contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such spaces. Following Grabiec's approach, Mishra et al. [4] obtained common fixed point theorems for asymptotically commuting mappings on fuzzy metric spaces. In 1998, Vasuki [5] established a generalization of Grabiec's fuzzy contraction theorem wherein he proved a common fixed point theorem for a sequence of mappings in a fuzzy metric space. Thereafter, Cho [2] extended the concept of compatible mappings of type (alpha)

Our objective of this paper is to prove a common fixed point theorem by removing the assumption of continuity, relaxing compatibility to compatible maps of type (α) or (β). weak compatibility and replacing the completeness of the space with a set of alternative conditions for functions satisfying an implicit relation in FM-space.

In this paper the following implicit relation: Let $I = [0,1]$,$*$ be a continuous t-norm and F be the set of all real continuous functions $F : I^6 \rightarrow R$ satisfying the following conditions

1.1 F is no increasing in the fifth and sixth variables,

1.2 if, for some constant $k \in (0,1)$ we have

1.2(a) $F\left(u(kt), v(t), v(t), u(t), 1, u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right)\right) \geq 1,$

or

1.2(b) $F\left(u(kt), v(t), u(t), v(t), u\left(\frac{t}{2}\right) * v\left(\frac{t}{2}\right), 1\right) \geq 1$

for any fixed $t > 0$ and any nondecreasing functions $u, v : (0,\infty) \rightarrow I$ with $0 \leq u(t), v(t) \leq 1$ then there exists $h \in (0,1)$ with $u(ht) \geq v(t) * u(t)$,

1.3 if, for some constant $k \in (0,1)$ we have $F(u(kt), u(t), 1, 1, u(t), u(t)) \geq 1$

for any fixed $t > 0$ and any nondecreasing function $u : (0,\infty) \rightarrow I$ then $u(kt) \geq u(t)$.

2. PRELIMINARIES
Definition − 2.1 A triplet $(X, M, \star)$ is said to be a Fuzzy 2-metric space if X is an arbitrary set, $\star$ is a continuous t − norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following condition for all $x, y, z, s, t > 0$,

(FM − 1) $M(x, y, \theta, t) > 0$

(FM − 2) $M(x, y, \theta, t) = 1$ if and only if $x = y = \theta$.

(FM − 3) $M(x, y, \theta, t) = M(y, \theta, x, t) = M(\theta, x, y, t)$

(FM − 4) $M(x, y, \theta, t) \star M(y, z, \theta, s) \star M(z, x, \theta, q) \leq M(x, y, z, t + s + q)$

(FM − 5) $M(x, y, \theta, \bullet) : (0,\infty) \rightarrow (0,1]$ is continuous.

Then M is called a Fuzzy 2- metric on X. The function $M(x, y, \theta, t)$ denote the degree of nearness between x , y and θ with respect to t.

Example : Let (X, d) be a metric space. Define $a * b = \min\{a, b\}$ and $M(x, y, \theta, t) = \frac{t}{t+d(x,y,\theta)}$

For all $x, y \in X$ and all $t > 0$. Then $(X, M, \star)$ is a Fuzzy 2-metric space.

It is called the Fuzzy 2- metric space induced by d.We note that, $M(x, y, \theta, t)$ can be realized as the measure of nearness between x and y with respect to t. It is known that $M(x, y, \cdot)$ is non decreasing for all $x, y \in X$. Let $M(x, y, \star)$ be a Fuzzy 2- metric space for $t > 0$, the open ball $B(x, r, \theta, t) = \{y \in X: M(x, y, \theta, t) > 1 - r\}$.

Now, the collection $\{B(x, r, \theta, t): x \in X, 0 < r < 1, t > 0\}$ is a neighborhood system for a topology τ on X induced by the Fuzzy 2- metric M. This topology is Housdroff and first countable.

Definition 2.2 A sequence $\{x_n\}$ in a Fuzzy 2- metric space $(X, M, \star)$ is said to be a converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_n, x, \theta, t) > 1 - \varepsilon$ for all $n \geq n_0$.

Definition 2.3 A sequence $\{x_n\}$ in a Fuzzy 2- metric space $(X, M, \star)$ is said to be a G- Cauchy sequence converges to x iff for each $\varepsilon > 0$ and each $t > 0$, $n_0 \in N$ such that $M(x_m, x_n, \theta, t) > 1 - \varepsilon$ for all $m, n \geq n_0$.

A Fuzzy 2- metric space $(X, M, \star)$ is said to be complete if every G- Cauchy sequence in it converges to a point in it.

3. MAIN RESULT
Theorem 3.1 Let $(X, M, \star)$ be a complete Fuzzy 2- metric space and let A, B, S, T, P and Q be mappings from X into itself such that the following conditions are satisfied:

3.1(a) $P(X) \subset ST(X)$ and $Q(X) \subset AB(X)$, 3.1 (b) (P, AB) is compatible of type (β) and (Q, ST) is weak compatible,

3.1(c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$\int_0^{F\left(\begin{array}{c} M^2(Px,Qy,\theta,kt),M^2(ABx,STy,\theta,t),M^2(Px,ABx,\theta,t), \\ M^2(Qy,STy,\theta,t),M^2(Px,STy,\theta,t),M^2(ABx,Qy,\theta,t)\end{array}\right)} \xi(v)\ dv \geq 1$$

Where $\xi : [0, +\infty] \to [0, +\infty]$ is a lebgesgue integrable mapping which is summable on each compact subset of $[0, +\infty]$ non negative and such that $\forall\ \varepsilon > 0$, $\int_0^{\varepsilon} \xi(v)\ dv > 0$. Then A, B,S,T, P and Q have a unique common fixed point in X.

Proof : Let $x_0 \in X$, then from 3.1 (a) we have $x_1, x_2 \in X$ such tha $Px_0 = STx_1$ and $Qx_1 = ABx_2$

Inductively, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that for $n \in N$

$Px_{2n-2} = STx_{2n-1} = y_{2n-1}$ and $Qx_{2n-1} = ABx_{2n} = y_{2n}$

put $x = x_{2n}$ and $y = x_{2n+1}$ in 3.1(c) then we have

$$\int_0^{F\left(\begin{array}{c} M^2(Px_{2n},Qx_{2n+1},\theta,kt),M^2(ABx_{2n},STx_{2n+1},\theta,t),M^2(Px_{2n},ABx_{2n},\theta,t) \\ ,M^2(Qx_{2n+1},STx_{2n+1},\theta,t),M^2(Px_{2n},STx_{2n+1},\theta,t),M^2(ABx_{2n},Q_{2n+1},\theta,t)\end{array}\right)} \xi(v)\ dv > 1$$

$$\int_0^{F\left(\begin{array}{c} M^2(y_{2n+1},y_{2n+2},\theta,kt),M^2(y_{2n},y_{2n+1},\theta,t),M^2(y_{2n+1},y_{2n},\theta,t), \\ M^2(y_{2n+2},y_{2n+1},\theta,t),M^2(y_{2n+1},y_{2n+1},\theta,t),M^2(y_{2n},y_{2n+2},\theta,t)\end{array}\right)} \xi(v)\ dv > 1$$

$$\int_0^{F\left(\begin{array}{c} M^2(y_{2n+1},y_{2n+2},\theta,kt),M^2(y_{2n},y_{2n+1},\theta,t),M^2(y_{2n+1},y_{2n},\theta,t), \\ M^2(y_{2n+2},y_{2n+1},\theta,t),M^2(y_{2n+1},y_{2n+1},\theta,t), \\ M^2\left(y_{2n},y_{2n+1},\theta,\frac{t}{2}\right)\star M^2\left(y_{2n+1},y_{2n+2},\theta,\frac{t}{2}\right)\end{array}\right)} \xi(v)\ dv > 1$$

From condition 3.1 (a) we have

$$\int_0^{M^2(y_{2n+1},y_{2n+2},\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2\left(y_{2n},y_{2n+1},\theta,\frac{t}{2}\right)\star M^2\left(y_{2n+2},y_{2n+1},\theta,\frac{t}{2}\right)} \xi(v)\ dv$$

we have

$$\int_0^{M^2(y_{2n+1},y_{2n+2},\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2\left(y_{2n},y_{2n+1},\theta,\frac{t}{2}\right)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function so we have

$$M(y_{2n+1}, y_{2n+2}, \theta, kt) \geq M\left(y_{2n}, y_{2n+1}, \theta, \frac{t}{2}\right)$$

Similarly we have

$$M(y_{2n+2}, y_{2n+3}, \theta, kt) \geq M\left(y_{2n+1}, y_{2n+2}, \theta, \frac{t}{2}\right)$$

Thus we have

$$M(y_{n+1}, y_{n+2}, \theta, kt) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{2}\right)$$

$$M(y_{n+1}, y_{n+2}, \theta, t) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{2^k}\right)$$

$$M(y_n, y_{n+1}, \theta, t) \geq M\left(y_0, y_1, \theta, \frac{t}{2^{nk}}\right) \to 1 \text{ as } n \to \infty,$$

and hence $M(y_n, y_{n+1}, \theta, t) \to 1$ as $n \to \infty$ for all $t > 0$.

For each $\epsilon > 0$ *and* $t > 0$, we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, \theta, t) > 1 - \epsilon$ for all $n > n_0$.

For any $m, n \in N$ we suppose that $m \geq n$. Then we have

$$M(y_n, y_m, \theta, t) \geq M\left(y_n, y_{n+1}, \theta, \frac{t}{m-n}\right) \star M\left(y_{n+1}, y_{n+2}, \theta, \frac{t}{m-n}\right) \star \ldots. \star M\left(y_{m-1}, y_m, \theta, \frac{t}{m-n}\right)$$

$M(y_n, y_m, \theta, t) \geq (1-\epsilon) \star (1-\epsilon) \star \ldots\ldots \star (1-\epsilon)(m-n)$times

$$M(y_n, y_m, \theta, t) \geq (1-\epsilon)$$

And hence $\{y_n\}$ is a Cauchy sequence in X.

Since $(X, M, \star)$ is complete, $\{y_n\}$ converges to some point $z \in X$. Also its subsequences converges to the same point $z \in X$.

That is $\{Px_{2n+2}\} \to z$ and $\{STx_{2n+1}\} \to z$

$\{Qx_{2n+1}\} \to z$ and $\{ABx_{2n}\} \to z$

As (P, AB) is compatible pair of type (β), we have

$M(PPx_{2n}, (AB)(AB)x_{2n}, \theta, t) = 1$, for all $t > 0$

Or $M(PPx_{2n}, ABz, \theta, t) = 1$ Therefore, $PPx_{2n} \to ABz$.

Put $x = (AB)x_{2n}$ and $y = x_{2n+1}$ in 3.1(c)we have

$$\int_0^{F\left(\begin{array}{c} M^2(P(AB)x_{2n},Qy,\theta,kt),M^2(AB(AB)x_{2n},STx_{2n+1},\theta,t), \\ M^2(P(AB)x_{2n},AB(AB)x_{2n},\theta,t),M^2(Qx_{2n+1},STx_{2n+1},\theta,t), \\ M^2(P(AB)x_{2n},STx_{2n+1},\theta,t),M^2(AB(AB)x_{2n},Qx_{2n+1},\theta,t)\end{array}\right)} \xi(v)\ dv > 1$$

Taking $n \to \infty$ and 3.1(a) we get

$$\int_0^{M^2((AB)z,z,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2((AB)z,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function which implies

$$M((AB)z, z, \theta, kt) \geq M((AB)z, z, \theta, t)$$

we have $ABz = z$.

Put $x = z$ and $y = x_{2n+1}$ in 3.1(c) we have

$$\int_0^{F\left(\begin{array}{c} M^2(Pz,Q\,x_{2n+1},\theta,kt),M^2(ABz,ST\,x_{2n+1},\theta,t)\star M^2(Pz,ABz,\theta,t) \\ M^2(Q\,x_{2n+1},ST\,x_{2n+1},\theta,t),M^2(Pz,ST\,x_{2n+1},\theta,t), \\ M^2(ABz,Q\,x_{2n+1},\theta,t)\end{array}\right)} \xi(v)\ dv > 1$$

Taking $n \to \infty$ 3.1 (a) That is $\int_0^{M^2(Pz,z,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(Pz,z,\theta,t)} \xi(v)\ dv$

Since $\xi(v)$ is a lebesgue integrable function so we have

$$M(Pz, z, \theta, kt) \geq M(Pz, z, \theta, t)$$

we get $Pz = z$

So we have $ABz = Pz = z$.

Putting $x = Bz$ and $y = x_{2n+1}$ in 3.1(c), we get

$$\int_0^{F\left(\begin{array}{c} M^2(PBz,Qx_{2n+1},\theta,kt),M^2(ABBz,STx_{2n+1},\theta,t),M^2(PBz,ABBz,\theta,t) \\ M^2(Qx_{2n+1},STx_{2n+1},\theta,t),M^2(PBz,STx_{2n+1},\theta,t),M^2(ABBz,Qx_{2n+1},\theta,t)\end{array}\right)} \xi(v)\ dv > 1$$

Taking $n \to \infty$, 3.1(a)

$$\int_0^{M^2(Bz,z,,\theta kt)} \xi(v)\ dv \geq \int_0^{M^2(Bz,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function which follows

$$M(Bz, z, \theta, kt) \geq M(Bz, z, \theta, t)$$

we have $Bz = z$

And also we have $ABz = z$ implies $Az = z$

Therefore $Az = Bz = Pz = z$.

As $P(X) \subset ST(X)$ there exists $u \in X$ such that

$z = Pz = STu$

Putting $x = x_{2n}$ and $y = u$ in 3.1(c) we get

$$\int_0^{F\left(\substack{M^2(Px_{2n},Qu,\theta,kt),M^2(ABx_{2n},STu,\theta,t),M^2(Px_{2n},ABx_{2n},\theta,t)\\ M^2(Qu,STu,\theta,t),M^2(Px_{2n},STu,\theta,t),M^2(ABx_{2n},Qu,\theta,t)}\right)} \xi(v)\ dv > 1$$

Taking $n \to \infty$ we get

$$\int_0^{F\left(\substack{M^2(z,Qu,\theta,kt),M^2(z,STu,\theta,t),M^2(z,z,\theta,t)\\ M^2(Qu,STu,\theta,t),M^2(z,STu,\theta,t),M^2(z,Qu,\theta,t)}\right)} \xi(v)\ dv > 1$$

$$\int_0^{M^2(z,Qu,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(z,Qu,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function which implies

$M(z, Qu, \theta, kt) \geq M(z, Qu, \theta, t)$

we have $Qu = z$

Hence $STu = z = Qu$.

Hence (Q, ST) is weak compatible, therefore, we have

$QSTu = STQu$

Thus $Qz = STz$.

Putting $x = x_{2n}$ and $y = z$ in 3.1© we get

$$\int_0^{F\left(\substack{M^2(Px_{2n},Qz,\theta,kt),M^2(ABx_{2n},STz,\theta,t),M^2(Px_{2n},ABx_{2n},\theta,t)\\ M^2(Qz,STz,\theta,t),M^2(Px_{2n},STz,\theta,t),M^2(ABx_{2n},Qz,\theta,t)}\right)} \xi(v)\ dv > 1$$

Taking $n \to \infty$ we get

$$\int_0^{F\left(\substack{M^2(z,Qz,\theta,kt),M^2(z,STz,\theta,t),M^2(z,z,\theta,t)\\ M^2(Qz,STz,\theta,t),M^2(z,STz,\theta,t),M^2(z,Qz,\theta,t)}\right)} \xi(v)\ dv > 1$$

$$\int_0^{M^2(z,Qz,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(z,Qz,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function and hence

$M(z, Qz, \theta, kt) \geq M(z, Qz, \theta, t)$

we get $Qz = z$.

Putting $x = x_{2n}$ and $y = Tz$ in 5.3.2(c) we get

$$\int_0^{F\left(\substack{M^2(Px_{2n},QTz,\theta,kt),M^2(ABx_{2n},STTz,\theta,t),M^2(Px_{2n},ABx_{2n},\theta,t)\\ M^2(QTz,STTz,\theta,t),M^2(Px_{2n},STTz,\theta,t),M^2(ABx_{2n},QTz,\theta,t)}\right)} \xi(v)\ dv > 1$$

As $QT = TQ$ and $ST = TS$ we have

$QTz = TQz = Tz$

And $ST(Tz) = T(STz) = TQz = Tz$.

Taking $n \to \infty$ we get

$$\int_0^{F\left(\substack{M^2(z,Tz,\theta,kt),M^2(z,Tz,\theta,t),M^2(z,z,\theta,t)\\ M^2(Tz,Tz,\theta,t),M^2(z,Tz,\theta,t),M^2(z,Tz,\theta,t)}\right)} \xi(v)\ dv > 1$$

$$\int_0^{M^2(z,Tz,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(z,Tz,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function therefore

$M(z, Tz, \theta, kt) \geq M(z, Tz, \theta, t)$

we have $Tz = z$

Now $STz = Tz = z$ implies $Sz = z$.

Hence $Sz = Tz = Qz = z$

Combining $Az = Bz = Pz = Sz = Tz = Qz = z$

Hence z is the common fixed point of A, B, S, T, P and Q.

Uniqueness Let u be another common fixed point of A,B,S,T,P and Q. Then $Au = Bu = Su = Tu = Pu = Qu = u$ Putting $x = u$ and $y = z$ in 3.1(c) then we get

$$\int_0^{F\left(\substack{M^2(Pu,Qz,\theta,kt),M^2(ABu,STz,\theta,t),M^2(Pu,ABu,\theta,t)\\ M^2(Qz,STz,\theta,t),M^2(Pu,STz,\theta,t),M^2(ABu,Qz,\theta,t)}\right)} \xi(v)\ dv > 1$$

Taking limit both side then we get

$$\int_0^{F\left(\substack{M^2(u,z,\theta,kt),M^2(u,z,\theta,t),M^2(u,u,\theta,t)\\ M^2(z,z,\theta,t),M^2(u,z,\theta,t),M^2(u,z,\theta,t)}\right)} \xi(v)\ dv > 1$$

$$\int_0^{M^2(u,z,\theta,kt)} \xi(v)\ dv \geq \int_0^{M^2(u,z,\theta,t)} \xi(v)\ dv$$

Since $\xi(v)$ is a lebesgue integrable function so we have

$M(u, z, \theta, kt) \geq M(u, z, \theta, t)$

we get $z = u$.

That is z is a unique common fixed point of A,B, S, T, P and Q in X.

Corollary 3.2 Let $(X, M, \star)$ be a complete Fuzzy 2- metric space and let A, S, P and Q be mappings from X into itself such that the following conditions are satisfied:

(a) $P(X) \subset S(X)$ and $Q(X) \subset A(X)$,

(b) (P, A) is compatible of type (β) and (Q, S) is weak compatible,

(c) there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$

$$\int_0^{F\left(\substack{M^2(Px,Qy,\theta,kt),M^2(Ax,Sy,\theta,t),M^2(Px,Ax,\theta,t),\\ M^2(Qy,Sy,\theta,t),M^2(Px,Sy,\theta,t),M^2(Ax,Qy,\theta,t)}\right)} \xi(v)\ dv \geq 1$$

Where $\xi : [0, +\infty] \to [0, +\infty]$ is a lebgesgue integrable mapping which is summable on each compact subset of $[0, +\infty]$, non negative, and such that, $\forall\ \varepsilon > 0$, $\int_0^{\varepsilon} \xi(v)\ dv > 0$. Then A, S,P and Q have a unique common fixed point in X.

Proof If we take B = T = I (identity mapping) in Theorem 3.1 then we get the result.

4. REFERENCES

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