On the Termination Problem for String Rewrite Systems

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ABSTRACT
Based on some result given in[**2**], concerning the termination of a **semi-Thue systeme**, we give some results illustrated by some examples to give some Noetherian **semi-Thue systems**.

Keywords
Free monoid, morphism of monoids, closure of a binary relation, rewriting systems of words, well-founded (Noetherian), weight function.

1. INTRODUCTION
The central idea of rewriting is to impose directionality on the use of equations in proofs. A **semi-Thue system** is a pair (A, R) where A is an alphabet and R is a non-empty finite binary relation on A^*. We write $usv \to_R utv$ whenever u, v $\in A^*$ and $(s, t) \in R$. We write $u \underset{R}{\to}^* v$ if there words $u_0, u_1, \dots, u_n \in A^*$ such that:

$$u_0 = u,\, u_i \to_R u_{i+1}, \forall 0 \le i \le n-1 \text{ and } u_n = v.$$

If $n = o$, then $u = v$, and if $n = 1$, we have $u \to_R v$.

$\underset{R}{\to}^*$ is the reflexive transitive closure of $\to_R$.

The **semi-Thue system** (A, R) is terminating if there does not exist an infinite chain $u_1 \to_R u_2 \to_R u_3 \to_R \dots$ in A^*.

Define the accessibility problem as follow: given a **semi-Thue system** (A, R) and two words u and v over the alphabet of A, decide whether $u \underset{R}{\to}^* v$.

The remainder of this paper is organized as follows. The Section 2, is devoted to the preliminaries notions. In Section 3, we give some results concerning the termination problem for a rewriting system. Finally, we draw our conclusions in Section 4.

2. PRELIMINARIES
Let A be a set, which we call an alphabet. A word w on the alphabet A is a finite sequence of elements of A

$u = (a_1, a_2, \dots, a_n) \quad a_i \in A.$

The set of all words on the alphabet A is denoted by $A^*$and is equipped with the associative operation defined by the concatenation of two sequences

$$(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_m) = (a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m)$$

This operation is associative. This allows us to write

$u = a_1 a_2 \dots a_n$. The string consisting of zero letters is called the empty word, written ε. Thus, ε, 0, 1, 011, 1111 are words over the alphabet $\{0, 1\}$. The set A^* of words is equipped with the structure of a monoid. The monoid A^* is called the free monoid on A. The length of a word u, denoted $|u|$, is the number of letters in u where each letter is counted as many times as it occurs. Again by definition, |ε|=0. The length function possesses some of the formal properties of logarithm: $|uv| = |u| + |v|, |u^i| = i|u||$, for any words u and v and integers $i \ge 0$. For example $|011| = 3$ and $|1111| = 4$. For a subset B of A, we let $|u|_B$ denote the number of letters of u which are in B . Thus $|\, u| = \sum_{a \in A} |u|_a$. A language L over A^* is any subset of $A^*, i.e, L \subseteq A^*$.

A mapping $h: A^* \longrightarrow \Delta^*$, where A and Δ are alphabets, satisfying the condition

$h(uv) = h(u)h(v)$, for all words u and v of A^*

is called a morphism. It is noted here, to define an alphabetic morphism$h$, it suffices to list all the words $h(a)$, where $a \in A$. If M is a monoid, then any mapping $f: A \to M$ can be extends to a unique morphism

$\hat{f}: A^* \longrightarrow M$ For instance, if M is the additive monoid ℕ, and f is defined by $f(a) = 1$ for each $a \in A$, then $\hat{f}(u)$ is the length $|u|$ of the word u.

Let $h: A^* \longrightarrow \Delta^*$ be a morphism of monoids. if h is one-to-one and onto, then h is an isomorphism and the monoids A^* and Δ^* are isomorphic. we denote by Hom(A^*, Δ^*) the set of morphisms from A^* to Δ^* and Isom(A^*, Δ^*) the set of all isomorphism's from A^* to Δ^*.

A binary relation on X is a subset $R \subseteq X \times X$. If $(x, y) \in R$, then we denote xRy and we say that x is related to y by R. The inverse relation of R is the binary relation

$R^{-1} \subseteq X \times X$. Defined by: $yR^{-1}x \Leftrightarrow (x, y) \in R$.

The relation $I_X = \{(x, x), x \in X\}$ is called the identity relation. The relation $(X)^2$ is called the complete relation.

Let $R \subseteq X \times X$ and $S \subseteq X \times X$ two a binary relations, the composition of R and S is a binary relation $S \circ R \subseteq X \times X$ defined by $xS \circ Rz \Leftrightarrow \exists y \in X$ such that xRy and ySz.

A binary relation R on a set X is said to be

1. Reflexive if xRx for all x in X;
2. Symmetric if xRy implies yRx;
3. Transitive if xRy and yRz imply xRz.
4. Ant symmetry if xRy and yRx imply $x = y$.

A binary relation $\ge$ on a set X is **partial order** (or partial ordering) iff it is reflexive, transitive and ant symmetric.

A **strict partial ordering** $>$ on a set X is irreflexive, ant symmetric and transitive relation on X.

A **strict partial ordering** $>$ is called **well-founded (Noetherian)**, if there is no infinite descending chain $x_0 > x_1 > x_2 \dots$

Let $(X, >)$ be a **well-founded ordering**, let P be property of elements of X, if for all $x \in X$ the implication:

If $P(x')$, for all $x' \in X$ such that $x > x'$ then $P(x)$.

Let R be a relation on a set X. The reflexive closure of R is the smallest reflexive relation R^0 on X that contains R; that is,

1. $R \subseteq R^0$
2. if R' is a reflexive relation on X and $R \subseteq R'$, then $R^0 \subseteq R'$.

The symmetric closure of R is the smallest symmetric relation R^+ on X that contains R; that is,

3. $R \subseteq R^+$

4. *if R' is a symmetric relation on X and $R \subseteq R'$, then $R^+ \subseteq R'$.*

The transitive closure of R is the smallest transitive relation R^* on X that contains R; that is,

1. $R \subseteq R^*$,
2. if R' is a transitive relation on A^* and $R \subseteq R'$, then $R^* \subseteq R'$.

Let R be a relation on a set X. Then

1. $R^0 = R \cup I_X$.
2. $R^+ = R \cup R^{-1}$.
3. $R^* = \bigcup_{k=1}^{k=+\infty} R^k$.

A **semi-Thue system** R over A, for briefly STS, is a finite set $R \subseteq A^* \times A^*$, whose elements are called rules. A rule (s,t) will also be written as $s \to t$. The set $dom(R)$ of all left-hand sides and $ran(R)$ of all right-hand sides are defined by:

$dom(R) = \{s \in A^*, \exists t \in A^* : (s,t) \in R\}$ and

$ran(R) = \{t \in A^*, \exists s \in A^* : (s,t) \in R\}$.

If R is finite, then the size of R is defined to be $\sum_{(s,t)\in R}(|s| + |t|)$ and is denoted by $\|R\|$.

We define the binary relation $\to_R$ as follows, where u, v $\in A^*$: $u \to_R v$ if there exist x, y $\in A^*$ and $(r,s) \in R$ with $u = xry$ *and* $v = xsy$. We write $u \underset{R}{\to}^* v$ if there words $u_0, u_1, \dots, u_n \in A^*$ such that:

$u_0 = u, u_i \to_R u_{i+1}, \forall 0 \le i \le n-1$ and $u_n = v$.

If $n = o$, then $u = v$, and if $n = 1$, we have $u \to_R v$.

Note that $\underset{R}{\to}^*$ is the reflexive transitive closure of $\to_R$.

The set of irreducible words with respect to R is

$IRR(R) = A^* - \{xsy : x, y \in A^*, s \in dom(R)\}$.

We say that R is Noetherian if there does not exist an infinite sequence of words $u_i \in A^* (i \in \mathbb{N})$ such that $u_0 \to_R u_1 \to_R \dots$

Let $>$ be a binary relation on A^*. The relation $>$ is **admissible,** if for all $u, v, x, y \in A^*$,

$u > v$ *implies* $xuy > xvy$.

Let $A = \{a_1, a_2, \dots, a_n\}$, with $a_n > a_{n-1} > \dots > a_1$, in the following cases, we give some of **admissible partial orderings** on A^*.

1. Define $x > y$ as follows: $x > y$ if $|x| > |y|$ is the **length ordering** on A^*.
2. Let $w: A \longrightarrow \mathbb{N}$ be a mapping that associates a positive integer (a **weight**) with each letter. Define the **weight ordering** $>_w$ induced by w as follows: $x >_w y$ if $w(x) > w(y)$.

Here w is extended to a mapping from A^* into $\mathbb{N}$ by taking $w(\varepsilon) = 0$ and $w(xa) = w(x) + w(a)$ for all $x \in A^*, a \in A$.

3. The **lexicographical ordering** $>_{lex}$ on A^* is defined as follows: $x >_{lex} y$ if there is a non-empty string z such that

$x = yz, or\ x = ua_i v\ and\ y = ua_j z,$

For some $x, v, z \in A^*$ and $i, j \in \{1, \dots, n\}$ satisfying $i > j$.

4. The **length-lexicographical ordering** $>_{ll}$ is a combination of the **length ordering** and **lexicographical ordering** : $\boldsymbol{x >_{ll} y}$ if $|x| > |y|$ or $|x| = |y|$ and $x >_{lex} y$.

3. STUDY OF CASES WHERE THE WORD REWRITING SYSTEM IS NOETHERIAN

In this section, we use the following theorem from[2], to giving some results concerning a Noetherian **semi-Thue system**.

Theorem 3.1 [2]

Let (A,R) be a **semi-Thue system**. Then the following two statements are equivalent:

1. The reduction relation $\to_R$ is Noetherian.
2. There exists an **admissible well-founded partial ordering** $>$ on A^* such that $x > y$ holds for each $(s,t) \in R$.

Corollary 3.2

Let (A,R) be a **semi-Thue system**, with

$R = \{(a_i, b_i), 0 \le i \le n, n \in \mathbb{N}\}$, if $\forall 0 \le i \le n$, $|a_i| > |b_i|$, then (A,R) is Noetherian.

Proof

To obtain the desired result, we show that there exists an **admissible well-founded partial ordering** $>$ on A^* such that $x > y$ holds for each $(s,t) \in R$. It suffices to take the **length ordering** $>$ defined by $x > y$ if $|x| > |y|$.

Example 3.3

Consider the **semi-Thue system** (A,R) with $A = \{a,b\}$ and $R = \{(aa,b)\}$. We have $|aa|$=2, $|b| = 1$, then $|aa| > |b|$, consequently (A,R) is Noetherian.

Corollary 3.4

Let (A,R) be a **semi-Thue system**, with

$A = \{a_0, a_1, \dots, a_n\}$ and

$R = \{(a_j, b_j), 0 \le j \le m, m \in \mathbb{N}\}$. Consider the mapping $w: A \longrightarrow \mathbb{N}, a_i \longmapsto w(a_i)$ *and*

$w: A^* \longrightarrow \mathbb{N}, w(x) = \sum_{i=0}^{i=n} w(a_i)|x|_{a_i}$.

If for all $j \in \{0, \dots, m\}$: $w(a_i) > w(b_i)$ then

(A,R) is Noetherian.

Proof

To obtain the desired result, we show that there exists an **admissible well-founded partial ordering** $>$ on A^* such that $x > y$ holds for each $(s,t) \in R$. It suffices to take the **weight ordering** $>_w$ induced by w as follows:

$x >_w y$ if $\sum_{i=0}^{i=n} w(a_i)|x|_{a_i} > \sum_{i=0}^{i=n} w(a_i)|y|_{a_i}$.

Example 3.5

Let $A = \{a,b,c\}$, $R = \{(bb,aa),(cb,ab)\}$ and

$w: \{a,b,c\} \longrightarrow \mathbb{N}$, with $w(a) = 1, w(b) = 2, w(c) = 3$.

We check that $w(bb) > w(aa)$ *and* $w(cb) > w(ab)$.

We have $w(bb) = w(a)|bb|_a + w(b)|bb|_b + w(c)|bb|_c$

$= 1 \times 0 + 2 \times 2 + 3 \times 0 = 4$.

A similar argument we have

$$w(aa) = w(a)|aa|_a + w(b)|aa|_b + w(c)|aa|_c$$
$$= 1 \times 2 + 2 \times 0 + 3 \times 0 = 2.$$

On the other hand we have,

$$w(cb) = w(a)|cb|_a + w(b)|cb|_b + w(c)|cb|_c$$
$$= 1 \times 0 + 2 \times 1 + 3 \times 1 = 5.$$

And $w(ab) = w(a)|ab|_a + w(b)|ab|_b + w(c)|ab|_c$

$$= 1 \times 1 + 2 \times 1 + 3 \times 0 = 3.$$

Finally (A,R) is Noetherian.

Corollary 3.6

Let $A = \{a_1, a_2, \dots, a_n\}$, with $a_n > a_{n-1} > \cdots > a_1$. Let (A,R) be a **semi-Thue system**.

If for all $(s,t) \in R, s >_{lex} t$, then (A,R) is Noetherian.

Proof

To obtain the desired result, we show that there exists an **admissible well-founded partial ordering** $>$ on A^* such that $x > y$ holds for each $(s,t) \in R$. It suffices to take the **lexicographical ordering** $>_{lex}$.

Example 3.7

Let $A = \{a,b,c\}$, with $c > b > a$.
R= $\{(ba,ab),(cb,bc)\}$. We have $ba >_{lex} ab$ and $cb >_{lex} bc$. Then (A,R) is Noetherian.

Corollary 3.8

Let (A,R) be a **semi-Thue system**. Consider the morphism of monoids $f: (A^*,\cdot) \longrightarrow (\mathbb{N},+)$.

If for all $(s,t) \in R, f(s) > f(t)$, then Then (A,R) is Noetherian.

Proof

To obtain the desired result, we show that there exists an **admissible well-founded partial ordering** $>$ on A^* such that $x > y$ holds for each $(s,t) \in R$. It suffices to take the **weight ordering** $>_f$ induced by f as follows:

$x >_f y$ if $f(x) > f(y)$.

Example 3.9

Let $A = \{a,b,c\}$, R= $\{(ba,bc),(ab,ac)\}$.

Consider the morphism of monoids $f: (A^*,\cdot) \longrightarrow (\mathbb{N},+)$, with $f(a) = 2, f(b) = 1, f(c) = 0$. We check that $f(ba) > f(bc)$ and $f(ab) > f(ac)$.

We have $f(ba) = 3, f(bc) = 1$ and $f(ab) = 3$,

$f(ac) = 2$. Finally (A,R) is Noetherian.

4. CONCLUSION

In this paper, we have given au an **admissible well-founded partial ordering** $>$ on the free monoid A^* with a finite alphabet$A$, in order to assure that the **semi-Thue system** is Noetherian.

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