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Best Proximity Point on a Class of Multiplicative $\mathcal{MT}$-Cyclic Contraction Mapping

Ibrahim Aliyu Fulatan
Department of Mathematics
Ahmadu Bello University, Zaria

Sirajo Yahaya
Department of General Studies
Federal College of Education, Zaria

ABSTRACT

In this paper, by using a concept of $\mathcal{MT}$-function we introduce a new class of cyclic contraction mappings and consider the best proximity points theorem in the context of multiplicative metric space

General Terms

Best proximity point, Contraction maps

Keywords

Cyclic map, Best proximity point, $\mathcal{MT}$-function, Multiplicative metric, multiplicative $\mathcal{MT}$-cyclic contraction maps

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we denote by R, R^+, and N the sets of real numbers, non-negative real numbers and natural numbers respectively. Consistent with [1, 2], the following definitions and results will be needed in the sequel.

DEFINITION 1. *[1, 2] The multiplicative absolute value function* $|\cdot|^* : R^+ \to R^+$ *is defined as*

$$|x|^* = \begin{cases} x, & x \geq 1, \\ \frac{1}{x}, & x < 1. \end{cases}$$

Using the above definition of multiplicative absolute value function, [3] deduce the following proposition.

PROPOSITION 1. *[3] For arbitrary* $x, y \in R^+$, *the following hold:*

i. $|x| \geq 1$,
ii. $\frac{1}{|x|} \leq x \leq |x|$,
iii. $|\frac{1}{x}| = |x|$.
iv. $|x| \leq y$ *if and only if* $\frac{1}{y} \leq x \leq y$,
v. $|x \cdot y| \leq |x||y|$.

With the help of multiplicative absolute value function, they define the multiplicative distance between two non-negative real numbers as well as between two positive square matrices. This provides the basis for multiplicative metric spaces.

DEFINITION 2. *[1] Let X be a non-empty set. A function d^* : $X^2 \to R^+$ is said to be a multiplicative metric on X if for any $x, y, z \in X$, the following conditions holds:*

m_1. $d^*(x, y) \geq 1$ *and* $d^*(x, y) = 1 \iff x = y$,
m_2. $d^*(x, y) = d^*(y, x)$,
m_3. $d^*(x, y) \leq d^*(x, z) \cdot d^*(z, y)$.

The pair (X, d^*) is called *multiplicative metric space*.

EXAMPLE 1. *[1] Let $\mathcal{R}_+$ be the collection of all n-tupples of positive real numbers. And let $d(x, y) = |\frac{x_1}{y_1}| \cdot |\frac{x_2}{y_2}| \cdots |\frac{x_n}{y_n}|$ where $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in \mathcal{R}_+$. Then, clearly, $d(x, y)$ is a multiplicative metric space.*

EXAMPLE 2. *[1] Let (X, d) be a metric space, then the mapping d_a defined on X as follows is multiplicative metric, $d_a = a^{d(x,y)}$ where $a > 1$ is a real number. For discrete metric d the corresponding mappings d_a called discrete multiplicative metric is defined as:*

$$d_a(x, y) = a^{d(x,y)} = \begin{cases} 1, & x = y, \\ a, & x \neq y. \end{cases}$$

EXAMPLE 3. *[1] Let $d : \mathcal{R} * \mathcal{R} \to [\infty, \infty)$ be defined as $d_a = a^{d(x,y)}$ where $x, y \in \mathcal{R}$ and $a > 1$. Then $d(x, y)$ is a multiplicative metric.*

EXAMPLE 4. *[1] Let $X = C^*[a, b]$ be a collection of all real-valued multiplicative continuous functions over $[a, b] \subset \mathcal{R}^+$. Then (X, d) is a multiplicative metric with d defined by $d(f, g) = \sup_{x \in [a,b]} |\frac{f(x)}{g(x)}|$ for arbitrary $f, g \in X$*

Remark
Neither every metric is multiplicative metric nor every multiplicative metric is metric. The mapping in example 1 defined above is multiplicative metric but not metric as it doesn't satisfy triangular inequality. Consider, $d(2, 3) + d(3, 5) = \frac{3}{2} + \frac{5}{3} = 3.2 > 2.5 = d(2, 5)$ On the other hand the usual metric on $\mathcal{R}$ is not multiplicative metric as it doesn't satisfy multiplicative triangular inequality. As $d(4, 7).d(7, 9) = 6 > 5 = d(4, 9)$

DEFINITION 3. *[1] Let $(X.d)$ be a multiplicative metric space. Then we have the following $|\frac{d(x,z)}{d(y,z)}| \leq d(x, y)$ for all $x, y \in X$ Which is called multiplicative reversed triangular inequality.*

DEFINITION 4. *[2] Let x_0 be an arbitrary point in a multiplicative metric space X and $\epsilon > 1$. A multiplicative open ball $B(x_0, \epsilon)$ of radius ϵ centered at x_0 is the set $\{y \in X : d(y, x_0) < \epsilon\}$.*

A sequence $\{x_n\}$ in a multiplicative metric space X is said to be multiplicative convergent to a point $x \in X$ if for any given $\epsilon > 1$, there is $N_0 \in N$ such that $x_n \in B(x, \epsilon)$ for all $n \geq N_0$. If $\{x_n\}$ converges to x, we write $x_n \to x$ as $n \to \infty$. A sequence x_n in X is multiplicative convergent to x in X if and only if $d(x_n, x) \to 1$ as $n \to \infty$ [2].

DEFINITION 5. *[2] Let(X, d) be a multiplicative metric space.*

a) A sequence $\{x_n\}$ in X is said to be multiplicative Cauchy if for any $\epsilon > 1$, there exists $N_0 \in N$ such that $d(x_n, x_m) < \epsilon$ for all $n, m > N_0$,
b) A multiplicative metric space (X, d) is said to be complete if every Cauchy sequence $\{x_n\}$ in X is multiplicative convergent to a point $x \in X$.

A sequence $\{x_n\}$ in X is multiplicative Cauchy if and only if $d(x_n, x_m) \to 1$ as $n, m \to \infty$ [2].

DEFINITION 6. *[2](Multiplicative generalization of Supremum): Let A be a nonempty subset of R^+. Then, $s = \sup A$ if and only if*

i. $a \leq s$ for all $a \in A$
ii There exists atleast a point $a \in A$ such that $|\frac{s}{a}| < \epsilon$ for all $\epsilon > 1$

DEFINITION 7. *[2](Multiplicative generalization of Infimum): Let A be a nonempty subset of $\mathcal{R}^+$. Then, $m = Inf A$ if and only if*

i. $m \leq s \forall a \in A$
ii. There exists atleast a point $a \in A$ such that $|\frac{a}{m}| < \epsilon$ for all $\epsilon > 1$

THEOREM 8. *[2](Multiplicative Bolzano-Weierstrass): Every multiplicative bounded sequence $\{x_n\} \in (\mathcal{R}^+, |\cdot|)$ has a multiplicative convergent subsequence.*

Banach contraction mapping principle has been a very advantageous and efficient means in nonlinear analysis. Various authors have generalized Banach Contraction principle in different spaces. Recently, Ozavsar and Cervikel [2] generalized the celebrated Banach contraction principle in the set up of multiplicative metric spaces.

DEFINITION 9. *[2] Let X be a multiplicative metric space. A mapping $T : X \to X$ is said to be multiplicative contractive if there exists $\lambda \in [0, 1)$ such that $d(Tx, Ty) \leq d(x, y)^\lambda$, for all $x, y \in X$.*

THEOREM 10. *[4] Let X be a multiplicative metric space and $T : X \to X$ a multiplicative contractive mapping. Then T has a unique fixed point.*

PROPOSITION 2. *[4] Let $\psi : [0, \infty) \to [0, 1)$ be a function. Then, ψ is an $\mathcal{R}$-function if and only if for any nonincreasing sequence $\{x_n\}_{n \in (0,\infty)}$, we have $0 \leq sup\psi\{x_n\} < 1$*

On the other hand, [5] established a classical Best proximity point theorems which generalizes the concept of fixed point theorems. For more information about multiplicative metric space and best proximity point theorems see [6, 7, 8, 9]
In this paper, we established some new existence and convergence theorems of iterates of best proximity points for multiplicative $\mathcal{MT}$-cyclic contractions.

2. MAIN RESULTS

Motivated by the concept of multiplicative cyclic contractions and $\mathcal{MT}$-functions, we first introduce the concept of multiplicative $\mathcal{MT}$-cyclic contractions

DEFINITION 11. *Let A and B be non-empty subsets of a multiplicative metric space (X, d). If a map $T : A \cup B \to A \cup B$ satisfies*

M_1^* *$T(A) \subset B$ and $T(B) \subset A$*
M_2^* *there exists an $\mathcal{MT}$-function $\psi : [1, \infty) \to [0, 1)$ such that*

$d(Tx, Ty) \leq (d(x, y))^{\psi(d(x,y))} \cdot d(A, B)^{(1-\psi(d(x,y)))}$, *for all $x \in A$ and $y \in B$,*

then T is called a multiplicative $\mathcal{MT}$-cyclic contraction with respect to ψ on $A \cup B$.
Remark 2.1 It is obvious that (M_2^*) implies that T satisfies $d(Tx, Ty) \leq d(x, y)$ for any $x \in A$ and $y \in B$.

THEOREM 12. *Let A and B be non-empty subsets of a multiplicative metric space (X, d) and $T : A \cup B \to A \cup B$ be a multiplicative $\mathcal{R}$-cyclic contraction with respect to ψ.*
Then there exists a sequence $\{x_n\}_{n \in N} \subset A \cup B$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in N} d(x_n, x_{n+1}) = d(A, B)$$

PROOF. Let $x_1 \in A$ be given. Define an iterative sequence $\{x_n\}_{n\in N}$ by $x_{n+1} = Tx_n$ for $n \in N$
then $d(A, B) \leq d(x_n, x_{n+1})$ for all $n \in N$.
If there exists $j \in N$ such that $x_j = x_{j+1} \in A \cap B$,
then by definition $Tx_j = x_{j+1} = x_j$
also, $x_{j+2} = Tx_{j+1} = T(Tx_j) = Tx_j = x_j$
so $x_j = x_{j+1} = x_{j+2} = ...$ and
therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 1$.
So,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in N} d(x_n, x_{n+1}) = d(A, B) = 1$$

and it suffices to consider the case $x_{n+1} \neq x_n$ for all $n \in N$. Since the sequence $\{d(x_n, x_{n+1})\}$ is non-increasing in $(1, \infty)$,then $t_1 = \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in N} d(x_n, x_{n+1}) \geq 1$.
Since φ is an $\mathcal{MT}-$function we have,
$0 \leq \sup_{n\in N} \varphi(d(x_n, x_{n+1})) < 1$.
Letting $\lambda := \sup_{n\in N} \varphi(d(x_n, x_{n+1}))$,
then $0 \leq \varphi(d(x_n, x_{n+1})) \leq \lambda < 1$ for all $n \in N$.
If $x_1 \in A$, then by the cyclic nature of T, we have $x_{2n-1} \in A$ and $x_{2n} \in B$ for all $n \in N$.
Since T is a multiplicative cyclic $\mathcal{MT}$-contraction, we have

$$\begin{aligned} d(x_2, x_3) &= d(Tx_1, Tx_2) \\ &\leq d(x_1, x_2)^{\psi d(x_1,x_2)} \cdot d(A, B)^{1-\psi d(x_1,x_2)} \\ &\leq d(x_1, x_2)^\lambda \cdot d(A, B) \end{aligned}$$

and

$$\begin{aligned} d(x_3, x_4) &= d(Tx_2, Tx_3) \\ &\leq d(x_2, x_3)^{\psi d(x_2,x_3)} \cdot d(A, B)^{1-\psi d(x_2,x_3)} \\ &\leq (d(x_1, x_2)^\lambda \cdot d(A, B))^{\psi d(x_2,x_3)} \cdot d(A, B)^{1-\psi d(x_2,x_3)} \\ &= d(x_1, x_2)^{\lambda^2} \cdot d(A, B) \end{aligned}$$

also,

$$\begin{aligned} d(x_4, x_5) &= d(Tx_3, Tx_4) \\ &\le d(x_3, x_4)^{\psi d(x_3,x_4)} \cdot d(A,B)^{1-\psi d(x_3,x_4)} \\ &\le (d(x_1,x_2)^{\lambda^2} \cdot d(A,B))^{\psi d(x_3,x_4)} \cdot d(A,B)^{1-\psi d(x_3,x_4)} \\ &= d(x_1,x_2)^{\lambda^3} \cdot d(A,B) \end{aligned}$$

Hence, continuing in this fashion, one can obtain

$$d(A,B) \le d(x_{n+1}, x_{n+2}) \le d(x_1, x_2)^{\lambda^n} \cdot d(A,B) \qquad (1)$$

for all $n \in N$.
Since $\lambda \in [0, 1)$, $\lim_{n\to\infty} \lambda^n = 0$. Using inequality (1) and the non-increasing nature of $\{d(x_n, x_{n+1})\}$, we obtain

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in N} d(x_n, x_{n+1}) = d(A,B).$$

□

Next, we give an existence theorem for a class of cyclic mappings.

THEOREM 13. *Let A and B be non-empty subsets of a multiplicative metric space (X, d) and $T : A \cup B \to A \cup B$ be a multiplicative $\mathcal{MT}$-cyclic contraction. Let $x_1 \in A$ be given. Define an iterative sequence $\{x_n\}_{n\in N}$ by $x_{n+1} = Tx_n$ for $n \in N$. Suppose further that $\{x_{2n-1}\}$ has a multiplicative convergent subsequence in A, then there exists $x \in A$ such that $d(x, Tx) = d(A, B)$.*

PROOF. Let $\{x_{2n_k-1}\}$ be a subsequence of $\{x_{2n-1}\}$ such that $x_{2n_k-1} \longrightarrow x$ as $k \longrightarrow \infty$. Now we observe that,

$$d(A,B) \le d(x, x_{2n_k}) \le d(x, x_{2n_k-1}) \cdot d(x_{2n_k-1}, x_{2n_k})$$

for all $k \in N$.
since, $\lim_{k\to\infty} d(x, x_{2n_k-1}) = 1$, from theorem 12 and

$$d(A,B) \le d(x, x_{2n_k}) \le d(x, x_{2n_k-1}) d(x_{2n_k-1}, x_{2n_k})$$

we have, (taking limit as $k \to \infty$)

$$d(A,B) \le \lim_{k\to\infty} d(x, x_{2n_k}) \le \lim_{k\to\infty} d(x_{2n_k-1}, x_{2n_k}) = d(A,B)$$

hence, $\lim_{k\to\infty} d(x, x_{2n_k}) = d(A, B)$.
Now,

$$\begin{aligned} d(A,B) &\le d(x_{2n_k+1}, Tx) \\ &= d(x_{2n_k}, Tx) \\ &\le d(x_{2n_k}, x)^{\varphi(d(x_{2n_k},x))} \cdot d(A,B)^{1-\varphi(d(x_{2n_k},x))} \\ &\le d(x_{2n_k}, x)^{\varphi(d(x_{2n_k},x))} \cdot d(x_{2n_k}, x)^{1-\varphi(d(x_{2n_k},x))} \\ &\le d(x_{2n_k}, x) \end{aligned}$$

hence,

$$d(A,B) \le d(x_{2n_k+1}, Tx) \le d(x_{2n_k}, x)$$

Taking limit as $k \to \infty$ we obtain

$$d(A,B) \le d(x, Tx) \le d(A,B)$$

and so $d(x, Tx) = d(A, B)$. □

3. CONCLUSIONS

In this paper, Best Proximity Point on a Class of Multiplicative $\mathcal{MT}$-Cyclic Contraction Mapping is investigated under some suitable conditions. Since Best proximity point results generalizes fixed point results, these results extend the results of [7]. Furthermore, the best proximity point results for cyclic contraction mapping satisfying another contractive conditions in multiplicative metric spaces is still open for interested researchers.

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