Every Complete Binary Tree is Prime

Abstract

A graph with a vertex set $V$ is said to have a prime labeling if its vertices can be labeled with distinct integers $1, 2, \ldots, |V|$ such that for every edge $xy$, the labels assigned to $x$ and $y$ are relatively prime. A tree is prime if it has at least one prime labeling. Around 1980, Entringer conjectured that every tree is prime. After three decades, this conjecture remains open. Nevertheless, a few special classes of trees, specifically paths, stars, caterpillars, spiders, and small trees, have been shown to be prime. Among different types of trees, binary trees are probably the most frequently used in computer science. Fu and Huang showed that every perfect binary tree of order $2d - 1$ is prime. Although Fu and Huang ambiguously called perfect binary trees as complete binary trees in their paper, it has been verified that they only proved that perfect binary trees are prime. In this paper, the author looked beyond perfect binary trees and devised a two-step method to prove that every complete binary tree is prime. First, for the case of $2^k - 1$ vertices, a near prime labeling was constructed such that the co-prime requirement was satisfied for every edge, except possibly for the edges between right leaves and their parents. In order to successfully construct a prime labeling without co-prime violations, the original prime labeling problem was transformed into a complete (co-prime) matching problem between the right leaves and their parents. By applying Hall’s Theorem, we proved that a complete (co-prime) matching exists for the right leaves and their parents, thus proving that a prime labeling exists for every complete binary tree with $2k - 1$ vertices. Second, for the case of $2k$ vertices, we applied Bertrand-Chebyshev Theorem and
proved that a three-way child swap could be performed to construct a prime labeling for 2k vertices based on the prime labeling for 2k − 1 vertices, thus completing the proof that every complete binary tree of any number of vertices is prime. Our proof that all complete binary trees are prime broadens the coverage of the tree classes that are known to be prime and propels the research one step closer to prove Entringer’s conjecture.

References


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Keywords

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Every Complete Binary Tree is Prime