A Note on Acyclic Coloring of Central Graphs

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ABSTRACT
In this paper, we discuss the acyclic vertex colouring and acyclic chromatic number of central graph of Dutch-windmill graph and Sunlet graph and acyclic colouring of some other graphs.

General Terms
Central graph of Dutch-windmill graph and Sunlet graph are denoted by $C(D_3^m)$ and $C(S_n)$ respectively. The Crown graph is denoted by S_n^0

Keywords
Central graph, Dutch-windmill graph, Sunlet graph, Crown graph, acyclic colouring, acyclic chromatic number.

1. INTRODUCTION
Let G be a finite undirected graph with no loops and multiple edges. The central graph [17] of a graph G, $C(G)$ is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G. By the definition $p_{c\ G} = p+q$. For any p,q graph there exist exactly p vertices of degree $p-1$ and q vertices of degree 2 in $C(G)$.

The Dutch-Windmill graph [11,14], denoted D_3^m, also called a friendship graph, is a graph obtained by taking m-copies of the cycle C_3 with a vertex in common.

The n-sunlet graph S_n [12] is the graph on $2n$ vertices obtained by attaching n pendent edges to the cycle graph C_n.

The Crown graph S_n^0 [5] for an integer $n>2$ is the graph with the vertex set $\{u_1,u_2,u_3....u_n,v_1,v_2,v_3,...v_n\}$ and edge set $\{(u_i,v_i):1\le i,j\le n, i\ne j\}$.

A proper vertex colouring of a graph is acyclic if every cycle uses at least three colours [13]. The acyclic chromatic number of G, denoted by $a\ G$, is the minimum k such that G admits an acyclic k-colouring.

2. ACYCLIC COLOURING OF $C(D_3^m)$

2.1 Theorem
For the Dutch-windmill graph D_3^m, $a\left[C(D_3^m)\right] = 2m = \Delta$ where Δ is the maximum degree of G.

Proof
Consider the Dutch-Windmill graph D_3^m formed by m-copies of the cycle C_3 with vertices v_1^i, v_2^i, v_3^i, $i=1,2,3,....m,$ named in counter clockwise direction in which $v_1^1 = v_1^2 = v_1^3 = ... = v_1^m$ is the common vertex. Let $v_{h,k}^i$ represents the newly introduced vertex in the edge connecting v_h^i and v_k^i for $i=1,2...m$ and $x,y=1,2,3$. Now assign a proper colouring to these vertices as follows. Consider a colour class $C = c_1,c_2,c_3,....c_{2m}$ and let the pair $c_{h,}c_k$ represents an edge whose end points are coloured as c_h and c_k. Assign the color c_1 to the common vertex $v_1^i, i=1,2,..m$ and the color c_{2k-1} to the vertex v_3^k for $k=1,2,...m.$ Next assign the colour c_{2k} to the vertex v_2^{k+1} for $k=2,3,4...m-1$ if $m>2$ and c_{2m}, c_3 to the vertices v_2^1 and v_2^2 respectively. Such a colouring excludes the newly introduced vertices. Assign the colour c_2 to the newly introduced $3m$ vertices $v_{h,k}^i$ for $i=1,2,3,..m$ and $x,y=1,2,3$. Now we prove that the above said colouring is acyclic. That is it does not contain any bichromatic cycles. It is obvious that to form a 2-chromatic cycle both colours should occur at least twice. So in the above said colouring, the color classes $c_k, 4\le k\le 2m$ never induce a 2-chromatic cycle. Now we need to examine the subgraphs induced by $\langle c_i,c_j\rangle$ for $i=1,2$ and $j=2,3$ with $i\ne j$ whether they induce a 2-chromatic cycle or not.

Case 1. If $i=1$ and $j=2$, **t**hen subgraph induced by $\langle c_1,c_2\rangle$ is the union of a path tree with $\Delta-2$ pendent vertices and $m-1$ isolated vertices, which is clearly a forest.

Case 2. If $i=1$ and $j=3$, then subgraph induced by $\langle c_1,c_3\rangle$ is the union of a path P_2 and a single isolated vertex, which is a forest.

Case 3. If $i=2$ and $j=3$, then subgraph induced by $\langle c_2,c_3\rangle$ is the union of a path P_4 and $3(m-1)$ isolated vertices, which also forms a forest.

OR In each case we can easily verify the result $\varepsilon=\upsilon-\omega$, which is the necessary and sufficient condition for a forest.

Thus any pair of the colour class will never induce a 2-chromatic cycle in the graph. By the very construction the colouring is minimum.

Therefore, $a\left[C(D_3^m)\right]=2m=\Delta$.

Example

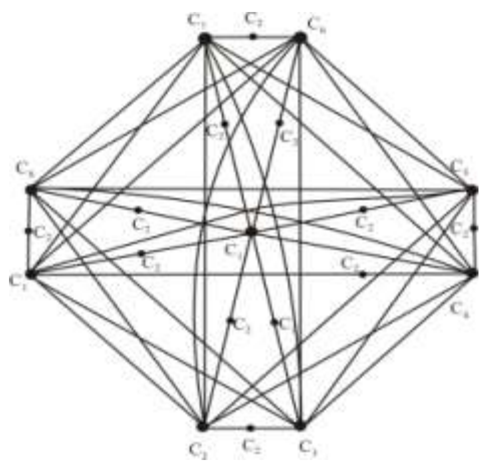

Figure 1

$a\left[C(D_3^4)\right]=8$

3. ACYCLIC COLOURING OF $C(S_n)$

3.1 Theorem

For the sunlet graph S_n, the acyclic chromatic number $a[C(S_n)]=2n-2$

Proof

Let S_n be the the n-sunlet graph having $2n$ vertices formed by attaching n pendent edges to the cycle graph C_n. Let $V(C_n)=\{u_1,u_2,u_3...u_n\}$. The pendent edge connected to the vertex u_i be u_iv_i, $i=1,2,3...n$. That is $v_1,v_2,v_3...v_n$ are the pendent vertices. Now consider $C(S_n)$, let the vertex introduced in the edge joining u_i and u_j be $u_{i,j}$, $i=1,2,3...n$, $j=2,3,4...n$ and vertex in the edge joining u_i and v_j be v_{ij}. Now in $C(S_n)$, $[v_1,v_2,v_3...v_n]$ is a complete graph on n vertices. Assign a colouring to $C(S_n)$ as follows. Colour the vertex v_i as c_i for $i=1,2,3...n$. Next the vertex u_i is assigned the colour c_i for $i=1,2$ and c_{n+i-2} for $3\le i\le n$. Such a colouring excludes the newly introduced vertices. Assign the colour c_{2n-2} to $v_{1,1}$ and $u_{1,2}$. Now c_{n+1} to $v_{2,2}$ and c_2 to $u_{n,1}$. All the remaining vertices are coloured c_1. Then clearly the above said colouring is acyclic, also the colouring is minimum. Because if we replace any colour which is minimum in number by a colour already used, the resulting colouring will be improper or cyclic.

Therefore $a[C(S_n)]=2n-2$.

Example

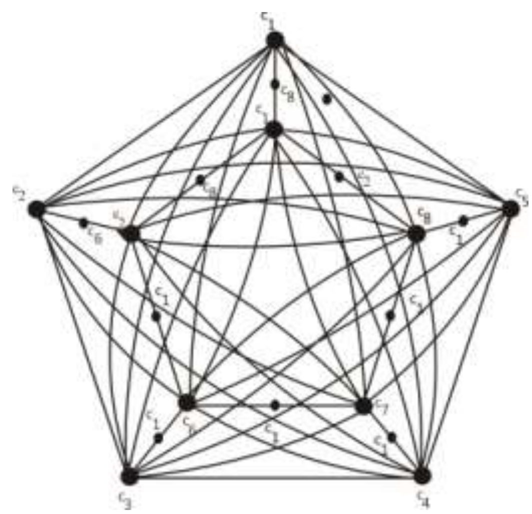

Figure 2

$a[C(S_5)]=8$

4. ACYCLIC COLOURING OF S_n^o

4.1 Theorem

The acyclic chromatic number for the Crown graph is $a[(S_n^o)]=n$ for every $n=3,4,5...$

Proof

Cosider a crown graph $G=S_n^o$ having $2n$ vertices.Let the vertices be $\{u_1,u_2,u_3....u_n,v_1,v_2,v_3,...v_n\}$ .Consider the colour class $C=\{c_1,c_2,c_3...c_n\}$. Assign the colour c_i to the vertices u_i and v_j for $i=1,2,3...n$. This colouring is clearly acyclic.because the subgraph induced by $<c_i,c_j>$ for all $i\neq j$ is the union of two paths P_1 always.thus the colouring is acyclic. Also this colouring is minimum. Because if we replace any colour class by another used colour class the colouring will become improper.

Thus $a[(S_n^o)] = n$ for every $n = 3,4,5...$

Example

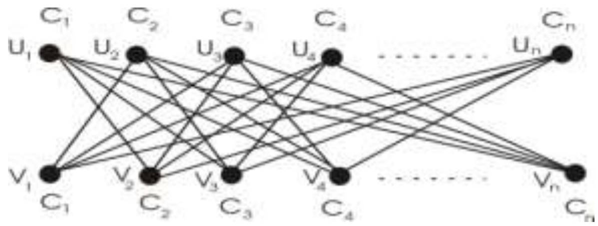

Figure 3

$$a[(S_n^o)] = n .$$

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