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A Novel Solution Approach using Linearization Technique for Nonlinear Programming Problems

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ABSTRACT

In this paper, a novel solution approach for solving the nonlinear programming (NLP) problems having m nonlinear algebraic inequality (equality or mixed) constraints with a nonlinear algebraic objective function in n variables using linearization technique is presented. This approach performs successive increments to find a solution of the NLP problem, based on the optimal solutions of linear programming (LP) problems, satisfying the nonlinear constraints oversensitively. In the proposed approach, the original problem is converted to the LP problem using increments in the linearization process and the impact of computational efficiency makes the performance of the solution well. It is presented that how the solution approach can be applied to solve the illustrated examples from the literature.

General Terms

Computational Mathematics, Optimization

Keywords

Linear Programming, Incremental Technique, Taylor Series, Linearization Algorithm

1. INTRODUCTION

Constructing a mathematical model for real life problems is an important issue in optimization theory. Optimization problems can be classified according to the nature of the objective function and constraints. An optimization problem can be defined as min (or max) of a single (or multi) objective function, subject to (or not to) single (or multi) nonlinear (or linear) inequality (or equality or mixed) constraints. If all objective function(s) and constraint(s) are linear, then the problem is known LP problem. NLP problems are a special version of LP, i.e., the objective function and/or constraint(s) are nonlinear, that is called general NLP problems. LP or NLP problems optimize the objective function(s) subject to finite number of constraints, considering whether subject to non-negativity constraints or not.

There is no effective method for solving the general NLP problems like simplex method in LP. When the number of variables or constraints increases, solving NLP problems numerically needs huge computational efforts by using special optimization algorithms [6]. Since 1951, there has been great progress for solving NLP problems. Hestenes [9] proposed augmented Lagrangian methods for solving equality constrained problems. This approach was extended in [12] to the constrained optimization problem with both equality and inequality constraints. Sannomiya et al. [13] proposed an effective method even if there is no feasible solution satisfying the approximate linear constraints.

Linearization methods can be used converting a NLP problem into a LP problem. In this process, extra variables and constraints are introduced to construct the original problem. Various methods have been proposed in the literature by linearizing a NLP problem [14], [11]. Sequential Linear Programming (SLP) which is one of the direct methods solves NLP problems approximately, and uses a series of LP problems generated by using first order Taylor series expansions of objective functions and constraints. Byrd and Nocedal [3] have presented a new active-set, trust-region algorithm for large-scale optimization using SLP techniques to solve the NLP problems approximately.

Sequential Quadratic Programming (SQP) is first proposed by Wilson in his PhD thesis in 1963 for solving constrained NLP problems. Many research papers have been produced using SQP-based techniques .Gill and Wong [7] reviewed some of the most prominent developments in SQP. An improved SQP algorithm with arbitrary initial iteration point for solving a class of general NLP problems with equality and inequality constraints is proposed in [8]. Conjugate-gradient methods (CG) are used to solve large-dimensional problems that arise in computational linear or nonlinear optimization problem. The linear CG method for solving the system of linear n equations in n unknowns was developed in [10].The method did not compete with direct method, Gauss elimination, but it is used in real-world applications. Albayrak et al. [1] proposed an iterative approach for solving the NLP problems having n nonlinear (or linear) algebraic equality constraints with nonlinear (or linear) algebraic objective function in $n + 1$ variables.

This iterative approach constructs different optimization problems corresponding to the parameter related with arbitrary points which are chosen satisfying the constraints.

In this paper, a novel solution approach for solving general NLP problems, having m nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with nonlinear (or linear) objective function in n variables is presented. The original problem is converted to the LP problem using increments in the linearization process and the impact of computational efficiency makes the performance of the solution well.

This paper is organized as follows: Section 2 presents brief required information used in this work. In Section 3, the proposed approach is handled. Section 4 and Section 5 consist of numerical examples and conclusions, respectively.

2. PRELIMINARIES

In this section, required information is presented.

DEFINITION 1. *A general constrained NLP problem can be defined as follows:*

$$\begin{array}{l} min \quad f(x) \\ s.t. \\ g_i(x) = b_i, i = 1, 2, ..., p \\ g_j(x) \le b_j, j = p+1, ..., m \end{array} \tag{1}$$

where $x = [x_1, x_2, ..., x_n] \in R^n$ *is a vector,* $g_i : R^n \to R, (i = 1, 2, ..., p)$, $g_j : R^n \to R, (j = p+1, ..., m)$. *In (1), if the objective function and all constraints are linear, it is known as an LP problem.*

DEFINITION 2. *[5] Any point x satisfying the constraints is called the feasible point. The set of all feasible points is called the feasible set such that* $X = \{x \in R^n : g_i(x) = b_i, (i = 1, 2, ..., p), g_j(x) \le b_j, (j = p+1, ..., m)\}$.

DEFINITION 3. *An optimal solution x^* to a LP problem is a feasible solution with the smallest objective function value for a minimization problem.*

DEFINITION 4. *A point x in the feasible set X is said to be an interior point if X contains some neighborhood of x.*

DEFINITION 5. *After converting NLP problem to LP problem, the obtained solution is called a linearization point.*

3. THE PROPOSED APPROACH

A novel solution approach for solving general NLP problems, having m nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with nonlinear (or linear) objective function in n variables is presented. The iterations present reasonable progress to convert the NLP problem using successive linearization process by means of Taylor series expansions.

Step 1 Load the NLP problem having m nonlinear constraints with a nonlinear objective function in n variables as given in (1).

Step 2 Construct Lagrangian function

$$\begin{aligned} L(x_k, \lambda_i, \lambda_j) = f(x_k) + \textstyle\sum_i \lambda_i (g_i(x_k) - b_i) \\ + \textstyle\sum_j \lambda_j (g_j(x_k) - b_j) \end{aligned} \tag{2}$$

where $k = 1, ..., n; i = 1, 2, ..., p; j = p+1, ..., m$.

Step 3 Construct a nonlinear system obtained from (2).

$$\begin{array}{l} \frac{\partial L}{\partial x_k} = 0, k = 1, ..., n \\ \frac{\partial L}{\partial \lambda_i} = 0, i = 1, ..., p \\ \lambda_j (g_j(x_1, ..., x_n) - b_j) = 0, j = p+1, ..., m \end{array} \tag{3}$$

Step 4 Choose any initial arbitrary nonzero point $(x_k, \lambda_i, \lambda_j), k = 1, ..., n; i = 1, ..., p; j = p+1, ..., m.$

Step 5 Linearize each equation in (3) by expanding Taylor series at the chosen point.

Step 6 Construct a linear system as follows:

$$\begin{array}{l} \left[\frac{\partial L}{\partial x_k}\right]_L = 0, k = 1, ..., n \\ \left[\frac{\partial L}{\partial \lambda_i}\right]_L = 0, i = 1, ..., p \\ \left[\lambda_j (g_j(x_1, ..., x_n) - b_j)\right]_L = 0, j = p+1, ..., m \end{array} \tag{4}$$

where the subscript L is used to show the linearization, and solve the system in (4).

Step 7 Analyze the solution of the system in (4):
If there is a solution as $\bar{x} = (\bar{x}_1, ..., \bar{x}_n)$, go to Step 8.
Else, go to Step 4.

Step 8 Check the types of constraint of (1):
If they are mixed or equality constraints, go to Step 9.
If they are inequality constraints, go to Step 13.

Step 9 Linearize each equation in (3) by expanding Taylor series at $\bar{x}$.

Step 10 Solve the reconstructed linear system.

Step 11 Analyze the solution obtained in Step 10:
If there is a solution as $\underline{x} = (\underline{x}_1, ..., \underline{x}_n)$, go to Step 12.
Else, go to Step 4.

Step 12 Check successive two objective function values:
For a given $\epsilon > 0$ if $|f(\bar{x}) - f(\underline{x})| \le \epsilon$, then it is a solution of the NLP problem in (1) and STOP.
Else, assign $\underline{x}$ to $\bar{x}$, and go to Step 9.

Step 13 Check the solution $\bar{x}$:
If it satisfies all constraints of (1) individually, go to Step 14.
Else, linearize each equation in (3) by expanding Taylor series at $\bar{x}$, and go to Step 6.

Step 14 Introduce new variables $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ by making increments

$$\hat{x}_k = \bar{x}_k + u_k - v_k \tag{5}$$

where $u_k, v_k, (k = 1, ..., n)$ are nonnegative variables defined as $0 \le u_k \le 1$ and $0 \le v_k \le 1$.

Step 15 Substitute $\hat{x}$ generated in (5) into objective function and constraints of (1).

Step 16 Linearize new objective function and each constraint obtained in Step 15 by expanding Taylor series.

Step 17 Construct a LP problem by adding new constraints

$$\begin{array}{l} min \quad f_L(u_1, ..., u_n, v_1, ..., v_n) \\ s.t. \\ g_{Lj}(u_1, ..., u_n, v_1, ..., v_n) \le b_j, j = p+1, ..., m \\ 0 \le u_k \le 1, k = 1, ..., n \\ 0 \le v_k \le 1, k = 1, ..., n \end{array} \tag{6}$$

and solve (6).

Step 18 Check the solution of (6):
If there is a feasible solution, go to Step 19.
Else, go to Step 4.

Step 19 Analyze the increments:

If all u, v are zero, $\hat{x}$ is a solution for the NLP problem (1) and STOP.
Else, determine $\hat{x}$. Assign $\hat{x}$ to $\bar{x}$, and go to Step 14.

The flow chart of the proposed approach is given in Figure1.

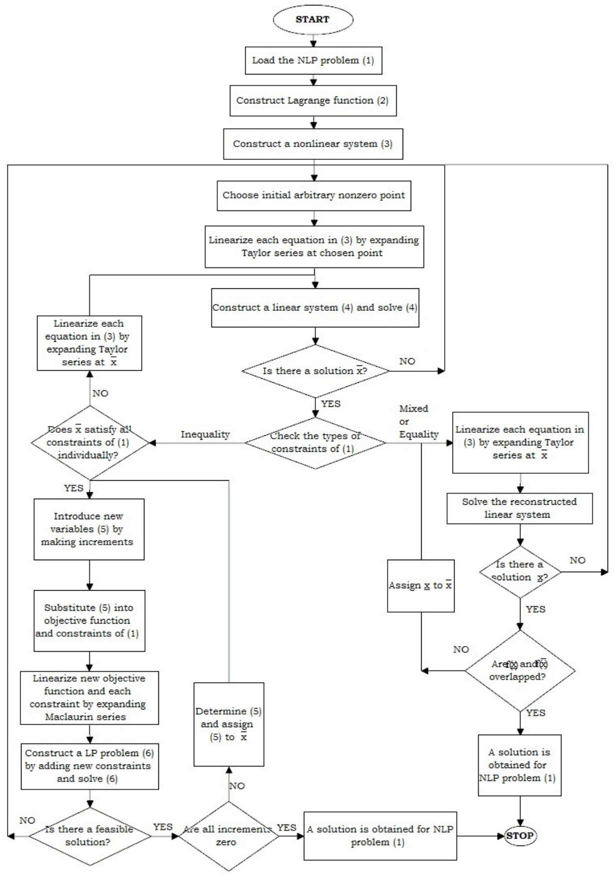

Fig. 1. The flow chart of the proposed approach

4. NUMERICAL EXAMPLES

Example 1 [2] Solve the NLP problem

$$\begin{array}{l} min \quad (x_1-2)^2+(x_2-2)^2 \\ s.t. \\ x_1^2+x_2^2-1=0 \\ x_2^2-x_1 \le 0 \end{array} \tag{7}$$

using the proposed approach where $\epsilon = 10^{-5}$.
Step 1-2. Load the NLP problem (7) and construct Lagrangian function:

$$\begin{aligned} L(x_1,x_2,\lambda_1,\lambda_2) &= (x_1-2)^2+(x_2-2)^2 \\ &+\lambda_1(x_1^2+x_2^2-1)+\lambda_2(x_2^2-x_1). \end{aligned} \tag{8}$$

Step 3. Construct the following nonlinear system obtained from (8).

$$\begin{array}{l} 2(x_1-2)+2\lambda_1 x_1-\lambda_2=0 \\ 2(x_2-2)+2\lambda_1 x_2+2\lambda_2 x_2=0 \\ x_1^2+x_2^2-1=0 \\ \lambda_2(x_2^2-x_1)=0. \end{array} \tag{9}$$

Step 4. Choose any initial arbitrary nonzero point: $x_1 = 3, x_2 = 3, \lambda_1 = 1, \lambda_2 = 1$.
Step 5-6. Linearize each equation in (9) by expanding Taylor series at chosen point, and construct the following linear system.

$$\begin{array}{r} 4x_1+6\lambda_1-\lambda_2=10 \\ 6x_2+6\lambda_1+6\lambda_2=16 \\ 6x_1+6x_2=19 \\ -x_1+6x_2+6\lambda_2=15 \end{array} \tag{10}$$

Step 7.The solution of (10) is obtained as $x_1 = 4.5455, x_2 = -1.3788, \lambda_1 = -0.5909$, $\lambda_2 = 4.6364$.
Summarized results of Example 1 using the proposed approach is given in Table 1. Basirzadeh also solved this problem in [2]. The comparison of solutions is presented in Table 2. The optimal solution obtained by the proposed approach satisfies the equality constraint of (7) oversensitively, however it is not verified with Basirzadeh's solution.

Table 1. Iteration Results of Example 1

Iterations	x_1	x_2	$f(x_1,x_2)$
1st	4.5455	-1.3788	17.8959
2nd	2.385	-0.682	7.3413
3rd	1.8908	1.3679	0.4115
4th	1.6591	0.063	3.8682
5th	1.0905	1.0966	1.6433
6th	0.5921	0.9577	3.0686
7th	0.633	0.7926	3.3265
8th	0.6182	0.7862	3.3827
9th	0.6181	0.7861	3.3832
10th	0.6181	0.7861	3.3832

Table 2. The Comparison of Solutions of Example 1

	Basirzadeh's Approach	Proposed Approach
x_1	0.7070	0.6181
x_2	0.7070	0.7861
z	3.3437	3.3832

Optimal solution of the NLP problem (9) is $x_1^* = 0.6181, x_2^* = 0.7861$ and the optimal value is $z^* = 3.3832$.
Example 2 [4] Solve the NLP problem

$$\begin{array}{l} max \quad 3x_1^3+2x_2^3 \\ s.t. \\ x_1^2+x_2^2-16 \le 0 \\ x_1-x_2-3 \le 0 \end{array} \tag{11}$$

using the proposed approach where $\epsilon = 10^{-5}$.
Step 1-2. Load the NLP problem (11) and construct Lagrangian function:

$$\begin{aligned} L(x_1,x_2,\lambda_1,\lambda_2) &= -3x_1^3-2x_2^3+\lambda_1(x_1^2+x_2^2-16) \\ &+\lambda_2(x_1-x_2-3). \end{aligned} \tag{12}$$

Step 3. Construct the following nonlinear system obtained from (12).

$$\begin{array}{l} -9x_1^2 + 2\lambda_1 x_1 + \lambda_2 = 0 \\ -6x_2^2 + 2\lambda_1 x_2 - \lambda_2 = 0 \\ \lambda_1(x_1^2 + x_2^2 - 16) = 0 \\ \lambda_2(x_1 - x_2 - 3) = 0. \end{array} \tag{13}$$

Step 4. Choose any initial arbitrary nonzero point: $x_1 = 5, x_2 = 1, \lambda_1 = 1, \lambda_2 = 2$.
Step 5-6. Linearize each equation in (13) by expanding Taylor series at chosen point, and construct the following linear system.

$$\begin{array}{r} 88x_1 - 10\lambda_1 - \lambda_2 = 215 \\ 10x_2 - 2\lambda_1 + \lambda_2 = 4 \\ 10x_1 + 2x_2 + 10\lambda_1 = 52 \\ 2x_1 - 2x_2 + \lambda_2 = 8 \end{array} \tag{14}$$

Step 7.The solution of (14) is obtained as

$$x_1 = 2.75, x_2 = 0.5161, \lambda_1 = 2.3468, \lambda_2 = 3.5323. \tag{15}$$

Step 8-13. Check if the solution obtained in (15), providing all the constraints in (11).
It is seen that this solution satisfies the constraints individually.
Step 14. Introduce new variables $\hat{x}_1, \hat{x}_2$ by making increments

$$\begin{array}{l} \hat{x}_1 = 2.75 + u_1 - v_1 \\ \hat{x}_2 = 0.5161 + u_2 - v_2 \end{array} \tag{16}$$

where u_1, u_2, v_1, v_2 are nonnegative variables.
Step 15-17. Substitute $\hat{x}_1, \hat{x}_2$ generated in (16) into objective function and constraints of (11), linearize new objective function and each constraint by expanding Taylor series, and construct the following LP problem by adding new constraints

$$\begin{array}{l} max \quad 68.0625(u_1 - v_1) + 1.5982(u_2 - v_2) \\ 5.5(u_1 - v_1) + 1.0322(u_2 - v_2) \le 8.1711 \\ (u_1 - v_1) - (u_2 - v_2) \le 0.7661 \\ 0 \le u_1 \le 1; 0 \le u_2 \le 1 \\ 0 \le v_1 \le 1; 0 \le v_2 \le 1 \end{array} \tag{17}$$

and solve (17).
Step 18-19. The solution of (17) is $u_1 = 1, v_1 = 0, u_2 = 1, v_2 = 0$. Because u_1, u_2, v_1, v_2 are different from zero, $\hat{x}_1 = 3.75$, $\hat{x}_2 = 1.5161$ are determined. $\hat{x}_1, \hat{x}_2$ are assigned to $\bar{x}_1 = 3.75$, $\bar{x}_2 = 1.5161$, respectively. Then, go to Step 14. Until all u_1, v_1, u_2, v_2 become zero, the process is continued from Step 14 to Step 19.
The proposed approach is applied to the problem solved in [4]. The approach is more efficient than Chiş and Cret's approach for maximizing (11). Summarized results and the comparison of solutions are shown in Table 3 and Table 4, respectively.

Table 3. Iteration Results of Example 2

Iterations	x_1	x_2	$f(x_1, x_2)$
1st	3.75	1.5161	165.1728
2nd	3.9363	0.9363	184.6141
3rd	3.8982	0.8982	179.16
4th	3.8979	0.8979	179.1175
5th	3.8979	0.8979	179.1175

Optimal solution of the NLP problem (11) is $x_1^* = 3.8979, x_2^* = 0.8979$ and the optimal value is $z^* = 179.1175$.

Table 4. The Comparison of Solutions of Example 2

	Chis and Cret's Approach	Proposed Approach
x_1	3.875	3.8979
x_2	0.875	0.8979
z	175.8965	179.1175

5. CONCLUSION

In this paper, a novel solution approach for solving general NLP problems, having m nonlinear (or linear) algebraic inequality (or equality or mixed) constraints with nonlinear (or linear) objective function in n variables is presented. This solution approach is applied according to structure of constraints:

This approach performs successive LP problems using increments after solving linear system(s) obtained from Lagrangian function to find a solution of the NLP problem subject to inequality constraints.

This approach also finds a solution of the NLP problem under mixed or equality constraints. The solution is based on the solutions of linear systems obtained from Lagrangian function.

After performing linearization approach at any initial arbitrary nonzero point for each type of structure of constraints, the obtained solution of the original NLP problem satisfies the constraints sensitively while making the objective function min or max.

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