Some Preserved Properties under Localization, S −Prime Radicals and S −Minimal Prime Ideals

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ABSTRACT
In this paper we prove some properties of ideals that are preserved under localization. Also, we establish several one to one correspondences between certain types of ideals in the ring and its localization at multiplicative systems. We introduce two concepts such as S −prime radical and S −minimal prime ideals and the relations of S −prime radical with the prime radical and Jacobson radical of a ring under localization are determined. Also, we prove a one to one correspondence between S −minimal primal ideals of a given ideal A of a ring R and the minimal prime ideals of the ideal A_P of R_P, where P is a prime ideal of R.

Keywords
Multiplicative system, Localization of a ring, S −prime radical, S −minimal prime ideal, S −semisimple ring and S −Jacobson radical($Srad(R)$).

1. INTRODUCTION
Let R be a commutative ring with identity. A nonempty subset S of R is called a multiplicative system in R, if $0 \notin S$ and $a, b \in S$ implies that $ab \in S$ [4]. An ideal A of R is called a principal ideal of R, if $A =< a >$, for some $a \in A$ and R is called a principal ideal ring if every ideal of R is a principal ideal [1]. The spectrum of R, is denoted by $Spec(R)$ and defined as $Spec(R) = \{P: P$ is a prime ideal of $R\}$ and the prime radical of R is denoted by $Rad(R)$ and defined as $Rad(R) = \bigcap_{P \in Spec(R)} P$ and R is said to be without prime radical if $Rad(R) = 0$ [1]. The maximal spectrum of R is denoted by $mSpec(R)$ defined as $mSpec(R) = \{P: P$ is a maximal ideal of $R\}$ and the Jacobson radical of R, denoted by $rad(R)$ (or $J(R)$ [5]), is defined as $rad(R) = \bigcap_{P \in mSpec(R)} P$ and R is called a semisimple ring if $rad(R) = 0$ [1]. The Nil Radical of an ideal A, is defined as $\sqrt{A} = \{x \in R: x^n \in A$, for some positive integer $n\}$ [1]. A prime ideal P of R is called a Minimal Prime Ideal of A if $A \subseteq P$ and R contains no prime ideal Q with $A \subseteq Q \subset P$ and P is called a minimal prime ideal of R if R contains no prime ideal Q with $Q \subset P$. Let A be an ideal of R and $S_R(A)$ be defined as $S_R(A) = \{r \in R: ra \in A$, for some $a \notin A\}$ [2,3].

2. SOME PROPERTIES OF IDEALS THAT ARE PRESERVED UNDER LOCALIZATION, S −RADICAL IDEALS AND S −MINIMAL PRIME IDEALS

2.1 Some properties of ideals that are preserved under localization
In this section, we prove some algebraic properties of ideals which are preserved under localization and we start with the following result which determines the radically property of ideals in the both rings R and R_S.

Proposition 2.1.1. Let R be a commutative ring with identity and A be an ideal of R. If S is a multiplicative system in R, then $\sqrt{A_S} = (\sqrt{A})_S$.

Proof. Let $\frac{r}{s} \in \sqrt{A_S}$, where $r \in R, s \in S$. Then, $(\frac{r}{s})^n \in A_S$, for some $n \in Z_+$, that is $\frac{r^n}{s^n} \in A_S$, there exists $t \in S$ such that $tr^n \in A$, and then $(tr)^n = t^{n-1}tr^n \in A$, so that $tr \in \sqrt{A}$ and thus we get $\frac{r}{s} = \frac{t}{t}\frac{r}{s} = \frac{tr}{ts} \in (\sqrt{A})_S$. Hence, $\sqrt{A_S} \subseteq (\sqrt{A})_S$. Next, let $\frac{r}{s} \in (\sqrt{A})_S$, where $r \in R, s \in S$, then $tr \in \sqrt{A}$, for some $t \in S$, so that $(tr)^n \in A$, for some $n \in Z_+$. Then, we get $(\frac{r}{s})^n = \frac{t^n}{t^n}\frac{r^n}{s^n} = \frac{(tr)^n}{t^n s^n} \in A_S$, so that $\frac{r}{s} \in \sqrt{A_S}$, thus we get$(\sqrt{A})_S \subseteq \sqrt{A_S}$. Hence, we have $\sqrt{A_S} = (\sqrt{A})_S$.

As a corollary to the above proposition we give the following.

Corollary 2.1.2. Let R be a commutative ring with identity and A be an ideal of R. If P is a prime ideal of R, then $\sqrt{A_P} = (\sqrt{A})_P$.

Proof. As P is a prime ideal, we have $R \setminus P$ is a multiplicative system in R, so by taking $S = R \setminus P$ in Proposition 2.1.1, the result follows at once.

Proposition 2.1.3. Let R be a commutative ring with identity and S be a multiplicative system in R. If A, B are ideals of R such that $A_S \subseteq B_S$ and $S_R(B) \cap S = \emptyset$, then $A \subseteq B$. In particular, if $A_S = 0$ and $S_R(0) \cap S = \emptyset$, then $A = 0$.

Proof. Let $a \in A$. Let $s \in S$ (this is possible since, $S \neq \emptyset$), then we have $\frac{a}{s} \in B_S$, so $ua \in B$, for some $u \in S$. If $a \notin B$, then $u \in S_R(B)$, which contradicts the fact that $S_R(B) \cap S = \emptyset$, so that we must have $a \in B$. Hence $A \subseteq B$. For the second part, if $a \in A$ is any element, then $\frac{a}{s} = 0$, so that $sa = 0$, for some $s \in S$. If $a \neq 0$, then $s \in S_R(0)$, which is a contradiction, so we must have $a = 0$. Hence, $A = 0$.

As a corollary to the above proposition, we give the following.

Corollary 2.1.4. Let R be a commutative ring with identity and P be a prime ideal of R. If A, B are ideals of R such that $A_P \subseteq B_P$ and $S_R(B) \subseteq P$, then $A \subseteq B$. In particular, if $A_P = 0$ and $S_R(0) \subseteq P$, then $A = 0$.

Proof. As $R \setminus P$ is a multiplicative system in R and $S_R(B) \cap (R \setminus P) = \emptyset$ if and only if $S_R(B) \subseteq P$, so by taking $S = R \setminus P$ in Proposition 2.1.3, the proof will follows at once.

In the following result, we prove that every proper ideal of R_S is a localization of some proper ideal of R.

Proposition 2.1.5. Let R be a commutative ring with identity and S be a multiplicative system in R. If $\bar{Q}$ is an ideal of R_S, then there exists an ideal Q of R with $\bar{Q} = Q_S$. Furthermore, if $\bar{Q}$ is a proper ideal of R_S, then Q is a proper ideal of R.

Proof. As $S \neq \emptyset$, let $t \in S$. Now, let $Q = \{a \in R: \frac{ta}{t} \in \bar{Q}\}$. One can easily show that Q is an ideal of R. Next, we will show that $\bar{Q} = Q_S$. Let, $\frac{r}{s} \in \bar{Q}$, where $r \in R, s \in S$. Then $\frac{tr}{t} = \frac{ts}{t}\frac{r}{s} \in \bar{Q}$, so that $r \in Q$, so we have $\frac{r}{s} \in Q_S$. Hence, $\bar{Q} \subseteq Q_S$. Let $\frac{r}{s} \in Q_S$, where $r \in R, s \in S$, then we have $qr \in Q$, for some $q \in S$. Hence, we get $\frac{tqr}{t} \in \bar{Q}$. Then, $\frac{r}{s} = \frac{1}{sq}\frac{tqr}{t} \in \bar{Q}$, so that $Q_S \subseteq \bar{Q}$. Hence, $\bar{Q} = Q_S$. Next, if $Q = R$, then $1 \in Q$, so that $\frac{t}{t} \in \bar{Q} = Q_S$, this implies that $\bar{Q} = R_S$, which is a contradiction. Hence, Q is a proper ideal of R.

Now, we prove that, under certain condition a localization of a primary ideal is also a primary ideal.

Proposition 2.1.6. Let R be a commutative ring with identity and S be a multiplicative system in R. If Q is a primary ideal of R such that $Q \cap S = \emptyset$, then Q_S is a primary ideal of R_S.

Proof. As $Q \cap S = \emptyset$, by [4], we have $Q_S \neq R_S$, so that Q_S is a proper ideal of R_S. Let $\frac{a}{s}\frac{b}{t} \in Q_S$, where $a, b \in R$ and $s, t \in S$, then $\frac{ab}{st} \in Q_S$, so that $pab \in Q$, for some $p \in S$. If $\frac{a}{s} \notin Q_S$, then $a \notin Q$ and as Q is primary, we get $(pb)^n \in Q$, for some $n \in Z_+$. Then, $(\frac{b}{t})^n = \frac{b^n}{t^n} = \frac{p^n}{p^n}\frac{b^n}{t^n} = \frac{(pb)^n}{p^n t^n} \in Q_S$, so that Q_S is a primary ideal of R_S.

Next, we prove that every primary ideal of R_S is a localization of a primary ideal of R disjoint from S.

Proposition 2.1.7. Let R be a commutative ring with identity and S be a multiplicative system in R. If $\bar{Q}$ is a primary ideal of R_S, then there exists a primary ideal Q of R with $Q \cap S = \emptyset$ and $\bar{Q} = Q_S$.

Proof. By Proposition 2.1.5, we have $\bar{Q} = Q_S$, where $Q = \{a \in R: \frac{ta}{t} \in \bar{Q}\}$ is an ideal of R and $t \in S$ is some fixed element. As, $Q_S \neq R_S$, by [4], we get $Q \cap S = \emptyset$ and as $\bar{Q}$ is proper, we get Q is a proper ideal of R. Let for $a, b \in R$, we have $ab \in Q$ but $a \notin Q$, then $\frac{ta}{t}\frac{tb}{t} = \frac{tab}{t} \in Q_S$. If $\frac{ta}{t} \in Q_S = \bar{Q}$, then $a \in Q$, which is a contradiction, thus $\frac{ta}{t} \notin Q_S$ and as Q_S is primary, we get $(\frac{tb}{t})^n \in Q_S$, for some $n \in Z_+$, then we get $\frac{tb^n}{t} = \frac{t^{n-1}}{t^{n-1}}\frac{tb^n}{t} = \frac{t^n b^n}{t^n} = (\frac{tb}{t})^n \in Q_S = \bar{Q}$, so that $b^n \in Q$. Hence, Q is a primary ideal of R.

By combining Proposition 2.1.6 and Proposition 2.1.7, we get the following theorem.

Theorem 2.1.8. Let R be a commutative ring with identity and S be a multiplicative system in R. Then, there is a one to one correspondence between the primary ideals of R_S and the primary ideals of R which does not meet S.

Proof. Let $F = \{Q: Q$ is a primary ideal of R with $Q \cap S = \emptyset\}$ and $H = \{\bar{Q}: \bar{Q}$ is a primary ideal of $R_S\}$. Define $f: F \to H$ as follows: If $Q \in F$, then Q is a primary ideal of R and $Q \cap S = \emptyset$, then by Proposition 2.1.6, we get Q_S is a primary ideal of R_S, so that $Q_S \in H$, so we define $f(Q) = Q_S$. One can easily show that f defines a one to one correspondence between F and H.

As a corollary to Theorem 2.1.8, we give the following corollary.

Corollary 2.1.9. Let R be a commutative ring with identity and P be a prime ideal of R, then there is a one to one correspondence between the primary ideals of R_P and the primary ideals of R which contained in P.

Proof. As $R \setminus P$ is a multiplicative system in R and since for any primary ideal Q of R, we have $Q \cap (R \setminus P) = \emptyset$ if and only if $Q \subseteq P$, so by taking $S = R \setminus P$ in Theorem 2.1.8, the proof will follows at once.

In the next two results, we establish a one to one correspondence between the maximal ideals of R_S and maximal ideals of R which are disjoint from S.

Proposition 2.1.10. Let R be a commutative ring with identity and S be a multiplicative system in R. If Q is a maximal ideal of R with the property that, $Q \cap S = \emptyset$, then Q_S is a maximal ideal of R_S.

Proof. As $\mathrm{Q} \cap \mathrm{S} = \emptyset$, by [4], we get $\mathrm{Q_S} \neq \mathrm{R_S}$. Now, let $\bar{\mathrm{J}}$ be any ideal of $\mathrm{R_S}$ such that $\mathrm{Q_S} \subseteq \bar{\mathrm{J}} \subset \mathrm{R_S}$ with $\bar{\mathrm{J}} \neq \mathrm{R_S}$, then by Proposition 2.1.5, we have $\bar{\mathrm{J}} = \mathrm{J_S}$, for the proper ideal $\mathrm{J} = \{\mathrm{x} \in \mathrm{R}: \frac{\mathrm{tx}}{\mathrm{t}} \in \bar{\mathrm{J}}\}$, where $\mathrm{t} \in \mathrm{S}$. As, $\bar{\mathrm{J}} = \mathrm{J_S} \neq \mathrm{R_S}$, by [4], we get $\mathrm{J} \cap \mathrm{S} = \emptyset$, then we have $\mathrm{Q_S} \subseteq \mathrm{J_S} \subset \mathrm{R_S}$. Next, let $\mathrm{x} \in \mathrm{Q}$, then $\mathrm{tx} \in \mathrm{Q}$, so that $\frac{\mathrm{tx}}{\mathrm{t}} \in \mathrm{J_S} = \bar{\mathrm{J}}$, so that $\mathrm{x} \in \mathrm{J}$. Hence, we get $\mathrm{Q} \subseteq \mathrm{J} \subset \mathrm{R}$ and as Q is maximal, we get $\mathrm{Q} = \mathrm{J}$, which gives $\mathrm{Q_S} = \mathrm{J_S}$. Hence, $\mathrm{Q_S}$ is a maximal ideal of $\mathrm{R_S}$

Proposition 2.1.11. Let R be a commutative ring with identity and S be a multiplicative system in R. If $\bar{Q}$ is a maximal ideal of R_S, then $\bar{Q} = Q_S$, for some maximal ideal Q of R with $Q \cap S = \emptyset$.

Proof. By Proposition 2.1.5, we have $\bar{Q} = Q_S$, for the proper ideal $Q = \{x \in R: \frac{tx}{t} \in \bar{Q}\}$ and some fixed $t \in S$. As $\bar{Q} = Q_S \neq R_S$, by [4], we get $Q \cap S = \emptyset$. To show Q is maximal, let J be any ideal of R such that $Q \subseteq J \subset R$ $(J \neq R)$ with $J \cap S = \emptyset$, then $\bar{Q} = Q_S \subseteq J_S \subseteq R_S$. If $Q \neq J$, then there exists $x \in J$ but $x \notin Q$. Then, $tx \in J$, so that, $\frac{tx}{t} \in J_S$, but $\frac{tx}{t} \notin \bar{Q}$ and as $\bar{Q}$ is maximal, we get $J_S = R_S$, then by [4], we get $J \cap S \neq \emptyset$, which is a contradiction, so that $Q = J$. Hence, Q is maximal with respect to the disjoin-ness property from S.

In the next few results, we prove that there is a one to one correspondence between the nil (nilpotent) ideals of the rings R and R_S.

Lemma 2.1.12. Let R be a commutative ring with identity and S be a multiplicative system in R. If A is a nilpotent (resp. a nil) ideal of R, then A_S is a nilpotent (resp. a nil) ideal of R_S.

Proof. As A is nilpotent, we have $A^n = 0$, for some $n \in Z_+$, then $(A_S)^n = (A^n)_S = 0$, so that A_S is nilpotent. For the second part, let $\frac{a}{p} \in A_S$, where $a \in R, p \in S$, then we have $sa \in A$, for some $s \in S$, that gives $(sa)^n = 0$, for some $n \in Z_+$ and then $(\frac{a}{p})^n = \frac{s^n}{s^n}\frac{a^n}{p^n} = \frac{(sa)^n}{(sp)^n} = 0$. Hence, A_S is a nil ideal of R_S.

Corollary 2.1.13. Let R be a commutative ring with identity and P be a prime ideal of R. If A is a nilpotent (resp. a nil) ideal of R, then A_P is a nilpotent (resp. a nil) ideal of R_P.

Proof. By taking $S = R \setminus P$ in Lemma 2.1.12, the result will follow directly.

Lemma 2.1.14. Let R be a commutative ring with identity and S be a multiplicative system in R and $S_R(0)\cap S = \emptyset$. If $\bar{A}$ is a nilpotent (resp. a nil) ideal of R_S, then $\bar{A} = A_S$, for some nilpotent (resp. nil) ideal A of R.

Proof. By Proposition 2.1.5, we have $\bar{A} = A_S$, for the ideal $A = \{x \in R: \frac{tx}{t} \in \bar{A}\}$, where $t \in S$ is a fixed element and as $\bar{A}$ is nilpotent, we have $(\bar{A})^n = 0$, for some $n \in Z_+$, so that $(A^n)_S = (A_S)^n = (\bar{A})^n = 0$ and since, $S_R(0)\cap S = \emptyset$, so by Proposition 2.1.3, we get $A^n = 0$. Hence, A is nilpotent. For the proof of second part, let $a \in A$, then $\frac{ta}{t} \in \bar{A} = A_S$ and as $\bar{A}$ is nil, we get $(\frac{ta}{t})^n = 0$, for some $n \in Z_+$, then $\frac{ta^n}{t} = \frac{t^n a^n}{t^n} = (\frac{ta}{t})^n = 0$, so that $uta^n = 0$, for some $u \in S$ and then, $ut \in S$. If $a^n \neq 0$, then $ut \in S_R(0)$, which contradicts the fact that $S_R(0)\cap S = \emptyset$. Hence, $a^n = 0$, that means A is a nil ideal of R.

Now, we get the following corollary.

Corollary 2.1.15. Let R be a commutative ring with identity and S be a multiplicative system in R. If A is an ideal of R such that A_S is a nilpotent (resp. a nil) ideal of R_S and $S_R(0)\cap S = \emptyset$, then A is a nilpotent (resp. a nil) ideal of R. In particular, if P is a prime ideal of R such that A_P is a nilpotent (resp. a nil) ideal of R_P and $S_R(0) \subseteq P$, then A is a nilpotent (resp. a nil) ideal of R.

Proof. The proof of the first part follows directly as the same as in Lemma 2.1.14 and the proof of second part follows by taking $S = R \setminus P$ in Lemma 2.1.14 and from the fact that $S_R(0)\cap(R \setminus P) = \emptyset$ if and only if $S_R(0) \subseteq P$.

Corollary 2.1.16. Let R be a commutative ring with identity and P be a prime ideal of R such that $S_R(0) \subseteq P$. If $\bar{A}$ is a nilpotent (resp. a nil) ideal of R_P, then $\bar{A} = A_P$, for some nilpotent (resp. nil) ideal A of R.

Proof. Since, $R \setminus P$ is a multiplicative system in R and since, $S_R(0)\cap(R \setminus P) = \emptyset$ if and only if $S_R(0) \subseteq P$, so by taking $S = R \setminus P$ in Lemma 2.1.14, the proof will follows directly.

Lemma 2.1.17. Let R be a commutative ring with identity and S be a multiplicative system in R. If $r \in R, s \in S$ and A is an ideal of R, then $\frac{r}{s}A_S = (rA)_S$.

Proof. Let $\frac{a}{t} \in A_S$, where $a \in R, t \in S$, then $pa \in A$, for some $p \in S$, then $pra \in rA$ and that $\frac{r}{s}\frac{a}{t} = \frac{p}{p}\frac{r}{s}\frac{a}{t} = \frac{pra}{pst} \in (rA)_S$, so that $\frac{r}{s}A_S \subseteq (rA)_S$. Next, let $\frac{x}{t} \in (rA)_S$, where $x \in R, t \in S$. Then, $px \in rA$, for some $p \in S$, so $px = ra$, for some $a \in A$ and that $sa \in A$. Now, we have $\frac{x}{t} = \frac{p}{p}\frac{x}{t} = \frac{px}{pt} = \frac{ra}{pt} = \frac{r}{s}\frac{s}{p}\frac{a}{t} = \frac{r}{s}\frac{sa}{pt} \in \frac{r}{s}A_S$, so that $(rA)_S \subseteq \frac{r}{s}A_S$. Hence, $\frac{r}{s}A_S = (rA)_S$.

Corollary 2.1.18. Let R be a commutative ring with identity and P be a prime ideal of R. If $r \in R, s \notin P$ and A is an ideal of R, then $\frac{r}{s}A_P = (rA)_P$.

Proof. By taking $S = R \setminus P$ in Lemma 2.1.17, the result follows directly.

The remaining results of this section deal with the concept of principality of ideals in the both rings R and R_S. In fact, we prove some results concerning this concept and among these results, we prove that a localization of a principal ideal ring is also a principal ideal ring.

Proposition 2.1.19. Let R be a commutative ring with identity and S be a multiplicative system in R. If R is a principal ideal ring, then R_S is a principal ideal ring.

Proof. Let $\bar{A}$, be any ideal of R_S, then by Proposition 2.1.5, we have $\bar{A} = A_S$, for the ideal $A = \{a \in R: \frac{ta}{t} \in \bar{A}\}$ and a fixed $t \in S$. As R is a principal ideal ring, we get $A = < x >$, for some $x \in R$. Clearly, $x \in A$, so that $\frac{tx}{t} \in \bar{A}$, so that $< \frac{tx}{t} > \subseteq \bar{A}$. Let $\frac{r}{s} \in \bar{A} = A_S$, where $r \in R, s \in S$, then $pr \in A$, for some $p \in S$, so that $pr = ax$, for some $a \in R$. Now, we have $\frac{r}{s} = \frac{t}{t}\frac{p}{p}\frac{r}{s} = \frac{tpr}{tps} = \frac{t}{t}\frac{ax}{ps} = \frac{a}{ps}\frac{tx}{t} \in < \frac{tx}{t} >$, so that $\bar{A} \subseteq < \frac{tx}{t} >$, thus $\bar{A} = < \frac{tx}{t} >$. Hence, R_S is a principal ideal ring.

Corollary 2.1.20. Let R be a commutative ring with identity and P be a prime ideal of R. If R is a principal ideal ring, then R_P is a principal ideal ring.

Proof. Take $S = R \setminus P$ in Proposition 2.1.19, the result follows directly.

Proposition 2.1.21. Let R be a commutative ring with identity and S be a multiplicative system in R such that $S_R(< x >)\cap S = \emptyset$, for every $x \in R$. If R_S is a principal ideal ring, then R is a principal ideal ring.

Proof. Let A be any ideal of R, then A_S is an ideal of R_S, so that $A_S = < \frac{x}{s} >$, for some $\frac{x}{s} \in A_S$, where $x \in R, s \in S$, then $px \in A$, for some $p \in S$, so that $< px > \subseteq A$. Next, let $a \in A$, then $\frac{a}{s} \in A_S$, so that $\frac{a}{s} = \frac{r}{t}\frac{x}{s} = \frac{rx}{ts} = \frac{p}{p}\frac{rx}{ts} = \frac{rpx}{pts}$, for some $r \in R, t \in S$, so $uptsa = usrpx \in < px >$, for some $u \in S$, then, $upts \in S$. If $a \notin < px >$, then $upts \in S_R(< px >)$, which is a contradiction, so that $a \in < px >$, then $A \subseteq < px >$ and thus $A = < px >$. Hence, R is a principal ideal ring.

Corollary 2.1.22. Let R be a commutative ring with identity and P be a prime ideal of R such that $S_R(< x >) \subseteq P$, for every $x \in R$. If R_P is a principal ideal ring, then R is a principal ideal ring.

Proof. By taking $S = R \setminus P$ in Proposition 2.1.21, the result follows from the fact that, for every $x \in R$, we have $S_R(< x >)\cap S = \emptyset$ if and only if $S_R(< x >) \subseteq P$.

Lemma 2.1.23. Let R be a commutative ring with identity and S be a multiplicative system in R with $a \in R$ and $s \in S$, then

(1) $< a >_S = < \frac{a}{s} >$.

(2) If A is an ideal of R such that $S_R(A)\cap S = \emptyset$ and $\frac{a}{s} \in A_S$, then $a \in A$.

Proof. (1) Let $\frac{x}{t} \in < a >_S$, where $x \in R, t \in S$, then $px \in < a >$, for some $p \in S$. Thus, $px = ra$, for some $r \in R$. Now, $\frac{x}{t} = \frac{s}{s}\frac{p}{p}\frac{x}{t} = \frac{spx}{spt} = \frac{sra}{spt} = \frac{sr}{pt}\frac{a}{s} \in < \frac{a}{s} >$. Hence, $< a >_S \subseteq < \frac{a}{s} >$. Let, $\frac{x}{t} \in < \frac{a}{s} >$, where $x \in R, t \in S$, then $\frac{x}{t} = \frac{r}{p}\frac{a}{s} = \frac{ra}{ps} \in < a >_S$ (since, $ra \in < a >$), so that $< \frac{a}{s} > \subseteq < a >_S$. Hence, we get $< a >_S = < \frac{a}{s} >$.

(2) $\frac{a}{s} \in A_S$ implies that $pa \in A$, for some $p \in S$. If $a \notin A$, then $p \in S_R(A)$, which contradicts the fact that $S_R(A)\cap S = \emptyset$, so that we must have $a \in A$.

Corollary 2.1.24. Let R be a commutative ring with identity and P be a prime ideal of R with $a \in R$ and $p \notin P$, then

(1) $< a >_P = < \frac{a}{p} >$.

(2) If A is an ideal of R such that $S_R(A) \subseteq P$ and $\frac{a}{p} \in A_P$, then $a \in A$.

Proof. By taking $S = R \setminus P$ in Lemma 2.1.23 and using that fact that, $S_R(A) \cap (R \setminus P) = \emptyset$ if and only if $S_R(A) \subseteq P$, the result follows directly.

Proposition 2.1.25. Let R be a commutative ring with identity and S be a multiplicatively closed set in R with $a \in R$ and $s \in S$ such that $S_R(< a >) \cap S = \emptyset$. Then, $< a >$ is a prime ideal of R if and only if $< a >_S$ $(= < \frac{a}{s} >)$ is a prime ideal of R_S.

Proof. Let $< a >$ be a prime ideal of R. As $< a > \subseteq S_R(< a >)$, we have $< a > \cap S \subseteq S_R(< a >) \cap S = \emptyset$, so by [4], we have $< a >_S$ is a prime ideal of R_S. Now, let $< a >_S$ be a prime ideal of R_S. If $< a > = R$, then $1 \in R = < a >$, so that $ra = 1$, for some $r \in R$, then $s = sra \in < a >$, so we have $\frac{s}{s} \in < a >_S$, so that $< a >_S = R_S$, which is a contradiction, so that $< a >$ is a proper ideal of R. Let, for $x, y \in R$, we have $xy \in < a >$, then $\frac{x}{s}\frac{y}{s} = \frac{xy}{ss} \in < a >_S$, and as $< a >_S$ is prime, we get $\frac{x}{s} \in < a >_S$ or $\frac{y}{s} \in < a >_S$. Then, by Lemma 2.1.23, the former case gives $x \in < a >$ and the latter case gives $y \in < a >$. Hence, $< a >$ is a prime ideal of R.

Corollary 2.1.26. Let R be a commutative ring with identity and P be a prime ideal of R with $a \in R$ and $p \notin P$ such that $S_R(< a >) \subseteq P$. Then, $< a >$ is a prime ideal of R if and only if $< a >_P$ $(= < \frac{a}{p} >)$ is a prime ideal of R_P.

Proof. By taking $S = R \setminus P$ in Proposition 2.1.25, the proof follows directly from the fact that $S_R(< a >) \cap S = \emptyset$ if and only if $S_R(< a >) \subseteq P$.

2.2 S −radical ideals and S −minimal prime ideals

In this section, we introduce two concepts namely, S −radical ideals and S −minimal prime ideals in commutative rings and we study the relations that combining these concepts with prime radicals, Jacobson radicals and minimal prime ideals, but first, we introduce the following definition.

Definition 2.2.1. Let R be a commutative ring with identity and S be a multiplicative system in R. We define, $SSpec(R) = \{P : P$ is a prime ideal of R such that $P \cap S = \emptyset\}$ and $SRad(R) = \bigcap_{P \in SSpec(R)} P$. We say that R is without S −prime radical if $SRad(R) = 0$.

The first relation that we prove is that, the localization of the S −Radical of a ring is the same as the prime radical of the localization of the ring.

Proposition 2.2.2. Let R be a commutative ring with identity and S be a multiplicative system in R, then $Rad(R_S) = (SRad(R))_S$.

Proof. Let $\frac{r}{p} \in Rad(R_S)$, where $r \in R, P \in S$. Let $Q \in SSpec(R)$, so that Q is a prime ideal of R such that $Q \cap S = \emptyset$, then by [4], we get that Q_S is a prime ideal of R_S, that means $Q_S \in Spec(R_S)$. Hence, $\frac{r}{p} \in Q_S$, then $qr \in Q$, for some $q \in S$. Since, Q is prime and $Q \cap S = \emptyset$, we get that $r \in Q$, so that $r \in SRad(R)$ and that $\frac{r}{p} \in (SRad(R))_S$. Hence, we get $Rad(R_S) \subseteq (SRad(R))_S$. Next, let $\frac{r}{p} \in (SRad(R))_S$, so that $sr \in SRad(R)$, for some $s \in S$. Let $\bar{Q} \in Spec(R_S)$, so that $\bar{Q}$ is a prime ideal of R_S. Then, by [4], there is a prime ideal Q of R such that $Q \cap S = \emptyset$ and $\bar{Q} = Q_S$, so that $Q \in SSpec(R)$. Hence, $sr \in Q$ and then, $\frac{r}{p} = \frac{s}{s}\frac{r}{p} = \frac{sr}{sp} \in Q_S = \bar{Q}$, so that $\frac{r}{p} \in Rad(R_S)$, this gives $(SRad(R))_S \subseteq Rad(R_S)$. Hence, we get $Rad(R_S) = (SRad(R))_S$.

Now, we give the following corollary.

Corollary 2.2.3. Let R be a commutative ring with identity and P be a prime ideal of R, then $Rad(R_P) = (SRad(R))_P$.

Proof. By taking $S = R \setminus P$ in Proposition 2.2.2, the proof will follows directly.

In the following result, we prove that if R is without S −prime radical, then R_S is without prime radical, but the converse is true under the disjointness of S from $S_R(0)$.

Proposition 2.2.4. Let R be a commutative ring with identity and S be a multiplicative system in R. If R is without S −prime radical, then R_S is without prime radical. If, in addition to the above, we have $S_R(0) \cap S = \emptyset$ and R_S is without prime radical, then R is without S −prime radical as well as it is without prime radical.

Proof. We have, $SRad(R) = 0$, so by Proposition 2.2.2, we get $Rad(R_S) = 0$, so that R_S is without prime radical. To prove the second part, let $Rad(R_S) = 0$, then by Proposition 2.2.2, we have $(SRad(R))_S = 0$ and since, $S_R(0) \cap S = \emptyset$, so by Proposition 2.1.3, we get $SRad(R) = 0$. Hence, R is without S −prime radical. Now, as $SSpec(R) \subseteq Spec(R)$, so that we have $Rad(R) \subseteq SRad(R)$ and this implies that $Rad(R) = 0$, so that R is without prime radical.

Next, we give the following corollary.

Corollary 2.2.5. Let R be a commutative ring with identity and P be a prime ideal of R. If R is without S −prime radical, then R_P is without prime radical. If, in addition to the above, we have $S_R(0) \subseteq P$ and R_P is without prime radical, then R is without S −prime radical as well as it is without prime radical.

Proof. As, $R \setminus P$ is a multiplicative system in R and $S_R(0) \cap (R \setminus P) = \emptyset$ if and only if $S_R(0) \subseteq P$, so by taking $S = R \setminus P$ in Proposition 2.2.4, the proof will follows directly.

Now, we introduce the following definition.

Definition 2.2.6. Let R be a commutative ring with identity and S be a multiplicative system in R. The maximal spectrum of R, is denoted by $mSpec(R)$, and defined as $mSpec(R) = \{Q : Q$ is a maximal ideal of $R\}$. Also, we define $SmSpec(R) = \{Q : Q$ is a maximal ideal of R such that $Q \cap S = \emptyset\}$ and $Srad(R) = \bigcap_{Q \in SmSpec(R)} Q$. We say that R is S −semisimple (or R is without S −Jacobson radical), if $Srad(R) = 0$.

It is obvious that, an S −semisimple ring R is always a semisimple ring, since if $Srad(R) = 0$, then as $SmSpec(R) \subseteq mSpec(R)$, we get $rad(R) \subseteq Srad(R)$ and this gives $rad(R) = 0$.

Next, we prove that, for a multiplicative system S in R, the localization of the S −Jacobson radical of a ring is the same as the Jacobson radical of the localization of the ring.

Proposition 2.2.7. If R is a commutative ring with identity and S is a multiplicative system in R, then $rad(R_S) = (Srad(R))_S$.

Proof. Let $\frac{r}{s} \in rad(R_S)$, where $r \in R, s \in S$. Let $Q \in SmSpec(R)$, so that Q is a maximal ideal of R with $Q \cap S = \emptyset$. By Proposition 2.1.10, Q_S is a maximal ideal of R_S, that is, $Q_S \in mSpec(R_S)$, thus $\frac{r}{s} \in Q_S$. As Q is prime and $Q \cap S = \emptyset$, one can easily get that $r \in Q$, so that $r \in Srad(R)$, then we get $\frac{r}{s} \in (Srad(R))_S$. Hence, $rad(R_S) \subseteq (Srad(R))_S$. Next, let $\frac{r}{s} \in (Srad(R))_S$, for $r \in R, s \in S$. Then, $tr \in Srad(R)$, for some $t \in S$. Let $\bar{Q} \in mSpec(R_S)$, so that $\bar{Q}$ is a maximal ideal of R_S, then by Proposition 2.1.11, we have $\bar{Q} = Q_S$, for some maximal ideal Q of R with respect to the property $Q \cap S = \emptyset$, so that $Q \in SmSpec(R)$. Hence, $tr \in Q$. Then, $\frac{r}{s} = \frac{t}{t}\frac{r}{s} = \frac{tr}{ts} \in Q_S = \bar{Q}$, so that we get $\frac{r}{s} \in rad(R_S)$ and thus$(Srad(R))_S \subseteq rad(R_S)$. Hence, $rad(R_S) = (Srad(R))_S$.

As a corollary to this result, we give the following.

Corollary 2.2.8. If R is a commutative ring with identity and P is a prime ideal of R, then $rad(R_P) = (Srad(R))_P$.

Proof. By taking $S = R \setminus P$ in Proposition 2.2.7, the result follows directly.

Now, we introduce the following definitions.

Definition 2.2.9. Let R be a commutative ring with identity and S be a multiplicative system in R. We say that a prime ideal Q of R is an $S-$minimal prime ideal of an ideal A of R, if Q is minimal in the set of all prime ideals which contain A and do not meet S.

To make the above definition more clear, we say that Q is an $S-$minimal prime ideal of A, if Q is a prime ideal of R, $A \subseteq Q$, $Q \cap S = \emptyset$ and if B is any prime ideal of R such that $A \subseteq B$ and $B \cap S = \emptyset$, then $Q \subseteq B$.

The following result shows that, the localization an $S-$minimal prime ideal of an ideal is a minimal prime ideal of the given ideal, but the converse is true for minimal prime ideals which themselves are prime as we prove in Proposition 2.2.14.

Proposition 2.2.10. Let R be a commutative ring with identity and S be a multiplicative system in R. If A is an ideal of R and Q is an $S-$minimal prime ideal of A, then Q_S is a minimal prime ideal of A_S.

Proof. We have Q is a prime ideal of R with $A \subseteq Q$ and Q is minimal in the set of all prime ideals of R that contain A and $Q \cap S = \emptyset$. Then, $A_S \subseteq Q_S$ and Q_S is a prime ideal of R_S. Let $\bar{B}$ be any prime ideal of R_S such that $A_S \subseteq \bar{B}$. To show that $Q_S \subseteq \bar{B}$. By **[4]**, we have $\bar{B} = B_S$, for the prime ideal $B = \{r \in R : \frac{sr}{s} \in \bar{B}\}$, of R with $B \cap S = \emptyset$, where $s \in S$. Then, $A_S \subseteq B_S$. Clearly, we have $A \subseteq Q$ and we will show that $A \subseteq B$. As $S \neq \emptyset$, let $s \in S$. If $x \in A$ is any element, then $sx \in A$, thus $\frac{sx}{s} \in B_S = \bar{B}$, so that $x \in B$. Hence, $A \subseteq B$ and as Q is a minimal prime ideal of A, we get $Q \subseteq B$, which gives $Q_S \subseteq B_S = \bar{B}$. Hence, Q_S is a minimal prime ideal of A_S.

Now, we give the following corollary.

Corollary 2.2.11. Let R be a commutative ring with identity and P be a prime ideal of R. If A is an ideal of R and Q is a minimal prime ideal of A with $Q \subseteq P$, then Q_P is a minimal prime ideal of A_P.

Proof. By taking $S = R \setminus P$ and since $Q \cap (R \setminus P) = \emptyset$ if and only if $Q \subseteq P$, then the proof will follows directly.

Proposition 2.2.12. Let R be a commutative ring with identity and S be a multiplicatively

closed set in R. If $\bar{I}$ is an ideal of R_S, then for any $t \in S$, we have $I = \{x \in R : \frac{tx}{t} \in \bar{I}\}$ is an ideal of R with $\bar{I} = I_S$. Furthermore, if $\bar{I}$ is a prime ideal of R_S, then I is a prime ideal of R with $I \cap S = \emptyset$.

Proof. By Proposition 2.1.5, we have $\bar{I} = I_S$, for the proper ideal $I = \{a \in R : \frac{ta}{t} \in \bar{I}\}$ of R. Next, to prove I is a prime ideal of R. As $I_S = \bar{I} \neq R_S$, by [4], we get $I \cap S = \emptyset$, now, if possible suppose that $I = R$, then $1 \in I$, so that $\frac{t1}{t} \in \bar{I}$, that is $\frac{t}{t} \in \bar{I}$, so that $\bar{I} = R_S$, which is a contradiction. Hence, $I \neq R$. Let $ab \in I$, where $a, b \in R$, then $\frac{ta}{t}\frac{tb}{t} = \frac{tab}{t} \in \bar{I}$ and as $\bar{I}$ is prime, we get $\frac{ta}{t} \in \bar{I}$ or $\frac{tb}{t} \in \bar{I}$. The former case gives $a \in I$ and the latter case gives $b \in I$. Hence, I is a prime ideal of R. If $I \cap S \neq \emptyset$, then there exists $s \in S$ and $s \in I$, so $\frac{s}{s} \in I_S = \bar{I}$ and thus we get $\bar{I} = R_S$, which is a contradiction. Hence, $I \cap S = \emptyset$.

We give the following corollary.

Corollary 2.2.13. Let R be a commutative ring with identity and P be a prime ideal of R. If $\bar{I}$ is an ideal of R_P, then we have $I = \{x \in R : \frac{x}{1} \in \bar{I}\}$ is an ideal of R with $\bar{I} = I_P$. Furthermore, if $\bar{I}$ is a prime ideal of R_P, then I is a prime ideal of R with $I \subseteq P$.

Proof. As $S = R \setminus P$ is a multiplicative system in R, so by taking $t = 1 \in S = R \setminus P$ in Proposition 2.2.12, the proof follows directly.

Proposition 2.2.14. Let R be a commutative ring with identity and S be a multiplicative system in R. If A, Q are ideals of R with Q a prime ideal and Q_S a minimal prime ideal of A_S, then Q is an $S-$minimal prime ideal of A.

Proof. As Q_S is a minimal prime ideal of A_S, so that Q_S is prime and $A_S \subseteq Q_S$ and Q_S is minimal in the set of all prime ideals of R_S which contain A_S. Since, $Q_S \neq R_S$, so by [4, Proposition 3.5], we have $Q \cap S = \emptyset$ and as Q is prime, we get $A \subseteq Q$. Let B be any prime ideal of R such that $A \subseteq B$ and $B \cap S = \emptyset$, then we get B_S is a prime ideal of R_S and $A_S \subseteq B_S$ and as Q_S is a minimal prime ideal of A_S, we get $Q_S \subseteq B_S$. Let $x \in Q$, then for an $s \in S$, we have $\frac{x}{s} \in B_S$, then $tx \in B$, for some $t \in S$ and as $B \cap S = \emptyset$, we get $t \notin B$ and as B is p rime, we get $x \in B$, so that $Q \subseteq B$. Hence, Q is an $S-$minimal prime ideal of A.

Corollary 2.2.15. Let R be a commutative ring with identity and P be a prime ideal of R. If A, Q are ideals of R with Q a prime ideal and Q_P a minimal prime ideal of A_P, then Q is an $S-$minimal prime ideal of A.

Proof. By taking $S = R \setminus P$ in Proposition 2.2.14, the proof follows directly.

We give the following corollary, which shows that each minimal prime ideal of an ideal in R_S is a localization of some $S-$minimal prime ideal of the contraction of the given ideal in R.

Corollary 2.2.16. Let R be a commutative ring with identity and S be a multiplicative system in R. If $\bar{A}$ is an ideal of R_S and $\bar{Q}$ is a minimal prime ideal of $\bar{A}$, then there exist ideals

A, Q of R with Q an S −minimal prime ideal of A, for which $\bar{Q} = Q_S$.

Proof. By [4], there exists a prime ideal Q of R with $Q \cap S = \emptyset$ and such that $\bar{Q} = Q_S$ and by Proposition 2.2.12, we have $\bar{A} = A_S$, where $A = \{x \in R : \frac{tx}{t} \in \bar{A}\}$, for $t \in S$. That means, Q is prime and Q_S is a minimal prime ideal of A_S, so by Proposition 2.2.14, we get Q is an S −minimal prime ideal of A.

Corollary 2.2.17. Let R be a commutative ring with identity and P be a prime ideal of R. If $\bar{A}$ is an ideal of R_P and $\bar{Q}$ is a minimal prime ideal of $\bar{A}$, then there exist ideals A, Q of R with Q an S −minimal prime ideal of A.

Proof. By taking $S = R \setminus P$ in Corollary 2.2.16, the proof follows directly.

We mention that, Proposition 2.2.11 and Corollary 2.2.16 lead to the following theorem.

Theorem 2.2.18. Let R be a commutative ring with identity and S be a multiplicative system in R and A be an ideal of R. Then, there is a one to one correspondence between the S −minimal prime ideals of A and minimal prime ideals of A_S.

Proof. Let $K = \{Q : Q$ is an S −minimal prime ideal of $A\}$ and $L = \{\bar{Q} : \bar{Q}$ is a minimal prime ideal of $A_S\}$. Define $\vartheta : K \to L$ as follows: Let $Q \in K$, so Q is an S −minimal prime ideal of A, then by Proposition 2.2.10, Q_S is a minimal prime ideal of A_S and thus, $Q_S \in L$, so we define $\vartheta(Q) = Q_S$. Clearly, ϑ is a mapping. Now, let $Q, Q' \in K$ be such that $\vartheta(Q) = \vartheta(Q')$, then we get $Q_S = Q'_S$ and as Q, Q' are prime ideals and $Q \cap S = \emptyset = Q' \cap S$, one can easily get that $Q = Q'$. Hence, ϑ is one to one. Next, let $\bar{Q} \in L$, then $\bar{Q}$ is a minimal prime ideal of A_S. Then, by Proposition 2.2.12, we get $\bar{Q} = Q_S$, for the prime ideal $Q = \{x \in R : \frac{tx}{t} \in \bar{Q}\}$, for some fixed $t \in S$ and $Q \cap S = \emptyset$, so by Corollary 2.2.14, we get that Q is an S −minimal prime ideal of A, so that $Q \in K$ and that $\vartheta(Q) = Q_S = \bar{Q}$, so that ϑ is onto. Hence, ϑ defines a one to one correspondence.

Now, we give the following corollary.

Corollary 2.2.19. Let R be a commutative ring with identity and P be a prime ideal R and A is an ideal of R. Then, there is a one to one correspondence between the S −minimal prime ideals of A and minimal prime ideals of A_P.

Proof. By taking $S = R \setminus P$ in Theorem 2.2.18, the proof will follow at once.

3. CONCLUSION

1. There is a one to one correspondence between the primary ideals of R_P and the primary ideals of R which contained in P. Where P be a prime ideal of a ring R.
2. Under certain condition, there is a one-one correspondence between the nilpotent (resp. nil) ideals of R and R_S.
3. Under certain condition, the principality of Rings is a localization property.
4. The Radical of the localization of a ring is the same as the localization of the S −radical of the ring.
5. There is a one-one correspondence between the S −minimal primes of A_P. Where A is an ideal of R and P is a prime ideal of R.

4. REFERENCES

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