Square Difference 3-Equitable Labeling of Paths and Cycles

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ABSTRACT

A square difference 3-equitable labeling of a graph $G$ with vertex set $V$ is a bijection $f$ from $V$ to $\{1, 2, \ldots, |V|\}$ such that if each edge $uv$ is assigned the label $-1$ if $|f(u)^2 - f(v)^2| \equiv -1(\text{mod } 4)$, the label $0$ if $|f(u)^2 - f(v)^2| \equiv 0(\text{mod } 4)$ and the label $1$ if $|f(u)^2 - f(v)^2| \equiv 1(\text{mod } 4)$, then the number of edges labeled with $i$ and the number of edges labeled with $j$ differ by atmost $1$ for $-1 \leq i, j \leq 1$. A graph which admits square difference 3-equitable labeling is called square difference 3-equitable graph.

EXAMPLE 1. Consider the following graph $G$.

![Fig 1. A square difference 3-equitable graph](image)

We see that $e_f(-1) = e_f(1) = 2$ and $e_f(0) = 3$.

Thus $|e_f(i) - e_f(j)| \leq 1$ for all $-1 \leq i, j \leq 1$ and hence $G$ is square difference 3-equitable.

2. MAIN RESULTS

THEOREM 1. The path $P_n$ admits square difference 3-equitable labeling.

PROOF. Let $P_n : u_1 u_2 \ldots u_n$ be the path. If $n \leq 4$, the following table gives the square difference 3-equitable labeling of $P_n$. 

<table>
<thead>
<tr>
<th>$e$</th>
<th>$f(e_{uv})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$uv$</td>
<td>${-1, 0, 1}$ is defined by $f^*(e_{uv})$</td>
</tr>
</tbody>
</table>

We see that $e_f(-1) = e_f(1) = 2$ and $e_f(0) = 3$. Thus $|e_f(i) - e_f(j)| \leq 1$ for all $-1 \leq i, j \leq 1$ and hence $G$ is square difference 3-equitable.
If $n \geq 5$, we consider the following cases.

Case(i): $n \equiv 1(\text{mod } 6)$

Define

$$f(u_1) = 2$$
$$f(u_2) = 3$$
$$f(u_3) = 1$$
$$f(u_4) = 4,$$

for $1 \leq i \leq \frac{n-2}{6}$.

$$f(u_{6i-1}) = 6i + 1$$
$$f(u_{6i}) = 6i - 1$$
$$f(u_{6i+1}) = 6i$$
$$f(u_{6i+2}) = 6i + 2$$
$$f(u_{6i+3}) = 6i + 3$$
$$f(u_{6i+4}) = 6i + 4$$

and

$$f(u_{n-2}) = n$$
$$f(u_{n-1}) = n - 2$$
$$f(u_n) = n - 1.$$

Then

$$||f(u_1)||^2 - ||f(u_2)||^2 \equiv 1(\text{mod } 4)$$
$$\Rightarrow f'(u_1u_2) = 1$$
$$||f(u_2)||^2 - ||f(u_3)||^2 \equiv 0(\text{mod } 4)$$
$$\Rightarrow f'(u_2u_3) = 0$$
$$||f(u_3)||^2 - ||f(u_4)||^2 \equiv -1(\text{mod } 4)$$
$$\Rightarrow f'(u_3u_4) = -1,$$

for $1 \leq i \leq \frac{n-2}{6}$.

$$||f(u_{6i-2})||^2 - ||f(u_{6i-1})||^2 \equiv 1(\text{mod } 4)$$
$$\Rightarrow f'(u_{6i-2}u_{6i-1}) = 1$$
$$||f(u_{6i-1})||^2 - ||f(u_{6i})||^2 \equiv 0(\text{mod } 4)$$
$$\Rightarrow f'(u_{6i-1}u_{6i}) = 0$$
$$||f(u_{6i})||^2 - ||f(u_{6i+1})||^2 \equiv -1(\text{mod } 4)$$
$$\Rightarrow f'(u_{6i}u_{6i+1}) = -1$$

$$||f(u_{6i+1})||^2 - ||f(u_{6i+2})||^2 \equiv 0(\text{mod } 4)$$
$$\Rightarrow f'(u_{6i+1}u_{6i+2}) = 0$$
$$||f(u_{6i+2})||^2 - ||f(u_{6i+3})||^2 \equiv 1(\text{mod } 4)$$
$$\Rightarrow f'(u_{6i+2}u_{6i+3}) = 1$$

for $1 \leq i \leq \frac{n-2}{6}$. And

$$||f(u_{n-3})||^2 - ||f(u_{n-2})||^2 \equiv 1(\text{mod } 4)$$
$$\Rightarrow f'(u_{n-3}u_{n-2}) = 1$$
$$||f(u_{n-2})||^2 - ||f(u_{n-1})||^2 \equiv 0(\text{mod } 4)$$
$$\Rightarrow f'(u_{n-2}u_{n-1}) = 0$$
$$||f(u_{n-1})||^2 - ||f(u_n)||^2 \equiv -1(\text{mod } 4)$$
$$\Rightarrow f'(u_{n-1}u_n) = -1.$$
and

\[ f(u_{n-4}) = n - 2 \]
\[ f(u_{n-3}) = n - 4 \]
\[ f(u_{n-2}) = n - 3 \]
\[ f(u_{n-1}) = n - 1 \]
\[ f(u_n) = n. \]

Then \( e_f(-1) = \frac{n-2}{3} \) and \( e_f(0) = e_f(1) = \frac{n}{3} \).

**Case(iv):** \( n \equiv 4 \pmod{6} \)

Define

\[ f(u_1) = 2 \]
\[ f(u_2) = 3 \]
\[ f(u_3) = 1 \]
\[ f(u_4) = 4 \]

and for \( 1 \leq i \leq \frac{n-4}{6} \),

\[ f(u_{6i-1}) = 6i + 1 \]
\[ f(u_{6i}) = 6i - 1 \]
\[ f(u_{6i+1}) = 6i \]
\[ f(u_{6i+2}) = 6i + 1 \]
\[ f(u_{6i+3}) = 6i + 3 \]
\[ f(u_{6i+4}) = 6i + 4. \]

Then \( e_f(-1) = e_f(0) = e_f(1) = \frac{n}{3} \).

**Case(v):** \( n \equiv 5 \pmod{6} \)

Define

\[ f(u_1) = 2 \]
\[ f(u_2) = 3 \]
\[ f(u_3) = 1 \]
\[ f(u_4) = 4, \]

for \( 1 \leq i \leq \frac{n-5}{6} \),

\[ f(u_{6i-1}) = 6i + 1 \]
\[ f(u_{6i}) = 6i - 1 \]
\[ f(u_{6i+1}) = 6i \]
\[ f(u_{6i+2}) = 6i + 1 \]
\[ f(u_{6i+3}) = 6i + 3 \]
\[ f(u_{6i+4}) = 6i + 4. \]

and

\[ f(u_n) = n. \]

Then \( e_f(-1) = e_f(0) = \frac{n-2}{3} \) and \( e_f(1) = \frac{n+1}{3} \).

**Case(vi):** \( n \equiv 0 \pmod{6} \)

Define

\[ f(u_1) = 2 \]
\[ f(u_2) = 3 \]
\[ f(u_3) = 1 \]
\[ f(u_4) = 4, \]

for \( 1 \leq i \leq \frac{n-6}{6} \),

\[ f(u_{6i-1}) = 6i + 1 \]
\[ f(u_{6i}) = 6i - 1 \]
\[ f(u_{6i+1}) = 6i \]
\[ f(u_{6i+2}) = 6i + 2 \]
\[ f(u_{6i+3}) = 6i + 3 \]
\[ f(u_{6i+4}) = 6i + 4. \]

and

\[ f(u_{6i+5}) = n - 1 \]
\[ f(u_{6i+6}) = n. \]

Then \( e_f(-1) = e_f(1) = \frac{n}{3} \) and \( e_f(0) = \frac{n-3}{3} \).

Thus in all cases, \( |e_f(i) - e_f(j)| \leq 1 \) for all \( -1 \leq i, j \leq 1 \) and therefore \( P_n \) is a square difference 3-equitable graph. \( \Box \)

**EXAMPLE 2.** The square difference 3-equitable labeling of \( P_{10} \) is shown below.

Fig 2. Square difference 3-equitable labeling of \( P_{10} \)

**THEOREM 2.** The cycle \( C_n \) admits square difference 3-equitable labeling.

**PROOF.** Let \( C_n : u_1u_2...u_nu_1 \) be the cycle.

The square difference 3-equitable labeling of \( C_3 \) is given as follows.

Fig 3. Square difference 3-equitable labeling of \( C_3 \)

If \( n \geq 4 \), we consider the following cases.

**Case(i):** \( n \equiv 1 \pmod{6} \)

Define

\[ f(u_1) = 2 \]
\[ f(u_2) = 3 \]
\[ f(u_3) = 1 \]
\[ f(u_4) = 4, \]
for $1 \leq i \leq \frac{n-7}{6}$,

\[
\begin{align*}
f(u_{i-1}) &= 6i + 1 \\
f(u_i) &= 6i - 1 \\
f(u_{i+1}) &= 6i \\
f(u_{i+2}) &= 6i + 2 \\
f(u_{i+3}) &= 6i + 3 \\
f(u_{i+4}) &= 6i + 4 \\
\end{align*}
\]

and

\[
\begin{align*}
f(u_{n-2}) &= n \\
f(u_{n-1}) &= n - 2 \\
f(u_n) &= n - 1.
\end{align*}
\]

Then $e_f(-1) = e_f(1) = \frac{n+1}{3}$ and $e_f(0) = \frac{n-2}{3}$.

Case(ii): $n \equiv 2 \pmod{6}$

Define

\[
\begin{align*}
f(u_1) &= 2 \\
f(u_2) &= 1 \\
f(u_3) &= 4 \\
f(u_4) &= 5 \\
f(u_5) &= 3.
\end{align*}
\]

for $1 \leq i \leq \frac{n-8}{6}$,

\[
\begin{align*}
f(u_{6i}) &= 6i + 2 \\
f(u_{6i+1}) &= 6i \\
f(u_{6i+2}) &= 6i + 1 \\
f(u_{6i+3}) &= 6i + 3 \\
f(u_{6i+4}) &= 6i + 4 \\
f(u_{6i+5}) &= 6i + 5 \\
\end{align*}
\]

and

\[
\begin{align*}
f(u_{n-2}) &= n \\
f(u_{n-1}) &= n - 2 \\
f(u_n) &= n - 1.
\end{align*}
\]

Then $e_f(-1) = e_f(1) = \frac{n+1}{3}$ and $e_f(0) = \frac{n-2}{3}$.

Case(iii): $n \equiv 3 \pmod{6}$

Define

\[
\begin{align*}
f(u_1) &= 2 \\
f(u_2) &= 3 \\
f(u_3) &= 5 \\
f(u_4) &= 4 \\
f(u_5) &= 1 \\
f(u_6) &= 6,
\end{align*}
\]

for $1 \leq i \leq \frac{n-9}{6}$,

\[
\begin{align*}
f(u_{6i+1}) &= 6i + 3 \\
f(u_{6i+2}) &= 6i + 1 \\
f(u_{6i+3}) &= 6i + 2 \\
f(u_{6i+4}) &= 6i + 4 \\
f(u_{6i+5}) &= 6i + 5 \\
f(u_{6i+6}) &= 6i + 6
\end{align*}
\]

and

\[
\begin{align*}
f(u_{n-2}) &= n \\
f(u_{n-1}) &= n - 2 \\
f(u_n) &= n - 1.
\end{align*}
\]

Then $e_f(-1) = e_f(0) = e_f(1) = \frac{2}{3}$.

Case(iv): $n \equiv 4 \pmod{6}$

Define

\[
\begin{align*}
f(u_1) &= 2 \\
f(u_2) &= 3 \\
f(u_3) &= 1 \\
f(u_4) &= 4
\end{align*}
\]

and for $1 \leq i \leq \frac{n-4}{6}$,

\[
\begin{align*}
f(u_{6i-1}) &= 6i + 1 \\
f(u_6) &= 6i - 1 \\
f(u_{6i+1}) &= 6i \\
f(u_{6i+2}) &= 6i + 2 \\
f(u_{6i+3}) &= 6i + 3 \\
f(u_{6i+4}) &= 6i + 4
\end{align*}
\]

Then $e_f(-1) = e_f(1) = \frac{n+1}{3}$ and $e_f(0) = \frac{n+2}{3}$.

Case(v): $n \equiv 5 \pmod{6}$

Define

\[
\begin{align*}
f(u_1) &= 2 \\
f(u_2) &= 1 \\
f(u_3) &= 4 \\
f(u_4) &= 5 \\
f(u_5) &= 3
\end{align*}
\]

and for $1 \leq i \leq \frac{n-5}{6}$,

\[
\begin{align*}
f(u_{6i}) &= 6i + 2 \\
f(u_{6i+1}) &= 6i \\
f(u_{6i+2}) &= 6i + 1 \\
f(u_{6i+3}) &= 6i + 3 \\
f(u_{6i+4}) &= 6i + 4 \\
f(u_{6i+5}) &= 6i + 5
\end{align*}
\]

Then $e_f(-1) = e_f(1) = \frac{n+1}{3}$ and $e_f(0) = \frac{n+2}{3}$. 
Case (vi): $n \equiv 0 \pmod{6}$

Define

\[
\begin{align*}
  f(u_1) &= 2 \\
  f(u_2) &= 3 \\
  f(u_3) &= 5 \\
  f(u_4) &= 4 \\
  f(u_5) &= 1 \\
  f(u_6) &= 6
\end{align*}
\]

and for $1 \leq i \leq \frac{n-6}{6}$,

\[
\begin{align*}
  f(u_{6i+1}) &= 6i + 3 \\
  f(u_{6i+2}) &= 6i + 1 \\
  f(u_{6i+3}) &= 6i + 2 \\
  f(u_{6i+4}) &= 6i + 4 \\
  f(u_{6i+5}) &= 6i + 5 \\
  f(u_{6i+6}) &= 6i + 6
\end{align*}
\]

Then $e_f(-1) = e_f(0) = e_f(1) = \frac{3}{2}$.

Thus in all cases, $|e_f(i) - e_f(j)| \leq 1$ for all $-1 \leq i, j \leq 1$ and therefore $C_n$ is a square difference 3-equitable graph.

**Example 3.** The square difference 3-equitable labeling of $C_9$ is shown below.

![Square difference 3-equitable labeling of $C_9$](image)

Fig 4. Square difference 3-equitable labeling of $C_9$

3. **Conclusion**

We have discussed here a new labeling called square difference 3-equitable labeling and we have investigated for paths and cycles only. The results reported here are new and expected to add new dimension to the theory of 3-equitable graphs. It is possible to investigate similar results for other graph families.

4. **References**


