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Statistical Inference for Pareto Distribution based on Progressive Type-I Hybrid Censoring Scheme

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ABSTRACT

In this paper, the maximum likelihood and Bayesian estimations are developed based on progressive Type-I hybrid censored sample from the Pareto distribution. The Bayesian estimators for the unknown parameters are computed using the squared error loss function. Also, the point and interval Bayesian predictions for the unobserved failures from the same sample and that from the future sample are derived. Moreover, a Monte Carlo simulation study is carried out to compare the performance of the maximum likelihood and the Bayesian estimators. Finally, numerical example is presented for illustrating all the inferential procedures developed here.

Keywords

Bayesian estimation, Bayesian prediction, Pareto distribution, Maximum likelihood estimation, progressive hybrid censoring sample

1. INTRODUCTION

In reliability analysis, experiments are often terminated before all units on test fail based on cost and time considerations. In such cases, failure information is available and only partial information is available on all units that have not failed. Such data are called censored data. There are several forms of censored data. Two commonly used right censoring schemes are the Type-I and Type-II censoring, wherein the test terminates, respectively, at a pre-determined time T and upon observing certain number of failures. If n is the number identical units on a life-testing experiment, in the Type-I censoring scheme the experiment is terminated when a pre-fixed censoring time T is reached. In the Type-II censoring scheme, the experiment gets terminated when a pre-specified number $m \leq n$ of failures is observed. The mixture of Type-I and Type-II censoring schemes is known as a hybrid censoring scheme.
Progressive Type-II censoring scheme is a generalization of Type-II censoring scheme, where $n - m$ units are withdrawn from the life-test at different time points (rather than all at the time $X_{m:n}$). When the first failure is observed, R_1 of the $n - 1$ surviving units are randomly selected and removed. At the second observed failure, R_2 of the $n - R_1 - 2$ surviving units are randomly selected and removed. The experiment finally terminates at the time of the m^{th} failure when all remaining $R_m = n - R_1 - \ldots - R_{m-1} - m$ surviving units are removed. The censoring numbers $\{R_i, i = 1, .., m\}$ are prefixed. The resulting m ordered values which are obtained from this type of censoring are referred to as progressively Type-II right censored order statistics. Several authors have studied progressive Type-II censoring and properties of order statistics arising from such a progressively censored life-test. Some key references are Aggarwala and Balakrishnan [1], Cramer and Iliopoulos [2], Raqab et al. [3], Mohie El-Din and Shafay [4], and Balakrishnan and Cohen [5].
The disadvantages of the progressive Type-II censoring scheme are that the time of the experiment can be very long if the units are highly reliable. Therefore, Kundu and Joarder in [6] and Childs et al. in [7] proposed a progressive Type-I hybrid censoring scheme (HCS), in this life-testing the experiment is terminated at time $\min\{X_{m:m:n}, T\}$, where $T \in (0, \infty)$ pre-fixed in advance. Under progressive Type-I HCS, the time on experiment will be no more than T. Some recent studies on progressive hybrid censored sample have been carried out by many authors including Lin et al. [8], Lin and Huang. [9], and Hemmati and Khorram [10].
In this paper, the underlying distribution is assumed to be the Pareto distribution which introduced by Pareto in [11] as a model for the distribution of income, with the probability density function (*PDF*) and cumulative distribution function (*CDF*) given by

$$f(x|\alpha, \sigma) = \frac{\alpha}{\sigma}\left(\frac{\sigma}{x}\right)^{(\alpha+1)}, x \geq \sigma, \tag{1}$$

and

$$F(x|\alpha, \sigma) = 1 - \left(\frac{\sigma}{x}\right)^{\alpha}. \tag{2}$$

where $\alpha > 0$ and $\sigma > 0$.
In recent years, its models in several different forms have been studied by many authors including Davis and Feldstein[12], Cohen and Whitten[13], and Grimshaw [14].
The rest of this paper is organized as follows. In Section 2, the description of the model of the Type-I PHCS is presented. The maximum likelihood (ML) estimator and the Bayesian estimator under the squared error loss function for the unknown parameters are derived in Section 3. In Section 4, the Bayesian prediction is derived for the failure times of all units that are removed in all stages of censoring. Bayesian prediction for progressive order statistics from an unobserved future sample from the same

distribution is derived in Section 5. Finally, in Section 6, a Monte Carlo simulation study is carried out to compare the performance of the ML and the Bayesian estimates, for illustrating all the inferential methods developed here, numerical example is then presented.

2. THE MODEL DESCRIPTION

The progressive Type-I HCS can be described as follow: Suppose n identical items are put to test and the lifetime distributions of the n items are denoted by $X_1, X_2, ..., X_n$. The integer $m < n$ is fixed at the beginnig of the experiment, and $R = (R_1, R_2, ..., R_m)$ are m pre-fixed integers satisfying $n = m + R_1 + \ldots + R_m$. The time point T is also fixed beforehand. At the time of first failure $X_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure $X_{2:m:n}$, R_2 of the remaining units are removed and so on. If the m^{th} failure $X_{m:m:n}$ occurs before the time point T, the experiment stops at the time point $X_{m:m:n}$. On the other hand, suppose m^{th} failure does not occur before time point T and only k failures occur before the time point T , where $0 \leq k < m$; then, at the time point T all the remaining R^*_τ units are removed and the experiment terminates at the time point T . Note that $R^*_\tau = \sum\limits_{i=k+1}^{m} (R_i + 1)$. (see Fig. 1).

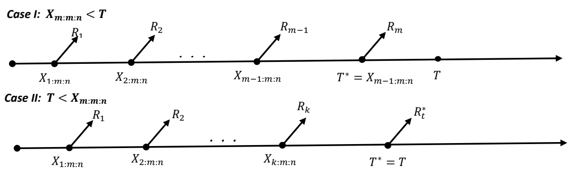

Fig. 1. Schematic representation of progressive Type-I hybrid censoring scheme

Therefore, in the presence of progressively Type-I HCS, one of the following types of observations is found:

(1) Suppose that the m^{th} failure occurs before T, then the experiment terminates at T and $\{X_{1:m:n} < \ldots\ldots < X_{k:m:n}\}$ are observed.

(2) Suppose that the m^{th} failure occurs before T, then the experiment terminates at $X_{m:m:n}$ and $\{X_{1:m:n} < \ldots\ldots < X_{m:m:n}\}$ are observed.

Given a progressively Type-I HCS, the joint density function for the different cases are as follows:

$$f_{\underline{\mathbf{x}}}(\underline{\mathbf{x}}) = \left[\prod_{i=1}^{D}\sum_{j=i}^{m}\left(R^*_j + 1\right)\right] \prod_{i=1}^{D} f\left(x_{i:D:n}\right)\left[\bar{F}(x_{i:D:n})\right]^{R_i}\left[\bar{F}(T)\right]^{R^*_\tau}, \quad (3)$$

where

$$D = \begin{cases} k & \text{if } X_{k:m:n} \leq T < X_{m:m:n}, \\ m & \text{if } X_{m:m:n} < T, \end{cases} \quad (4)$$

with R^*_τ is the number of surviving units that are removed at T, given by

$$R^*_\tau = \begin{cases} n - k - \sum\limits_{j=1}^{k} R_j & \text{if } X_{k:m:n} \leq T < X_{m:m:n}, \\ 0 & \text{if } X_{m:m:n} < T, \end{cases} \quad (5)$$

and

$$\underline{\mathbf{x}} = \begin{cases} (x_{1:m:n}, ..., x_{k:m:n}) & \text{if }, X_{k:m:n} \leq T < X_{m:m:n} \\ (x_{1:m:n}, ..., x_{m:m:n}) & \text{if } X_{m:m:n} < T, \end{cases} \quad (6)$$

Upon using (2) and (1) in (3), the likelihood function of α, σ based on progressively Type-I HCS can be obtained as

$$L\left(\alpha, \sigma|\underline{\mathbf{x}}\right) = \left[\prod_{i=1}^{D}\sum_{j=i}^{m}\left(R^*_j + 1\right)\right] \alpha^D\left(\prod_{i=1}^{D}\frac{1}{x_i}\right)\exp\left\{-\alpha\left[W\left(\underline{\mathbf{x}}\right) - n\ln\sigma\right]\right\}, \quad (7)$$

where $W\left(\underline{\mathbf{x}}\right) = \sum\limits_{i=1}^{D}\left(R^*_i + 1\right)\ln x_i + R^*_\tau \ln T$ and $x_i = x_{i:D:n}$ for simplicity of notation.

3. THE ML AND BAYESIAN ESTIMATIONS

It is clear that the likelihood function is monotone increasing function in σ, so its maximum value $\widehat{\sigma}_{ML}$ will be attained at the minimum value $x_{1:m:m}$ of σ. From (7), The log-likelihood function of (α, σ) is given by

$$\ln\left[L\left(\alpha, \sigma|\underline{\mathbf{x}}\right)\right] = const. + D\ln\alpha - \alpha\left[W\left(\underline{\mathbf{x}}\right) - n\ln\sigma\right], \quad (8)$$

to maximize relative to α, differentiate (8) with respect to α and solve the equation

$$\frac{\partial \ln\left[L\left(\alpha, \sigma|\underline{\mathbf{x}}\right)\right]}{\partial\alpha} = 0,$$

so the ML estimator $\widehat{\alpha}_{ML}$ of α is obtained as

$$\widehat{\alpha}_{ML} = \frac{D}{W\left(\underline{\mathbf{x}}\right) - n\ln x_{1:m:m}}. \quad (9)$$

For the Bayesian estimations, under the assumption that both parameters α and σ are unknown, the joint prior density function of α and σ, which was suggested by Lwin in [15] and generalized by Arnold and Press in [16], is considered. The generalized Lwin prior or the power-gamma prior is given by

$$\pi(\alpha, \sigma) \;\propto\; \alpha^a\sigma^{-1}\exp[-\alpha(\ln g - b\ln\sigma)], \;\; \alpha > 0,\; 0 < \sigma < h, \quad (10)$$

where $a,\ b,\ g,\ h$ are positive constants and $h^b < g$.
Upon combining (7) and (10), given progressively Type-I HCS, the posterior density function of α, σ is obtained as

$$\begin{aligned}\pi^*(\alpha, \sigma|\underline{\mathbf{x}}) &= L\left(\alpha, \sigma|\underline{\mathbf{x}}\right)\pi(\alpha, \sigma) / \int L\left(\alpha, \sigma|\underline{\mathbf{x}}\right)\pi(\alpha, \sigma)d\alpha d\sigma \\ &= I^{-1}\alpha^{D+a}\sigma^{-1}\exp\left\{-\alpha\left[W\left(\underline{\mathbf{x}}\right) - (n+b)\ln\sigma + \ln g\right]\right\},\end{aligned} \quad (11)$$

where

$$\begin{aligned} I &= \int_0^{x_0}\int_0^{\infty}\alpha^{D+a}\sigma^{-1}\exp\left\{-\alpha\left[W\left(\underline{\mathbf{x}}\right) - (n+b)\ln\sigma + \ln g\right]\right\}d\alpha d\sigma \\ &= \frac{\Gamma(D+a)}{n+b}\left[W\left(\underline{\mathbf{x}}\right) - (n+b)\ln x_0 + \ln g\right]^{-(D+a)}, \end{aligned} \quad (12)$$

with $x_0 = \min(x_1, g)$. By using (11), the Bayesian estimator of α under the squared error loss function is the mean of the posterior density function, given by

$$\begin{aligned}\widehat{\alpha}_B &= \int_0^{x_0}\int_0^{\infty} \alpha\pi^*(\alpha,\sigma|\underline{\mathbf{x}})d\alpha d\sigma \\ &= \frac{D+a}{W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g},\end{aligned} \quad (13)$$

and the Bayesian estimator of σ under the squared error loss function is obtained as

$$\begin{aligned}\widehat{\sigma}_B &= \int_0^{x_0}\int_0^{\infty} \sigma\pi^*(\alpha,\sigma|\underline{\mathbf{x}})d\alpha d\sigma \\ &= I^{-1}x_0\int_0^{\infty}\frac{\alpha^{D+a}}{\alpha(n+b)+1}\exp\{-\alpha[W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g]\}d\alpha \\ &= \frac{I^{-1}x_0}{(n+b)}[W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g]^{-(D+a)} \\ &\quad\times\int_0^{\infty}\frac{t^{D+a}e^{-t}}{t+[W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g]/(n+b)}dt \\ &= \frac{x_0}{\Gamma(D+a)}\Phi\left(D+a,\frac{[W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g]}{(n+b)}\right),\end{aligned} \quad (14)$$

where

$$\Phi(x,y)=\int_0^{\infty}\frac{t^x e^{-t}}{t+y}dt.$$

A partial tabulation of $\psi(x,y)=(y/\Gamma(x))\Phi(x-1,y)$ has been provided by Arnold and Press in [16].

4. ONE-SAMPLE BAYESIAN PREDICTION

For $\rho=1,2,...,R_j^*$, let $X_{\rho:R_j^*}$ denote the ρ^{th} order statistic out of R_j^* removed units at stage j. Then, the conditional density function of $X_{\rho:R_j^*}$, given the observed progressively Type-I HCS, is given, see Basak et al. [17], by

$$f_{X_{\rho:R_j^*}}(x|\underline{\mathbf{x}})=\frac{R_j^*!}{(\rho-1)!(R_j^*-\rho)!}\frac{[F(x)-F(x_j)]^{\rho-1}[1-F(x)]^{R_j^*-\rho}f(x)}{[1-F(x_j)]^{R_j^*}},\quad x>x_j, \quad (15)$$

where

$$j=\begin{cases}1,...,m & \text{if } X_{m:m:n}<T,\\ 1,...,k,\tau & \text{if } X_{k:m:n}\le T<X_{m:m:n},\end{cases}$$

with $x_\tau=T$.
By using (2) and (1) in (15), given Type-I PHCS, the conditional density function of $X_{\rho:R_j^*}$ is then given as follows:

$$f_{X_{\rho:R_j^*}}(x|\underline{\mathbf{x}})=\sum_{q=0}^{\rho-1}C_{1q}\frac{\alpha}{x}\exp\{-\alpha[\eta_q(\ln x-\ln x_j)]\},\ x>x_j, \quad (16)$$

where $C_{1q}=\frac{(-1)^q\binom{\rho-1}{q}R_j^*!}{(\rho-1)!(R_j^*-\rho)!}$ and $\eta_q=q+R_j^*-\rho+1$ for $q=0,...,\rho-1$. Upon combining (11) and (16), the Bayesian predictive density function of $X_{\rho:R_j^*}$, given progressively Type-I HCS, is obtained as

$$\begin{aligned}&f^*_{X_{\rho:R_j^*}}(x|\underline{\mathbf{x}}) \\ &=\int_0^{x_0}\int_0^{\infty} f_{X_{\rho:R_j^*}}(x|\underline{\mathbf{x}})\pi^*(\alpha,\sigma|\underline{\mathbf{x}})d\alpha d\sigma \\ &=\frac{I^{-1}\Gamma(D+a+1)}{(n+b)}\sum_{q=0}^{\rho-1}\frac{C_q}{x}[W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g+\eta_q(\ln x-\ln x_j)]^{-(D+a+1)}.\end{aligned} \quad (17)$$

The Bayesian predictive survival function of $X_{\rho:R_j^*}$, given progressively Type-I HCS, is given as

$$\begin{aligned}&\bar{F}^*_{X_{\rho:R_j^*}}(t|\underline{\mathbf{x}}) \\ &=\int_t^{\infty} f^*_{X_{\rho:R_j^*}}(x|\underline{\mathbf{x}})dx \\ &=\frac{I^{-1}\Gamma(D+a)}{(n+b)}\sum_{q=0}^{\rho-1}\frac{C_q}{\eta_q}[W(\underline{\mathbf{x}})-(n+b)\ln x_0+\ln g+\eta_q(\ln t-\ln x_j)]^{-(D+a)}.\end{aligned} \quad (18)$$

The Bayesian point predictor of $X_{\rho:R_j^*}$ under the squared error loss function is the mean of the predictive density, given by

$$\widehat{X}_{\rho:R_j^*}=\int_0^{\infty}xf^*_{X_{\rho:R_j^*}}(x|\underline{\mathbf{x}})dx, \quad (19)$$

where $f^*(x|\underline{\mathbf{x}})$ is given as in (17).
The Bayesian predictive bounds of $100(1-\gamma)\%$ two-sided equi-tailed (ET) interval for $X_{\rho:R_j^*}$ can be obtained by solving the following two equations:

$$\bar{F}^*_{X_{\rho:R_j^*}}(L_{ET}|\underline{\mathbf{x}})=\frac{\gamma}{2}\qquad\text{and}\qquad\bar{F}^*_{X_{\rho:R_j^*}}(U_{ET}|\underline{\mathbf{x}})=1-\frac{\gamma}{2}, \quad (20)$$

where $\bar{F}^*(t|\underline{\mathbf{x}})$ is given as in (18), and L_{ET} and U_{ET} denote the lower and upper bounds, respectively. On the other hand, for the highest posterior density (HPD) method, the following two equations need to be solved:

$$\bar{F}^*_{X_{\rho:R_j^*}}(L_{HPD}|\underline{\mathbf{x}})-\bar{F}^*_{X_{\rho:R_j^*}}(U_{HPD}|\underline{\mathbf{x}})=1-\gamma$$

and

$$f^*_{X_{\rho:R_j^*}}(L_{HPD}|\underline{\mathbf{x}})-f^*_{X_{\rho:R_j^*}}(U_{HPD}|\underline{\mathbf{x}})=0,$$

where $f^*(x|\underline{\mathbf{x}})$ is as in (17), and L_{HPD} and U_{HPD} denote the HPD lower and upper bounds, respectively.

5. TWO-SAMPLE BAYESIAN PREDICTION

Let $Y_{1:\ell:N}\le Y_{2:\ell:N}\le\ldots\le Y_{\ell:\ell:N}$ be a future independent progressive Type-II censored sample from the same population with censoring scheme $\underline{\mathbf{S}}=(S_1,\ldots,S_\ell)$. In this section, a general procedure for deriving the point and interval predictions for $Y_{s:\ell:N}$, $1\le s\le\ell$, based on the observed progressively Type-I HCS, is developed. The marginal density function of $Y_{s:\ell:N}$ is given by Balakrishnan et al. in [18] as

$$f_{Y_{s:\ell:N}}(y_s)=c(N,s)\sum_{q=0}^{s-1}c_{q,s-1}[1-F(y_s)]^{M_{q,s-1}-1}f(y_s), \quad (21)$$

where $1 \le s \le \rho$,
$c(N,s) = N(N - S_1 - 1)\ldots(N - S_1 \ldots - S_{s-1} + 1)$,
$M_{q,s} = N - S_1 - \ldots - S_{s-q-1} - s + q + 1$,
and $c_{q,s-1} = (-1)^q$
$$\left\{\left[\prod_{u=1}^{q}\sum_{v=s-q}^{s-q+u-1}(S_v+1)\right]\left[\prod_{u=1}^{s-q-1}\sum_{v=u}^{s-q-1}(S_v+1)\right]\right\}^{-1}.$$
Upon substituting (2) and (1) in (21), the marginal density function of $Y_{s:\ell:N}$ is then obtained as

$$f_{Y_{s:\ell:N}}(y_s) = c(N,s)\sum_{q=0}^{s-1} c_{q,s-1}\frac{\alpha}{y_s}\exp\left\{-\alpha\left[M_{q,s}\ln\left(\frac{y_s}{\sigma}\right)\right]\right\},\ \ y_s > 0. \tag{22}$$

Upon combining (11) and (22), given progressively Type-I HCS, the Bayesian predictive density function of $Y_{s:\ell:N}$ is obtained as

$$f^*_{Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}}) = \begin{cases} f^*_{1Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}}), & 0 < y_s \le x_0, \\ f^*_{2Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}}), & y_s > x_0, \end{cases} \tag{23}$$

where

$$\begin{aligned} f^*_{1Y_{s:\ell:N}}&(y_s|\underline{\mathbf{x}}) \\ &= \int_0^{y_s}\int_0^{\infty} f_{Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})\pi^*(\alpha,\sigma|\underline{\mathbf{x}})d\alpha d\sigma \\ &= I^{-1}\Gamma(D+a+1)c(N,s)\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{(n+b+M_{q,s})\,y_s} \\ &\quad\times\left[W(\underline{\mathbf{x}}) - (n+b)\ln y_s + \ln g\right]^{-(D+a+1)}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} f^*_{2Y_{s:\ell:N}}&(y_s|\underline{\mathbf{x}}) \\ &= \int_0^{x_0}\int_0^{\infty} f_{Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})\pi^*(\alpha,\sigma|\underline{\mathbf{x}})d\alpha d\sigma \\ &= I^{-1}\Gamma(D+a+1)c(N,s)\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{(n+b+M_{q,s})\,y_s} \\ &\quad\times\left[W(\underline{\mathbf{x}}) - (n+b+M_{q,s})\ln x_0 + M_{q,s}\ln y_s + \ln g\right]^{-(D+a+1)}. \end{aligned} \tag{25}$$

From (23), the predictive survival function of $Y_{s:\ell:N}$, given progressively Type-I HCS, is obtained as

$$\bar{F}^*_{Y_{s:\ell:N}}(t|\underline{\mathbf{x}}) = \begin{cases} \bar{F}^*_{1Y_{s:\ell:N}}(t|\underline{\mathbf{x}}), & 0 < t \le x_0, \\ \bar{F}^*_{2Y_{s:\ell:N}}(t|\underline{\mathbf{x}}), & t > x_0, \end{cases} \tag{26}$$

where

$$\begin{aligned} \bar{F}^*_{1Y_{s:\ell:N}}&(t|\underline{\mathbf{x}}) \\ &= \int_t^{x_0} f^*_{1Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})dy_s + \int_{x_0}^{\infty} f^*_{2Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})dy_s \\ &= I^{-1}\Gamma(D+a)c(N,s)\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{(n+b)(n+b+M_{q,s})M_{q,s}} \\ &\quad\times\Big\{(n+b+M_{q,s})\left[W(\underline{\mathbf{x}}) - (n+b)\ln x_0 + \ln g\right]^{-(D+a)} \\ &\quad - M_{q,s}\left[W(\underline{\mathbf{x}}) - (n+b)\ln t + \ln g\right]^{-(D+a)}\Big\}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \bar{F}^*_{2Y_{s:\ell:N}}&(t|\underline{\mathbf{x}}) \\ &= \int_t^{\infty} f^*_{2Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})dy_s \\ &= I^{-1}\Gamma(D+a)c(N,s)\sum_{q=0}^{s-1}\frac{c_{q,s-1}}{M_{q,s}(n+b+M_{q,s})} \\ &\quad\times\left[W(\underline{\mathbf{x}}) - (n+b+M_{q,s})\ln x_0 + M_{q,s}\ln t + \ln g\right]^{-(D+a)}. \end{aligned} \tag{28}$$

The Bayesian point predictor of $Y_{s:\ell:N}$, $1 \le s \le m$, under the squared error loss function is the mean of the predictive density, given by

$$\widehat{Y}_{s:\ell:N} = \int_0^{\infty} y_s f^*_{Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})dy_s, \tag{29}$$

where $f^*_{Y_{s:\ell:N}}(y_s|\underline{\mathbf{x}})$ is given as in (23).

6. NUMERICAL RESULTS

Before progressing further, how to generate the progressively Type-I HCS, for a given set n, m, $R_1, R_2, \ldots, R_m$ and T, is described. The following transformation, suggested in Balakrishnan and Aggarwala [19], is used:
$Z_1 = nX_{1:m:n}$
Z_2=$(n - R_1 - 1)(X_{2:m:n}$-$X_{1:m:n})$
.
.
.
Z_m=$(n - R_1 - 1)(X_{m:m:n}$-$X_{m-1:m:n})$.
It is known that if X_i's are i.i.d. , then the spacings Z_i's are also i.i.d. random variables. Then it follows that
$X_{1:m:n}$=$\frac{1}{n}Z_1$
$X_{2:m:n}$=$\frac{1}{n-R_1-1}Z_2 + \frac{1}{n}Z_1$
.
.
.
$X_{m:m:n}$=$\frac{1}{n-R_1-\ldots-R_{m-1}-m-1}Z_m + \ldots + \frac{1}{n}Z_1$.
Thus, the progressively Type-I HCS can be easily generated as follows. If $X_{m:m:n} < T$, then the progressive Type-I hybrid censored sample $X_{1:m:n}$,...,$X_{m:m:n}$ is obtained. If $T < X_{m:m:n}$, then the progressive Type-I hybrid censored sample $X_{1:m:n}, \ldots, X_{k:m:n}$ is obtained where $X_{k:m:n} < T < X_{k+1:m:n}$.

6.1 Monte Carlo Simulation

In this section, a Monte Carlo simulation study is carried out to compare the performance of the ML and the Bayesian estimates under different sampling schemes. Different values for n, m and T is used to generate 1000 generalized progressive Type-I hybrid censored samples from the Pareto distribution (with $\alpha = 4$ and $\sigma = 6$). For comparison, the estimated risk (ER) for each estimate, by using the root mean square error, and the estimated bias (EB) for each estimate are computed. Tables 1 and 2 present the values of EB and ER of the ML and Bayesian estimates for α and σ, respectively. A Monte Carlo simulation study is performed using different sample sizes (n), different effective samples sizes (m) and the following two censoring schemes

(1) Scheme 1: $R_i = \frac{2(n-m)}{m}$ if i is odd and $R_i = 0$ if i is even.
(2) Scheme 2: $R_i = \frac{2(n-m)}{m}$ if i is even and $R_i = 0$ if i is odd.
(3) Scheme 3: $R_1 = R_2 = ... = R_{m-1} = 1$ and $R_m = n-2m+1$.

All Bayesian results are computed based on two different choices of the hyperparameters (a, b, g, h), namely,

(1) Informative prior (IP) : $a = 1.78$, $b = 0.33$, $g = 3.48$ and $h = 11.40$.
(2) Noninformative prior (NIP) : $a = -1$, $b = 0$, $g = 1$ and $h = \infty$.

6.2 Numerical example

In this numerical example, a progressive Type-I hybrid censored sample from a sample of size $n = 25$ is generated. Suppose $m = 15$ and $R = (1, 0, 0, 0, 0, 3, 0, 0, 0, 3, 0, 0, 0, 0, 3)$, then the following generated data: 6.0034, 6.0641, 6.2204, 6.2437, 6.2638, 6.2939, 6.3944, 6.3949, 6.5048, 6.5280, 6.7037, 6.7572, 7.9157, 8.5940, and 9.1067 are obtained. Different values for k and T are considered to obtaine the following three different progressive Type-I HCS:

(1) Scheme 1: Suppose $T = 6.500$, since$T < X_{15:15:25}$, then the experiment would have terminated at $X_{8:15:25}$, with $R^*_\tau = 13$, and the following data: 6.0034, 6.0641, 6.2204, 6.2437, 6.2638, 6.2939, 6.3944, and 6.3949 are obtained.
(2) Scheme 2: Suppose $T = 10$, since $X_{15:15:25} < T$, then the experiment would have terminated at$X_{15:15:25}$, with $R^*_\tau = 0$, and the following data: 6.0034, 6.0641, 6.2204, 6.2437, 6.2638, 6.2939, 6.3944, 6.3949, 6.5048, 6.5280, 6.7037, 6.7572, 7.9157, 8.5940, and 9.1067 are obtained.

These data are assumed to have come from the Pareto distribution with α and σ are unknown. Based on the above generated progressive Type-I hybrid censored sample, Table 3 presents the point predictor and 95% ET and HPD prediction intervals of $X_{\rho:R^*_j}$ and Table 4 presents the point predictor and 95% ET and HPD prediction intervals of $Y_{s:\ell:N}$ from the future progressive censored sample of size $\ell = 10$ from a sample of size $N = 20$ with progressive censoring scheme $S = (2, 0, 2, 0, 2, 0, 2, 0, 2, 0)$.

6.3 Conclusions and discussion

From Tables 1 and 2, it can be seen that the performance of the ML estimators is quite close to that of the Bayesian estimators based noninformative priors, as expected. Thus, if no prior information on the unknown parameters is found, then it is always better to use the ML rather than the Bayesian estimators, because the Bayesian estimators are computationally more sensitive. Also, the Bayesian method with informative priors is the best method for estimation under all different censoring schemes. Moreover, mean-squared error decreases when n and m increase.
From the results in Tables 3 and 4, it can be seen that the point predictor of mean is between the upper and lower bounds of the prediction intervals. Also, the lower bounds are relatively insensitive while the upper bounds are more sensitive. Moreover, a comparison of the results for the informative priors with the corresponding ones for non-informative priors reveals that the former produce more precise results. Finally, the HPD prediction intervals seem to be more precise than the ET prediction intervals.
In this paper, the maximum likelihood and Bayesian estimators are derived for the two unknown parameters of the Pareto distribution based on progressive Type-I hybrid censored sample. Even though many results of interest in estimation and prediction have been addressed in this paper, there are many problems in this direction are open further study. One possible problem that will be of interest is to use the progressive Type-I hybrid censoring scheme for developing the estimation and prediction problems for some other continuous distribution such as (1) Weibull distribution, (2) Burr Type-XII distribution, (3) Inverted exponential distribution, and (4) Generalized inverted exponential distribution. Another possible problem that will be of interest is to consider the estimation and prediction problems based on some other forms of censoring schemes such as (1) Progressive Type-II hybrid censoring scheme, (2) Adaptive progressively Type-I censoring scheme, and (3) Unified hybrid censoring scheme.

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Table 1. The values of EB and ER of the ML and Bayesian estimates for α.

			$\widehat{\alpha}_{ML}$		$\widehat{\alpha}_B$			
					IP		NIP	
n	m	$Scheme$	ER	EB	ER	EB	ER	EB
				$T = 7.000$				
20	10	1	2.44575	1.11372	1.23717	0.27455	1.95453	0.40553
		2	2.40738	1.09626	1.23783	0.27863	1.93682	0.40716
		3	2.51717	1.14502	1.24940	0.27455	2.00024	0.41336
25	10	1	2.33048	1.11390	1.24673	0.31269	1.93416	0.46489
		2	2.24738	1.05504	1.23877	0.31073	1.88182	0.45005
		3	2.22559	1.02791	1.24640	0.32500	1.87576	0.46747
40	20	1	1.43949	0.52874	1.05784	0.19179	1.30707	0.21219
		2	1.40926	0.51900	1.04273	0.19192	1.28062	0.21086
		3	1.42540	0.52076	1.05283	0.18952	1.29593	0.20862
50	20	1	1.36742	0.49942	1.01794	0.19162	1.24093	0.21341
		2	1.33914	0.48348	1.00652	0.18737	1.21852	0.20676
		3	1.27661	0.43686	0.97987	0.17375	1.17202	0.18784
				$T = 8.000$				
20	10	1	2.25603	1.09667	1.23202	0.35875	1.86925	0.52784
		2	2.21011	1.05822	1.22587	0.35849	1.84251	0.52003
		3	2.22140	1.07310	1.22867	0.36060	1.84872	0.52342
25	10	1	2.24444	1.11382	1.22361	0.38062	1.85137	0.56178
		2	2.21236	1.08997	1.22564	0.37414	1.83850	0.54902
		3	2.21065	1.06875	1.21911	0.39370	1.82759	0.57908
40	20	1	1.28555	0.48218	0.96860	0.20748	1.16251	0.23242
		2	1.26620	0.47754	0.95948	0.21048	1.14627	0.23455
		3	1.27434	0.47836	0.96396	0.20818	1.15350	0.23235
50	20	1	1.28035	0.49801	0.96394	0.22785	1.15387	0.25777
		2	1.25753	0.48588	0.95272	0.22514	1.13542	0.25328
		3	1.25050	0.50064	0.94368	0.24261	1.12334	0.27530

Table 2. The values of EB and ER of the ML and Bayesian estimates for σ.

			$\widehat{\sigma}_{ML}$		$\widehat{\sigma}_B$			
					IP		NIP	
n	m	$Scheme$	ER	EB	ER	EB	ER	EB
				$T = 7.000$				
20	10	1	0.10765	0.07313	0.08297	0.01201	0.11201	0.03022
		2	0.10765	0.07313	0.08317	0.01124	0.11029	0.02731
		3	0.10765	0.07313	0.08307	0.01155	0.11054	0.02837
25	10	1	0.08571	0.05835	0.06561	0.00822	0.07406	0.01734
		2	0.08571	0.05835	0.06544	0.00772	0.07169	0.01564
		3	0.08571	0.05835	0.06555	0.00706	0.07107	0.01397
40	20	1	0.05456	0.03856	0.03956	0.00177	0.04058	0.00385
		2	0.05456	0.03856	0.03950	0.00163	0.04045	0.00360
		3	0.05456	0.03856	0.03952	0.00174	0.04050	0.00378
50	20	1	0.04356	0.03081	0.03132	0.00112	0.03183	0.00237
		2	0.04356	0.03081	0.03133	0.00107	0.03182	0.00228
		3	0.04356	0.03081	0.03127	0.00092	0.03167	0.00199
				$T = 8.000$				
20	10	1	0.10765	0.07313	0.08074	0.00705	0.08476	0.01404
		2	0.10765	0.07313	0.08064	0.00660	0.08413	0.01296
		3	0.10765	0.07313	0.08070	0.00676	0.08441	0.01338
25	10	1	0.08571	0.05835	0.06424	0.00530	0.06697	0.00982
		2	0.08571	0.05835	0.06439	0.00509	0.06686	0.00920
		3	0.08571	0.05835	0.06416	0.00448	0.06614	0.00797
40	20	1	0.05456	0.03856	0.03904	0.00051	0.03856	0.00149
		2	0.05456	0.03856	0.03900	0.00039	0.03932	0.00166
		3	0.05456	0.03856	0.03898	0.00045	0.03931	0.00158
50	20	1	0.04356	0.03081	0.03110	0.00016	0.03133	0.00089
		2	0.04356	0.03081	0.03109	0.00022	0.03132	0.00098
		3	0.04356	0.03081	0.03106	0.00009	0.03123	0.00053

Table 3. Bayesian point predictor and 95% ET and HPD prediction intervals for $X_{\rho:R_j^*}$.

			IP			NIP		
$Sch.$	R_j^*	ρ	$\widehat{X}_{\rho:R_j^*}$	ET interval	HPD interval	$\widehat{X}_{\rho:R_j^*}$	ET interval	HPD interval
1	1	1	8.070	(6.037,15.940)	(6.003,12.887)	8.247	(6.036,16.714)	(6.003,13.205)
	6	1	6.855	(6.306,8.715)	(6.294,8.119)	6.868	(6.305,8.854)	(6.294,8.185)
		2	7.838	(6.426,12.056)	(6.302,10.671)	7.898	(6.420,12.569)	(6.300,10.926)
		3	10.689	(6.756,23.831)	(6.227,18.682)	11.169	(6.730,26.135)	(6.239,19.733)
	9	1	6.624	(6.503,7.007)	(6.500,6.893)	6.626	(6.503,7.033)	(6.500,6.906)
		2	6.761	(6.478,7.354)	(6.503,7.189)	6.525	(6.525,7.414)	(6.502,7.222)
		3	6.915	(6.569,7.724)	(6.482,7.494)	6.923	(6.565,7.826)	(6.485,7.554)
		4	7.090	(6.422,8.139)	(6.557,7.865)	7.100	(6.429,8.294)	(6.548,7.954)
		5	7.288	(6.693,8.621)	(6.603,8.276)	7.306	(6.680,8.842)	(6.587,8.402)
		6	7.524	(6.773,9.194)	(6.658,8.758)	7.544	(6.754,9.500)	(6.633,8.935)
		7	7.796	(6.868,9.898)	(6.723,9.347)	7.831	(6.840,10.313)	(6.688,9.585)
		8	8.140	(6.981,10.792)	(6.798,10.079)	8.189	(6.944,11.356)	(6.750,10.396)
		9	8.574	(7.118,11.986)	(6.887,11.038)	8.631	(7.067,12.761)	(6.827,11.489)
		10	9.149	(7.289,13.696)	(6.995,12.396)	9.267	(7.223,14.795)	(6.911,12.996)
		11	10.023	(7.515,16.442)	(7.120,14.489)	10.197	(7.427,18.102)	(7.019,15.394)
		12	11.550	(7.840,21.928)	(7.279,18.492)	11.948	(7.722,24.813)	(7.145,19.988)
		13	16.155	(8.416,41.675)	(7.458,31.412)	17.727	(8.245,49.569)	(7.290,35.050)
2	1	1	8.147	(6.040,16.227)	(6.003,13.234)	8.222	(6.040,16.674)	(6.003,13.433)
	6	1	6.886	(6.307,8.767)	(6.294,8.191)	6.898	(6.307,8.847)	(6.294,8.241)
		2	7.918	(6.443,12.072)	(6.305,10.788)	7.958	(6.443,12.326)	(6.304,10.945)
		3	10.821	(6.820,23.735)	(6.393,19.053)	11.004	(6.818,24.812)	(6.387,19.655)
	10	1	7.142	(6.541,9.093)	(6.528,8.496)	7.155	(6.541,9.176)	(6.528,8.547)
		2	8.213	(6.682,12.521)	(6.539,11.190)	8.254	(6.682,12.785)	(6.538,11.352)
		3	11.223	(7.074,24.618)	(6.630,19.762)	11.413	(7.072,25.735)	(6.625,20.387)
	15	1	9.963	(9.125,12.685)	(9.107,11.852)	9.981	(9.125,12.801)	(9.107,11.923)
		2	11.457	(9.322,17.467)	(9.122,15.610)	11.515	(9.322,17.835)	(9.121,15.836)
		3	15.656	(9.868,34.343)	(8.985,27.479)	15.920	(9.866,35.900)	(8.991,28.355)

Table 4. Bayesian point predictor and 95% ET and HPD prediction intervals for $Y_{s:\ell:N}$.

		IP			NIP		
$Scheme$	s	$\widehat{Y}_{s:N}$	ET interval	HPD interval	$\widehat{Y}_{s:N}$	ET interval	HPD interval
1	1	6.020	(5.834,6.247)	(5.823,6.235)	6.019	(5.826,6.255)	(5.814,6.242)
	2	6.107	(5.892,6.462)	(5.864,6.418)	6.108	(5.889,6.487)	(5.857,6.435)
	3	6.201	(5.949,6.677)	(5.908,6.601)	6.204	(5.947,6.724)	(5.901,6.631)
	4	6.320	(6.007,6.956)	(5.954,6.835)	6.325	(6.005,7.034)	(5.946,6.882)
	5	6.452	(6.063,7.263)	(6.001,7.091)	6.459	(6.059,7.378)	(5.991,7.159)
	6	6.633	(6.131,7.712)	(6.051,7.460)	6.644	(6.122,7.883)	(6.038,7.560)
	7	6.844	(6.207,8.238)	(6.100,7.891)	6.862	(6.192,8.482)	(6.082,8.033)
	8	7.203	(6.315,9.242)	(6.152,8.690)	7.232	(6.292,9.621)	(6.127,8.908)
	9	7.686	(6.454,10.643)	(6.203,9.792)	7.737	(6.419,11.234)	(6.170,10.126)
	10	10.452	(6.799,22.781)	(6.250,17.985)	10.785	(6.738,25.128)	(6.208,19.092)
2	1	6.020	(5.827,6.255)	(5.816,6.243)	6.020	(5.821,6.261)	(5.810,6.248)
	2	6.113	(5.886,6.467)	(5.860,6.429)	6.114	(5.881,6.480)	(5.472,6.440)
	3	6.212	(5.944,6.676)	(5.907,6.613)	2.215	(5.941,6.699)	(5.902,6.630)
	4	6.338	(6.007,6.944)	(5.958,6.846)	6.343	(4.006,6.980)	(5.953,6.872)
	5	6.477	(6.071,7.236)	(6.012,7.098)	6.485	(6.070,7.287)	(6.007,7.135)
	6	6.668	(6.149,7.664)	(6.070,7.462)	6.680	(6.147,7.739)	(6.065,7.515)
	7	6.891	(6.238,8.163)	(6.130,7.886)	6.908	(6.234,8.266)	(6.124,7.958)
	8	7.268	(6.366,9.125)	(6.197,8.677)	7.295	(6.362,9.284)	(6.188,8.786)
	9	7.776	(6.530,10.456)	(6.267,9.763)	7.816	(6.524,10.701)	(6.255,9.927)
	10	10.626	(6.935,22.583)	(6.331,18.208)	10.799	(6.925,23.630)	(6.316,18.807)