Analysis of M^X/M/1/MWV/BD Queuing Systems

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ABSTRACT
In this paper, the batch arrival M^X/M/1 queuing system along with server breakdowns and multiple working vacations is analyzed under exponential distribution. For this model Stochastic Decomposition is obtained and particular cases are evaluated. Further numerical illustration is also given to justify the validity of the model.

Keywords
Batch Arrival, Multiple Working Vacations, Breakdowns, Probability Generating Function(PGF), Stochastic Decomposition.

1. INTRODUCTION
A classical queuing system may be described as one having a service facility at which units of some customer arrive for service and whenever there are more units in the system than the service facility can handle simultaneously, a queue or waiting line is developed. The waiting units take their turn for service according to as pre assigned rule and after service they leave the system. The study of classical queuing models are made by Saaty (1961), Gross & Harris (1985) and Medhi (2006).

The batch arrival is described as the flow of arrivals in batches. Gaver (1959) introduced bulk arrival queues, where the arrivals could be in batch. Choudhury and Templeton (1983) and Medhi (1984) discuss the subject at great length.

In N-policy the server does not start his service until there are N-customers in the queue. This policy is introduced by Yadin and Naor (1963) and is designed to minimize server switch over's and to avoid excessive frequent use of setups. Lee and Srinivasan (1989), Lee et al., (1994 and 1995) studied the behavioral characteristics of batch arrival queues with N-policy and server vacations. Lee et al., (1994) successively combined the batch arrival queues with N-policy.

Queuing systems with server classical vacations are characterized by the fact that the idle time of the server may be used for other secondary jobs. Allowing server to take vacation make queuing models more realistic and flexible in studying real world queuing situations. Applications arise naturally in call centers with multi task employees, maintenance activity, production and quality control problems etc.,.

In N-policy queuing models, with server vacation, as soon as the system empties, the server leaves the system for a vacation of random length. When the server returns from the vacation and finds N or more customers, he immediately starts his service. Otherwise he takes repeated number of vacations until he finds N or more customers. This policy is called a Multiple Vacations Policy.

Most of the classical queuing systems the server may fail and can be repaired. The performance of the system may be affected heavily by these breakdowns and limited repair capacity. Queuing systems with such unreliable stations are the topics of worth investigating from the performance prediction point of view. As a result of breakdowns, service facility becomes inoperative and the units demanding service can be served only when it is restored to operative state. Wang (1995) first proposed Markovian queuing system with removable service station. Ke J.C (2003) considered the control policy for batch arrival M^X/M/1 queuing system under N-policy in which the server is characterized by breakdowns and multiple vacations.

In working vacation queues, the server works at a lower service rate rather than completely stopping service during the vacation period. At the vacation termination epochs, if there are customers in the system, the server will start a new regular busy period. Otherwise, he takes another working vacation which follows multiple working vacations policy.

In 2002, Servi and Finn, introduced a class of semi vacation policies, in which servers work at a lower rate rather than completely stopping primary service during vacation. Such a vacation is called working vacation (WV). Tian et al., (2008), Li and Tian (2007), Xu et al., (2009) considered M/M/1 queue with different working vacation policies. Xu et al., (2009) studied the results of Liu et al., (2007) to bulk input model M^X/M/1/MWV. They have formulated the model as two dimensional Markovian chain and obtained the PGF of the stationary queue length and its stochastic decomposition result using the matrix analysis method. Their concept is motivated to combine the batch arrival queues under server breakdowns and multiple working vacations.

In this paper, with the help of available literature a batch arrival M^X/M/1 queuing system along with server breakdowns and multiple working vacations is analyzed under exponential distribution. The probability generating function (PGF) of the system size is obtained through the Chapman-Kolmogorov balanced equations satisfied by the steady state system size probabilities. The PGF is presented in closed form so that various performance measures are calculated easily.

With the aid of PGF stochastic decomposition is obtained. Further particular cases are evaluated and sensitivity analysis is discussed.

2. MODEL DISCRIPTION
Consider a batch arrival M^X/M/1 queue in which, the arrival stream forms a Poisson process and the actual number of customers in any arriving module is a random variable X, which may take on any positive integral value k($<\infty$) with probability g_k. If λ_k is the arrival rate of a Poisson process of batches of size k then $g_k=\lambda_k/\lambda$, k=1,2,3,… where λ is the composite arrival rate of all batches equal to $\sum_{i=1}^{\infty}\lambda_i$. This total process, which arises from the overlap of the set of Poisson processes with rates {λ_k, k=1,2,…} is a compound Poisson process. Let X(z),E(X) and E(X^2) denote the PGF , first and second moments of random variable X.

Then $X(z)=\sum_{k=1}^{\infty} g_k z^k$; $E(X^k) = \sum_{n=1}^{\infty} g_n n^k$

The server serves the customers at a time with exponential service rate μ in a regular busy period. Whenever the system becomes empty at service completion instant, the server starts a working vacation during which the service is done at a lower rate. The vacation duration V follows an exponential distribution with parameter η. During working vacations, arriving customers are served with exponential service rate $\mu_v(\leq\mu)$.

When a vacation terminates and the server finds the system is empty, then the server begins another working vacation. On the other hand, if the server finds the system is not empty at the vacation termination instant, then he switches to a regular service period. The distributions of the service times during regular busy period and working vacation period are both exponential but with different rates μ and μ_v respectively. It is assumed that, inter arrival times, service time and working vacation times are mutually independent of each other.

The server is subject to breakdowns at any time while working, with Poisson rate α. Whenever the system fails, the server is sent immediately for repair at a repair facility where the repair time is an independent and identically distributed random variable Br following exponential distribution $(1-e^{-\beta t})$. The customer, who is just the being served when the server breaks down, joins the head of the waiting line and resumes the service as soon as the server returns from the repair facility. This type of service continues until the system becomes empty again.

This model is denoted by **M^x/M/1/MWV/BD.**

3. SYSTEM SIZE DISTRIBUTION

Let $N_s(t)$ denote the number of customers in the system at time t and

$$J(t) = \begin{cases} 0 & \text{the system is in a working vacation period at time } t. \\ 1 & \text{the system is in regular busy period at time } t. \\ 2 & \text{the system is in break down period at time } t. \end{cases}$$

Then $\{N_s(t),J(t)\}$ is a Markov process.

Let $Q_n(t) = Pr\{N_s(t) = n ; J(t) = 0\}$, $n \geq 0$,

$P_n(t) = Pr\{N_s(t) = n ; J(t) = 1\}$, $n \geq 1$,

and $B_n(t) = Pr\{N_s(t) = n ; J(t) = 2\}$, $n \geq 1$.

denote the system size probability at time t.

Assuming the steady state system size probabilities as

$P_n = \underset{t\to\infty}{lt}\; P_n(t)$; $Q_n = \underset{t\to\infty}{lt}\; Q_n(t)$ and $B_n = \underset{t\to\infty}{lt}\; B_n(t)$ exists.

The steady state equations satisfied by P_n's, Q_n's and B_n's are given by

$$\lambda Q_0 = \mu_V Q_1 + \mu P_1 \tag{1}$$

$$(\lambda+\eta+\mu_V) Q_n = \lambda\sum_{k=1}^{n} Q_{n-k} g_k + \mu_V Q_{n+1} \quad ;n \geq 1 \tag{2}$$

$$(\lambda+\mu+\alpha) P_1 = \mu P_2 + \eta Q_1 + \beta B_1 \tag{3}$$

$$(\lambda+\mu+\alpha) P_n = \beta B_n + \lambda\sum_{k=1}^{n-1} P_{n-k} g_k + \mu P_{n+1} + \eta Q_n \quad ;n \geq 2 \tag{4}$$

$$(\lambda+\beta) B_1 = \alpha P_1 \tag{5}$$

$$(\lambda+\beta) B_n = \alpha P_n + \lambda\sum_{k=1}^{n-1} B_{n-k} g_k \quad ; n \geq 2 \tag{6}$$

To obtain the steady state distribution of the model, the partial PGFs, are defined as

$Q(z) = \sum_{n=0}^{\infty} Q_n z^n$; $P(z) = \sum_{n=1}^{\infty} P_n z^n$ and $B(z) = \sum_{n=1}^{\infty} B_n z^n$.

By multiplying equation (2) by z^n, summing over $n \geq 1$ and then adding with equation (1) we get

$$Q(z) = \frac{\mu_v (z - z_1) Q_0}{z_1(\lambda z (1 - X(z)) + \mu_v (z - 1) + \eta z)} \tag{7}$$

Similarly proceeding for equations (3) and (4) imply,

$$P(z)= \frac{\lambda z \mu_v Q_0}{z_1}\left(\frac{(z-1)z_1(X(z_1)-1)-(z_1-1)z(X(z)-1)}{\lambda z(1-X(z))+\mu_v(z-1)+\eta z}\right)\left(\frac{\lambda(1-X(z))+\beta}{[\lambda z(1-X(z))+\mu(z-1)+\alpha z][\lambda(1-X(z))]+\beta[\lambda z(1-X(z))+\mu(z-1)]}\right) \tag{8}$$

Similarly equations (5) and (6) imply,

$$B(z) = \frac{\alpha P(z)}{\lambda(1 - X(z)) + \beta} \tag{9}$$

Thus the total PGF $P_{MWV}^{Br}(z)$ is obtained by adding Q(z), P(z) and B(z)

(i.e)., $P_{MWV}^{Br}(z) =$ Q(z) + P(z) + B(z)

$$P_{MWV}^{Br}(z) = \frac{\mu_v(z-1)Q_0}{z_1(\lambda z(1-X(z))+\mu(z-1))}\left(\frac{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}{\lambda z(1-X(z))+\mu_v(z-1)+\eta z}\right)\left(\frac{\lambda(1-X(z))+\beta+\dfrac{\alpha\lambda z z_1(X(z_1)-X(z))}{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}}{\lambda(1-X(z))+\beta+\dfrac{\alpha\lambda z(1-X(z))}{\mu(z-1)+\lambda z(1-X(z))}}\right) \tag{10}$$

Now by using the normalizing condition $P_{MWV}^{Br}(1) = 1$, which yields

$$1=\left(\frac{\mu_v Q_0}{z_1(-\alpha\lambda E(x)+\beta\mu(1-\rho))}\right)\left\{\frac{[\mu\beta(1-z_1)+\lambda z_1(X(z_1)-1)(\alpha+\beta)]}{\eta}\right\}$$

Now, with the help of the normalizing condition, Q_0 can be evaluated

$$\text{i.e., } Q_0= \frac{\eta z_1[\beta\mu(1-\rho)-\alpha\lambda E(X)]}{\mu_v[(1-z_1)\beta\mu+\lambda z_1(X(z_1)-1)(\alpha+\beta)]} \tag{11}$$

where $\rho = \frac{\lambda}{\mu}$ E(X)

By substituting for Q_0 in equation (11), the PGF is given by

$$P_{MWV}^{Br}(z) = \frac{\mu(1-\rho)(z-1)}{(\lambda z(1-X(z))+\mu(z-1))}\,\frac{\left(\dfrac{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}{\lambda z(1-X(z))+\mu_v(z-1)+\eta z}\right)}{\left(\dfrac{\lambda z_1(X(z_1)-1)+\mu(1-z_1)}{\eta}\right)}$$

$$\frac{\left(1-\frac{\alpha\rho}{\beta(1-\rho)}\right)}{\left(1+\frac{\alpha\lambda z_1(X(z_1)-1)}{\beta[\lambda z_1(X(z_1)-1)+\mu(1-z_1)]}\right)}\left(\frac{\lambda(1-X(z))+\beta+\frac{\alpha\lambda z z_1(X(z_1)-X(z))}{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}}{\lambda(1-X(z))+\beta+\frac{\alpha\lambda z(1-X(z))}{\mu(z-1)+\lambda z(1-X(z))}}\right) \quad (12)$$

Therefore Total PGF can also be written as,

$$P_{MWV}^{Br}(z)= P_{M^X/M/1/MWV}(z)\frac{\left(1-\frac{\alpha\rho}{\beta(1-\rho)}\right)}{\left(1+\frac{\alpha\lambda z_1(X(z_1)-1)}{\beta[\lambda z_1(X(z_1)-1)+\mu(1-z_1)]}\right)}\left(\frac{\lambda(1-X(z))+\beta+\frac{\alpha\lambda z z_1(X(z_1)-X(z))}{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}}{\lambda(1-X(z))+\beta+\frac{\alpha\lambda z(1-X(z))}{\mu(z-1)+\lambda z(1-X(z))}}\right) \quad (13)$$

Thus the total PGF of $P_{MWV}^{Br}(z)$ is obtained.

4. DECOMPOSITION PROPERTY

Equation (13) implies that the total PGF of the system size probabilities of the system is the product of the PGF of two random variables one of which is

$$P_{M^X/M/1/MWV}(z)=\frac{\mu(1-\rho)(z-1)}{(\lambda z(1-X(z))+\mu(z-1))}\frac{\left(\frac{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}{\lambda z(1-X(z))+\mu_v(z-1)+\eta z}\right)}{\left(\frac{\lambda z_1(X(z_1)-1)+\mu(1-z_1)}{\eta}\right)}$$

This gives the PGF of the system size for the batch arrival $M^X/M/1/MWV$ queuing model with multiple working vacations of Julia Rose Mary, k. and Afthab Begum, M.I. (2010) and the other one is,

$$\frac{\left(1-\frac{\alpha\rho}{\beta(1-\rho)}\right)}{\left(1+\frac{\alpha\lambda z_1(X(z_1)-1)}{\beta[\lambda z_1(X(z_1)-1)+\mu(1-z_1)]}\right)}\left(\frac{\lambda(1-X(z))+\beta+\frac{\alpha\lambda z z_1(X(z_1)-X(z))}{\mu(z-z_1)+\lambda z z_1(X(z_1)-X(z))}}{\lambda(1-X(z))+\beta+\frac{\alpha\lambda z(1-X(z))}{\mu(z-1)+\lambda z(1-X(z))}}\right)$$

which gives the PGF of the conditional system size distribution during the break down period (break down + repair).

Thus the PGF of the $M^X/M/1/MWV$ with Breakdowns Queuing Model is decomposed into the product of two random variables one is the PGF of classical $M^X/M/1$ Multiple Working Vacations Queuing Model and the other is PGF of the additional Breakdowns in the Queue. This justifies the **Decomposition Property.**

5. THE EXPECTED SYSTEM SIZE OF THE MODEL

i.e., $L = \frac{d}{dz}(P_{MWV}^{Br}(z))_{z=1} = \frac{\left(1-\frac{\alpha\rho}{\beta(1-\rho)}\right)}{\left(1+\frac{\alpha\lambda z_1(X(z_1)-1)}{\beta[\lambda z_1(X(z_1)-1)+\mu(1-z_1)]}\right)}$

$$\left\{\frac{\rho\mu}{\beta C^2}\left[C^2+\alpha\left(z_1-\frac{D}{\lambda E(X)}\right)(D-C)+\alpha D\left(\frac{C}{\beta}-\frac{1}{\rho}\right)-1\right]+\left(1+\frac{\alpha D}{\beta C}\right)\left[\frac{\lambda(E(X)+E(X^2))}{2\mu(1-\rho)}+\frac{\lambda E(X)-\mu_V}{\eta}+\frac{z_1(\mu-\lambda E(X))}{C}\right]\right\} \quad (14)$$

where $C = \mu(1-z_1)+\lambda z_1(X(z_1)-1)$ and $D = \lambda z_1(X(z_1)-1)$

Thus the expected system size of the model is evaluated.

6. PARTICULAR CASES

Case 1: $M^x/M/1$ Multiple Working Vacation Model

The results of $M^X/M/1$ Multiple Working Vacation Model can be obtained by applying $\alpha,\beta=0$ in the corresponding equations of (7) and (8), then we get

$$Q(z)=\frac{\mu_v(z-z_1)Q_0}{z_1(\lambda z(1-X(z))+\mu_v(z-1)+\eta z)}$$

$$P(z)=\frac{\lambda z\mu_v Q_0}{z_1}\left(\frac{(z-1)z_1(X(z_1)-1)-(z_1-1)z(X(z)-1)}{[\lambda z(1-X(z))+\mu_v(z-1)+\eta z][\lambda z(1-X(z))+\mu(z-1)]}\right)$$

where, $Q_0 = \frac{\eta z_1\mu(1-\rho)}{\mu_v[\lambda z_1(X(z_1)-1)+\mu(1-z_1)]}$ (15)

also reduced.

Also by letting $\alpha=\beta=0$, the expected number of customers in the system is reduced to

$$L=\frac{\lambda(E(X)+E(X^2))}{2\mu(1-\rho)}+\frac{\lambda E(X)-\mu_V}{\eta}+\frac{z_1(\mu-\lambda E(X))}{\mu(1-z_1)+\lambda z_1(X(z_1)-1)}$$
(16)

Thus from the equations (15) and (16) the results obtained coincide with the results of (Julia Rose Mary and Afthab Begum (2011)).

Case 2: M/M/1 Multiple Working Vacation Model

The results of M/M/1 Multiple Working Vacation can be obtained by letting X(z)=z and $\alpha,\beta=0$ in the corresponding equations.

Let $z_1 < 1$ and $z_2 > 1$ be the two roots of the equation,

$$\lambda z(1-z)+\mu_v(1-z)+\eta z=0$$

Then, $\lambda z(1-z)+\mu_v(z-1)+\eta z=-\lambda(z-z_1)(z-z_2)$

and the relation $z_2=\frac{\mu_v}{\lambda z_1}=\frac{1}{r_v}$ (17)

implies $\lambda z(1-z)+\mu_v(z-1)+\eta z=-\lambda(z-z_1)(z-\frac{1}{r_v})$ (18)

then $Q(z) = \frac{Q_0}{(1-r_v z)}$ from equation (7)

$$=\sum_{n=0}^{\infty}(r_v)^n z^n Q_0$$

and $P(z) = \frac{\rho z(1-z_1)Q_0}{(1-\rho z)(1-r_v z)}$ from equation (8)

$$=\frac{\eta r_v Q_0}{\mu(\rho-r_v)(1-r_v)}\left(\frac{1}{1-\rho z}-\frac{1}{1-r_v z}\right)$$

$$= \frac{\eta r_v}{\mu(\rho - r_v)(1-r_v)} \sum_{n=0}^{\infty} (\rho^n - r_v^n) z^n Q_0$$

and here $Q_0 = \left((1-\rho) + \frac{\eta r_v}{\mu(1-r_v)} \right)^{-1} (1-\rho)(1-r_v)$ (19)

also $L = \frac{\rho}{(1-\rho)} + \left(1 - \frac{r_v \mu_v}{\mu}\right)^{-1} \left(1 - \frac{\mu_v}{\mu}\right) \left(\frac{r_v}{1-r_v}\right)$ (20)

Hence from the equations (19) and (20) the results we obtained coincide with the results of (Liu *et al*., 2007).

7. SENSITIVITY ANALYSIS

Consider the expected system size (L) of the model M^X/M/1/MWV/BD queuing system.

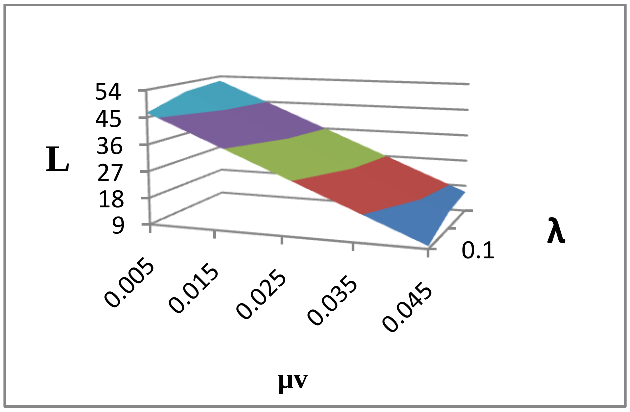
The values of the expected system size for the model are presented for different arrival rates λ and the service rates in vacation μ_v.

By considering the parameters as (μ,η,p,α,β)=(0.2,0.001,0.75,0.04,0.2) the expected system size of the model is calculated. The calculated values are tabulated and are represented in the following graph.

Table : Expected system size with respect to λ and μ_v.

λ \ μ_v	0.005	0.015	0.025	0.035	0.045
0.1	46.4917	37.3487	28.2057	19.0627	9.9197
0.2	50.9075	41.7644	32.6214	23.4784	14.3354
0.3	52.3527	43.2097	34.0667	24.9237	15.7807

Graph : Expected system size of the model M^X/M/1/MWV/BD queuing system.



It is noted from the table values and graph that the expected system size of the model (L) increases as λ increases and decreases as μ_v increases.

8. CONCLUSION

Thus in this paper the total PGF of the M^X/M/1 queuing system under server breakdowns and multiple working vacations is derived. With the help of PGF the stochastic decomposition property is verified, and moreover the particular cases are obtained. Further the validity of this M^x/M/1/MWV/BD model is justified with the help of sensitivity analysis. The Future scope of the idea of this model can be the application of fuzzy concept and finding out the crisp outputs.

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