# Trees in Disemigraphs

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## ABSTRACT
E. Sampathkumar [5] generalized the concept of graphs to semigraphs and that of digraphs to disemigraphs. Looking for an analogue of tree for the semigraph he introduced the concept of dendroid and tree, and also developed some of its significant characterizations. Analogous to his study of dendroids and trees in semigraphs, an attempt has been made to develop the concept of trees in disemigraph setting and derive some characterizations thereof. These results are primarily aimed at introducing the structural behaviour of the theory of disemigraphs.

## Keywords
Semigraph, disemigraph, dendroid, tree.

## 1. INTRODUCTION
The idea of tree was originated in Kirchhoff's work on electric network. Kirchhoff [9] developed the theory of trees in 1847 in order to solve the system of simultaneous linear equations involving the amount of current through each branch and around each circuit of an electric network. The term "tree" was coined in 1857 by the British mathematician Aurthur Cayley [1], who gave this name to a new kind of structural entity arising from the enumeration of organic chemical isomers. Trees appear in numerous instances. Perhaps what makes trees so useful is that they may be viewed in a variety of equivalent forms. The very simplicity of trees makes it possible to investigate conjectures for graphs in general by studying the properties of trees.

Disemigraphs are more natural and useful than semigraphs for describing situations in which order or direction is involved in the relationship between or among sets of objects. Similarly dendroids and trees with directed edges are of great importance for applications in various fields such as sorting, game theory, computer science, network analysis, phylogeny and genealogy etc. to name a few only.

## 2. PRELIMINARIES
A **graph** $G(V,X)$ consists of a finite nonempty set $V$ of points together with a prescribed set $X$ of unordered pairs of distinct points of $V$.

A **tree** is a connected acyclic graph.

A **digraph** $D(V,A)$ consists of a set $V$ of vertices and a set $A$ of arcs (directed edges). A **directed tree** is an oriented digraph whose underlying graph is a tree. A digraph $D$ is called a **forest** if all its components are trees. A **rooted tree** or **arborescence** is a directed tree $T$ with a vertex $r$ (called the **root**) such that $T$ contains an $r$-$v$ path for every vertex $v$ of $T$. Thus an arborescence is a tree with precisely one vertex of *in-degree* zero.

A **semigraph** $G$ is a pair $(V,E)$ where $V$ is a non-empty set whose elements are called vertices of $G$, and $E$ is a set of $n$-tuples , called edges of $G$, of distinct vertices, for various $n\geq 2$ satisfying the following conditions-

S.G.1- Any two edges have at most one vertex in common.

S.G.2- Two edges $(u_1,u_2,\ldots,u_n)$ and $(v_1,v_2,\ldots,v_m)$ are considered to be equal if and only if

i. $m=n$ and

ii. either $u_i=v_i$ for $1\leq i\leq n$, or $u_i=v_{n-i+1}$ for $1\leq i\leq n$.

A **subedge** of an edge $e=(u_1,u_2,\ldots,u_n)$ is a k-tuple $e'=(u_{i_1},u_{i_2},\ldots u_{i_k})$, where $1\leq i_1<i_2<\ldots<i_k\leq n$ or $1\leq i_k<i_{k-1}<\ldots<i_1\leq n$. A **partial edge** of $e=(u_1,u_2,\ldots,u_n)$ is a $(j-i+1)$-tuple $(u_i,u_j)=(u_i,u_{i+1},\ldots,u_j)$ where $1\leq i<j\leq n$. All vertices on an edge of a semigraph are considered to be adjacent to one another.

The concepts of subedge and partial edge lead to two different types of path in case of semigraphs. A $v_o$-$v_n$ path is an ***s*-path** (strong path) if all its subedges, if any, are partial edges. Otherwise, it is a ***w*-path** (weak path). A closed $s$-path ($w$-path) is called a ***s*-cycle** (***w*-cycle**).

For a vertex $v$ in a semigraph $G(V,X)$ the various types of degrees are as follows:

**Degree**: $degv$ is the number of edges having $v$ as an end vertex.

**Edge Degree**: $deg_ev$ is the number of edges containing $v$.

**Adjacent Degree**: $deg_av$ is the number of vertices adjacent to $v$.

**Consecutive Adjacent Degree**: $deg_{ca}v$ is the number of vertices which are consecutively adjacent to $v$.

A **dendroid** is a connected semigraph without $s$-cycles, and a **forest** is a semigraph in which every component is a dendroid.

The underlying graph of a dendroid is a tree which is acyclic but in semigraph if $(u_1,u_2,\ldots,u_n)$ is an edge, $n\geq 3$ then it contains a $w$-cycle $(u_1u_2\ldots u_nu_1)$. A dendroid $T$ is a **tree** if and only if $T$ contains no $w$-cycle.

A vertex $v$ in a semigraph $G$ is a **pendant vertex** if $degv=deg_ev=1$. A **pendant edge** is one having a pendant vertex. A **pendant dendroid** is a dendroid in which every edge is a pendant edge.

A **star** is a dendroid in which all edges have a common vertex.

A directed semigraph or **disemigraph** $D$ is a finite set of objects called vertices together with a (possibly empty) set of ordered $n$-tuples of distinct vertices of $D$ for various $n\geq 2$, called directed edges or arcs, satisfying the following condition-

"For any two distinct vertices $u$ and $v$ in a disemigraph $D$, there is at most one arc containing $u$ and $v$ such that $u$ is adjacent to $v$ and at most one arc containing $u$ and $v$ such that $v$ is adjacent to $u$".

While drawing a disemigraph in a plane, the initial (or terminal) vertex of an arc which is not a middle vertex of any arc is represented by a thick dot. Middle vertices of arcs are represented by small circles. If a middle vertex is also an initial (or terminal) vertex of an arc then a small tangent is drawn to the circle.

A disemigraph $D_1$ is a **subdisemigraph** of a disemigraph $D$ if $V(D_1) \subseteq V(D)$ and $E(D_1) \subseteq E(D)$.

The cardinality of the vertex set of a disemigraph $D$ is called the **order** of $D$ and is denoted by $p(D)$ or simply by $p$. Similarly, the cardinality of the arc set of a disemigraph $D$ is called its **size,** denoted by $q(D)$ or simply by $q$. A disemigraph of order $p$ and size $q$ is denoted by $D(p,q)$.

Subarc, partial arc, $s$-path, $w$-path, $s$-cycle and $w$-cycle are defined for disemigraphs in a manner similar to semigraphs. As in case of digraph, another kind of path can be found in disemigraph, namely semipath. A **semipath** connecting two vertices $v_0$ and $v_n$ in a disemigraph $D$ is a finite sequence of distinct vertices of $D$ viz, $v_0v_1v_2...v_n$ such that for each $i$, $0 \leq i \leq n-1$, either $v_i$ is adjacent to $v_{i+1}$ or $v_{i+1}$ is adjacent to $v_i$. A closed semipath is known as a **semicycle**.

If $a=(u_1,u_2,\ldots,u_n)$ is an arc in a disemigraph $D$ then

$u_i$ is adjacent to all other $u_j$, $1 \leq i<j \leq n$ and $u_j$ is adjacent from $u_i$, $1 \leq i<j \leq n$. The **out-degree** (**in-degree**) of a vertex $u$ of a disemigraph $D$ is the number of vertices of $D$ that are adjacent from (adjacent to) $u$. The **degree** $deg(u)$ or $d(u)$ of $u$ is defined to be the sum of the *in-degree* and the *out-degree* of $u$. That is, $d(u)=od(u)+id(u)$.

A vertex $u$ in a disemigraph $D$ is called a **source** if $id(u)=0$ and any other vertex $v$ of $D$ is reachable from $u$. A **sink** is the dual concept of a source.

The **adjacency digraph** $D_a$ of a disemigraph $D$ has $V(D)$ as vertex set where for any two vertices $u$ and $v$, $u$ is adjacent to $v$ if and only if it is so in $D$. For any vertex $v$ in $D$, $id(v)$ (respectively $od(v)$) in $D$ is the same as $id(v)$ (respectively $od(v)$) in $D_a$. The **distance** between any two vertices in $D$ is the distance between them in the underlying adjacency digraph $D_a$ of $D$.

The **consecutive adjacency digraph** $D_{ca}$ of a disemigraph $D$ has $V(D)$ as its vertex set where for any two vertices $u$ and $v$, $u$ is adjacent to $v$ if and only if,

(i) $u$ is adjacent to $v$ and

(ii) $u$ and $v$ are consecutive vertices of an arc in $D$.

A disemigraph $D$ is **simple** if any two arcs in $D$ either contain at most one vertex or all vertices in common. A disemigraph $D$ is **oriented** if $D$ contains no symmetric pair of arcs.

In the present context oriented and simple disemigraphs are only considered.

For any further concepts and terminology the readers are referred to [5], [6] and [8].

## 3. TREES IN DISEMIGRAPHS

Since any edge with $n$ vertices ($n \geq 3$) in a semigraph forms a $w$-cycle by virtue of adjacency condition of vertices, a logical contradiction arises in extending the concept of tree in graph to an analogous concept in semigraph.

However, apart from the $w$-cycle nature of the edges with more than two vertices, a natural extension of tree can be realised in semigraph in the form of a dendroid which is a connected semigraph without any $s$-cycle. Interestingly, the underlying graph of a dendroid is a tree confirming the justification of introducing such a concept. In fact, a dendroid itself is a tree if no edge in it has more than two vertices i.e. if the underlying semigraph coincides with a graph. The issue of tree again demands careful attention in case of a disemigraph. A connected disemigraph without any $s$-cycle does not naturally lead to a dendroid structure as in the case of semigraph unless its underlying digraph is acyclic which is possible only when the disemigraph is without semicycle. Also, it can be observed that a disemigraph has a $w$-cycle only when it has an $s$-cycle. Therefore, a connected disemigraph without $s$-cycle is also without a $w$-cycle. But in that case, it may contain a semi-cycle. Thus, if a connected disemigraph is without both $s$-cycle and semicycle then it exhibits a structure like tree rather than a dendroid. In fact, the concept of dendroid is non-existent in a disemigraph setting. In this context, the concept of tree has been introduced in disemigraph along with some of its characterizations.

**Illustrations:**

(i) **A disemigraph with *s*-cycle and semicycle**.

Here ($aeba$) is an s-cycle and ($bceb$) is a semicycle.

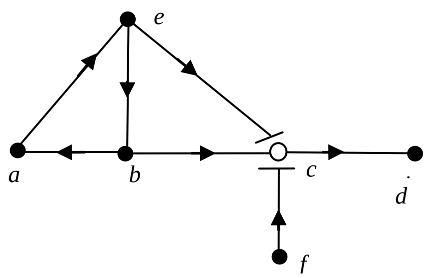


**Figure 1**

**(ii) A disemigraph without any s-cycle but with a semicycle.**

Here ($bceb$) is a semicycle.

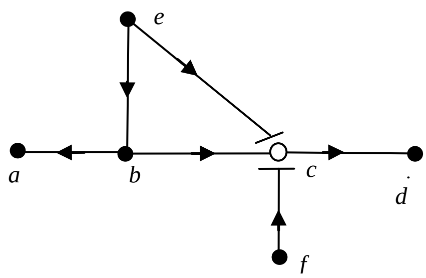


**Figure 2**

**(iii) A disemigraph without any semicycle i.e. an acyclic disemigraph.**

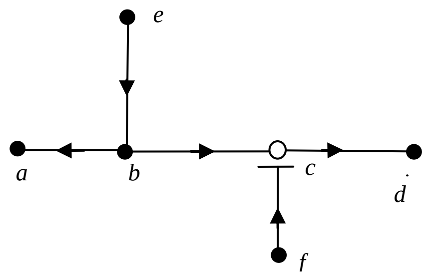


**Figure 3**

## 3.1 Tree

**Definitions 3.1.1**

A disemigraph *D* is said to be a **tree** (directed tree) if *D* is connected and without any semicycle.

A disemigraph *D* is said to be a **forest** if all its components are trees.

**Example 3.1.1**

A tree *T* is displayed in Figure 4.

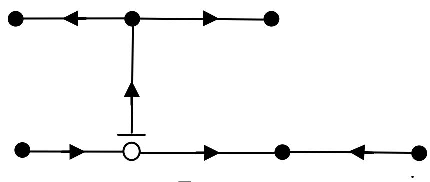


**Figure 4**

Few observations follow immediately-

1. A tree is a connected acyclic disemigraph
2. A tree is an asymmetric (oriented) disemigraph whose underlying consecutive adjacency digraph is a tree.
3. A tree is a simple disemigraph.
4. A tree *T* cannot be hamiltonian of any kind though it may possess a hamilton path.
5. All paths are trees in a disemigraph.

There are different ways to characterize trees. Some of these are mentioned in the following.

**Proposition 3.1.1**

A disemigraph *D* is a tree if and only if any two distinct vertices in *D* are joined by a unique semipath.

**Proof:**

If *D* is a tree, then *D* is connected and without any semicycle. Any two distinct vertices in *D* are joined by at least one semipath. If the semipath is not unique then a semicycle is produced in *D* contradicting the definition of a tree. Conversely, assume that *D* is a disemigraph in which any two distinct vertices are joined by a unique semipath. So, *D* is connected. If there is a semicycle in *D* containing *u* and *v* for any two vertices *u* and *v* in *D* then *u* and *v* are joined by at least two semipath contradicting the assumption. Hence *D* must be a tree. ■

**Proposition 3.1.2**

A connected disemigraph $D(p,q)$ is a tree if and only if $p-m=q+1$, where *m* is the number of middle vertices in *D*.

**Proof:**

Let $D(p,q)$ be a tree, then *D* is connected and any two distinct vertices in *D* are joined by a unique semipath (by proposition 3.1.1) . The result can be proved by induction on *p*.

If $p=1$, then $m=0$, $q=0$; so $p-m=q+1$.

If $p=2$, then $m=0$, $q=1$; so $p-m=q+1$.

For $p=3$, the possible cases are-

*Case i.* If $m=0$ then $q=2$; so $p-m=q+1$.

*Case ii.* If $m=1$ then $q=1$; so $p-m=q+1$.

Thus the result is true for $p=1,2$ and 3.

Now, if the result is true for any $n<p$ vertices and $q=t$ arcs.

Then $n-m=t+1$ i.e., $p-m=q+1$.

To show the result is true for $n+1$ vertices. Let *u* be added to the remaining *n* vertices. Now the possible subcases are-

*Subcase i.* If *u* is not a middle vertex then *u* is joined by a unique semipath to the existing *n* vertices in *D* and thus the number of arcs increased by one.

So $(n+1)-m=(t+1)+1$ i.e., $p-m=q+1$.

*Subcase ii.* If *u* is a middle vertex then $p=n+1$ and $q=t$.

So $(n+1)-(m+1)=t+1$.

$\Rightarrow (n+1)-m=(t+1)+1$ i.e., $p-m=q+1$.

Thus by induction hypothesis the result is true for any *p*. Hence $p-m=q+1$ whenever $D(p,q)$ is a tree. ■

**Proposition 3.1.3**

In a forest $D(p,q)$, $q=p-(n(D)-m)$, where $n(D)$ denotes the number of components in *D* and *m* is the number of middle vertices in *D*.

**Proof:**

Applying the proposition 3.1.2 to each component of *D*, the result follows immediately. ■

**Note:** The proposition 3.1.2 is a special case of proposition 3.1.3 as $n(D)=1$ for a tree.

Here, few results on dendroids in semigraph are reproduced.

**Proposition 3.1.4** [5]

Let $G(p,q)$ be a semigraph with *p* vertices and *q* edges $E_i$, $1\leq i\leq q$, and *k* components. Then G contains no cycles if and only if

$$p+q=\sum|E_i|+k,\ 1\leq i\leq q.$$

**Corollary 3.1.1:** [5] A connected semigraph *G* with *p* vertices and *q* edges $E_i$, $1\leq i\leq q$ is a dendroid if and only if

$$p+q=\sum|E_i|+1,\ 1\leq i\leq q.$$

**Proposition 3.1.5**
A connected disemigraph $D$ with $p$ vertices and $q$ arcs $E_i$, $1 \le i \le q$ is a tree if and only if

$p+q=\sum|E_i|+1$, $1 \le i \le q$.

**Proof:**

The proof is trivial as a consequence of Corollary 3.1.1 ■

The next two propositions are dual to each other.

**Proposition 3.1.6**

A tree $T$ has at least one vertex of out-degree zero.

**Proof:**

Let $u$ and $v$ be any two vertices in $T(p,q)$. Then $u$ and $v$ are joined by a unique semipath P in $T$ (by Proposition 3.1.1) and $P$ may be considered as maximal one between $u$ and $v$. If neither of $u$ and $v$ is with out-degree zero, then $P$ contains a vertex $w$ (say) with $od(w)=0$. If $od(w)\neq 0$ then, $w$ will be adjacent to some vertex of out-degree zero in $T$. Otherwise, by repeating the previous step a vertex of out-degree zero is obtained, since otherwise $T$ will contain a semicycle contradicting the fact that $T$ is a tree. Thus $T$ has at least one vertex of out-degree zero. ■

**Proposition 3.1.7**
A tree $T$ has at least one vertex of in-degree zero.
**Proof:**
Trivial. ■

Trees may be characterized in many different ways and each of them contributes to the structural understanding of disemigraphs in a different way. The following proposition includes some of them.

**Proposition 3.1.8**
Let $T(p,q)$ be a disemigraph with $m$ number of middle vertices. Then the following statements are equivalent.
(i) $T$ is a tree.
(ii) $T$ is connected and acyclic.
(iii) Any two distinct vertices in $T$ are joined by a unique semipath.
(iv) $q=p-(m+1)$.
(v) $T+e$ contains exactly one cycle or semicycle, where $e$ is a new arc added to $T$.
**Proof:**
The proof is immediate and the order of implications follow the sequence-
(i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(i). ■

## 3.2 Spanning tree
Here, the concept of spanning tree in disemigraph setting is developed analogous to its counterpart in digraph setting.

**Definitions 3.2.1**

A subdisemigraph $T$ of a disemigraph $D(V,A)$ is said to be a **spanning tree** if $T$ is a tree and $V(T)=V(D)$.

A spanning subdisemigraph $H$ of a disemigraph $D$ is said to be **distance preserving** from a vertex $u$ in $D$ if $d_H(u,v)=d_D(u,v)$ for every vertex $v$ in $D$.

Here few observations about spanning tree can be made.

(i) Every connected disemigraph $D(p,q)$ contains a spanning tree. If $D$ itself is a tree, then the observation is obvious; if $D$ is not a tree, then a spanning tree of $D$ may be obtained by removing arcs forming cycles, one at a time until the required tree is obtained. It is necessary to remove $q-p+(m+1)$ arcs in order to obtain a spanning tree.

(ii) The hamilton $s$-path (if any) of a disemigraph $D$ is the spanning tree of $D$.

(iii) If $D$ is $p$-hamiltonian then $D-e$ is a spanning tree, where $e$ is any arc in $D$.

The following is an application of spanning tree.

**Application 3.2.1**

Suppose it is required to construct a rail road system connecting certain cities. This situation can be modelled naturally by a connected disemigraph $D$. Finding a least expensive rail road system connecting all cities is equivalent to determining a spanning tree of $D$.

Now a result characterizing the spanning tree of a disemigraph is illustrated.

**Proposition 3.2.1**
For every vertex $u$ of a connected disemigraph $D$, there exists a spanning tree $T$ which is distance preserving from $u$.

**Proof:**
Let $S_i(u)=\{v\in V(D):d(u,v)=i\}$ for $i=1,2....$

For each $v\neq u$, $v\in S_i(u)$ for some $i$, and $v$ is adjacent with at least one vertex of $S_{i\pm j}(u)$, $j=1,2...$

If $v$ is a middle vertex, remove all but one arc containing $v$ as a middle vertex.

If $v$ is not a middle vertex, remove all but one arc of the type $(v,w)$ or $(w,v)$, where $w\in S_{i-1}(u)$. The resulting subdisemigraph is clearly connected, spanning and also distance preserving from $u$, which will be the required $T$ if $T$ is a tree. To verify that $T$ is a tree, it has to be checked that $T$ is without any semicycle. If possible, let $T$ contain a semicycle $C$. Let $x$ be a vertex on $C$ whose distance from $u$ is maximum, and let $w_1$ and $w_2$ be the vertices adjacent with $x$. Suppose $x\in S_k(u)$, then $w_1,w_2\in S_r(u)$, $r\le k$; this is in contradiction to our construction of $T$. So $T$ is a tree. Thus $T$ is a spanning tree which is distance preserving from $u$ in $D$.

## 3.3 Pendant tree
In a manner similar to digraphs and semigraphs the concepts of pendant vertex, pendant arc and pendant tree in disemigraphs are defined and illustrated.

**Definitions 3.3.1**
A vertex $u$ in a disemigraph $D$ is called a **pendant vertex** if $d_{ca}(u)=1$. A pendant vertex of a tree may also be called a **leaf** or **terminal vertex** of the tree. A vertex other than the pendant vertices is said to be an **internal vertex** or **branch vertex**. A pendant vertex $u$ is said to be **out-pendant** (**in-pendant**) **vertex** if $od(u)=1$ ($id(u)=1$). An out pendant vertex is always with in-degree zero and an in-pendant vertex is always with out-degree zero.
A **pendant arc** is an arc containing a pendant vertex. Any pendant arc has at least one pendant vertex as its end vertex.

A **pendant tree** is one having all its arcs as pendant arcs.

Now few of the preceding definitions are illustrated with the help of the following example.

**Example 3.3.1**
$T$ is a tree.
$a$, $d$, $e$, $g$ and $h$ are pendant vertices.

*b*, *c* and f are internal vertices.
*a* and *h* are out-pendant vertices, while *d*, *e* and *g* are in-pendant vertices.
All but (*b*, *f*) are pendant arcs.
*T* is a tree but not a pendant tree while $T–E_1$ is a pendant tree, where $E_1=(f,e)$ .

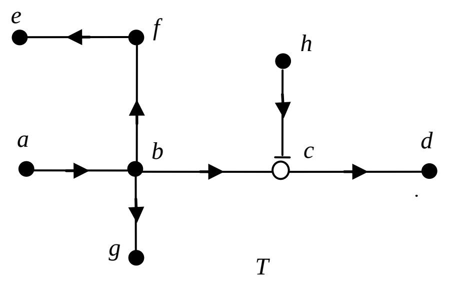


**Figure 5**

Few characterizations of the ideas incorporated in the preceding lines are in order.

**Proposition 3.3.1**
Every tree *T* contains at least two pendant vertices.
**Proof:**
*T* being a tree, by proposition 3.1.2, there is a smaller number of arcs than the number of vertices in *T*. So *T* has at least one pendant vertex *v* (say), otherwise T will contain a cycle or semicycle.
Let $u \neq v$ be any vertex in *T*. Then by proposition 3.1.1, *u* and *v* are joined by a unique semipath. If *u* is a pendant vertex then the result is obvious. If *u* is not a pendant vertex then *u* will be adjacent to some pendant vertex other than *v*, otherwise by repeating the previous step some vertex *w* in *T* can be found such that *u* is in the *v*-*w* semipath and *w* is the another pendant vertex in *T*. Thus *T* contains at least two pendant vertices. ■

**Proposition 3.3.2**
A tree $T(p,q)$ is a pendant tree if and only if $q \leq r$, where *r* is the number of pendant vertices in *T*.

**Proof:**
Let $T(p,q)$ be a pendant tree. Then every arc in *T* is a pendant arc. So each arc in *T* contains at least one pendant vertex. Thus $q \leq r$, where *r* is the number of pendant vertices in *T*. Conversely, suppose that $q \leq r$, where r is the number of pendant vertices in *T*. From the definition of a pendant vertex, each pendant vertex is an end vertex of some arc in *T*. So each pendant vertex is incident to at least one arc of *T*, since $r \geq q$. Thus every arc of *T* will contain at least one pendant vertex. Hence *T* is a pendant tree. ■

**Proposition 3.3.3**
If *T* is a pendant tree then $l \leq m+2$, where *l* is the length of a path between any two distinct vertices in *T* and *m* is the number of middle vertices in *T*.
**Proof:**
Given that *T* is a pendant tree, the following cases arise.
*Case i.* $m=0$
The pendant tree *T* becomes a simple digraph which is a pendant dendroid. Then obviously *T* is with the property $l \leq m+2$.
*Case ii.* $m \neq 0$ i.e. $m \geq 1$.
The result can be proved by induction on *m*.
For $m=1$ and 2, by inspection, the result can be seen to be true.
Next let the result be true for some $t<m$, then $l \leq t+2$.
Now for $(t+1)$ middle vertices in *T*, it can be seen that the length of the path becomes one more than that of the path having *t* middle vertices.
So $(l+1) \leq (t+1)+2$ i.e., $l \leq m+2$.
Thus the result is true for any *m*.
Hence $l \leq m+2$, whenever *T* is a pendant tree. ■

The foregoing result is independent of *s*-path or *w*-path.

## 3.4 Star

Now, it seems appropriate to make a short discussion on star.

**Definitions 3.4.1**

A tree is said to be a **star** if all its arcs have a common vertex. The common vertex is said to be the **centre** of the star.

A star is a tree consisting of one vertex adjacent to all of the other vertices.

**Example 3.4.1**

The following trees $T_1$, $T_2$ and $T_3$ are stars but the trees in Figure 4 and Figure 5 are not stars.

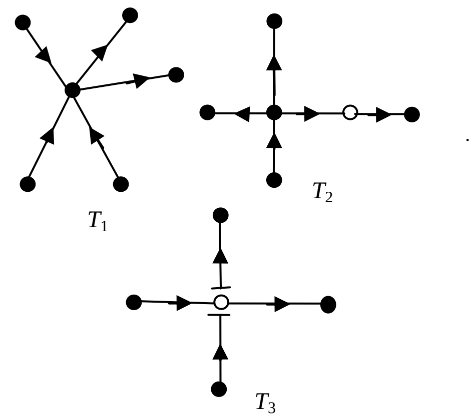


**Figure 6**

Each arc in a star is a pendant arc however each arc being pendant may not give a star.

**e.g.** In Figure 5, if the arc $E_1=(f,e)$ is removed then $T–E_1$ is a tree with all of its arcs as pendant arcs but it is not a star.

In the following an example is given highlighting the practical utility of a star.

**Application 3.4.1**
The situation of a classroom is represented by a disemigraph *D*, considering the teacher and students as vertices, and by drawing arcs between or among the teacher and students whenever an interaction occurs. One can easily judge the effectiveness of the teaching-learning process in the classroom by checking whether *D* is a star or *D* contains a spanning tree which is a star, where the teacher plays the role as the centre for the star.

**Proposition 3.4.1**
A tree $T(p,q)$ is a star if and only if *T* contains exactly one vertex of degree $p–1$.

**Proof:**

Let $T(p,q)$ be a star, then $T$ contains a vertex $w$ (say) adjacent to/from all other vertices of $T$. So obviously, $d(w)=p-1$. Thus $T$ contains exactly one vertex of degree $p-1$.

Next, suppose $T(p,q)$ is a tree with exactly one vertex w (say) of degree $p-1$, then $w$ is the common vertex in $T$ which is adjacent to all other vertices in $T$. Hence $T(p,q)$ is a star. ■

**Corollary 3.4.1:** A star $T(p,q)$ contains exactly one vertex with degree $q+m$, where $m$ is the number of middle vertices in $T$.

## 3.5 Arborescence

Arborescence [3] in digraphs is one of the most useful types of rooted tree and a well studied concept. There is a large number of synonyms for arborescence in graph theory, including directed rooted tree, out-tree, branching etc. The application of arborescence can be seen in computer algorithm, network analysis, enumeration etc. Here the idea of arborescence can be extended to disemigraph setting.

**Definitions 3.5.1**

A tree $T$ is said to be an **arborescence** if $T$ has a source. The source in $T$ may be called the **root** of the arborescence.

An arborescence is a tree directed out of the root, therefore an arborescence may equivalently be defined as an **out-tree** (or **rooted tree** or **branching**). Reversing of the direction of every arc in an arborescence will produce what may be called an **in-tree** (or **anti-arborescence**). In anti-arborescence the root is a sink. A tree without any root is said to be a **free tree**.

For any vertex $v$ in an arborescence (anti-arborescence) $T$ with the root at $r$ (say), the **depth/level** of $v$ is the length of the path from $r$ to $v$ ($v$ to $r$). The **height** of a rooted tree is the maximum depth in the tree. Thus the height is the length of a longest path from the root to any other vertices. A vertex $y$ in a rooted tree is said to be a **descendant** of a vertex $x$ if $x$ is on the unique path from the root to $y$, and here $x$ is called an **ancestor** of $y$.

In Figure 6, $T_1$ and $T_3$ are free trees, and $T_2$ is an arborescence.

From the above definitions of arborescence the following observations can be made.

i. Every tree may not have a root.

ii. (ii)A tree cannot have more than one root though it may have any number of pendant vertices / leaves.

iii. A root $r$ of arborescence has depth zero, $id(r)=0$ and itself is a pendant vertex.

iv. A root is the ancestor of all other vertices of the tree.

v. An arborescence cannot have a sink.

vi. An arborescence always has a root and all other vertices are reachable from the root but the root is not reachable from any other vertex and also the path from the root to any other vertex is unique.

Now few examples showing applications of arborescence are in order.

**Applications 3.5.1**

1. A playoff scheme can be developed by an arborescence where the root represents champion.

2. An arborescence may be used in a variety of search programmes. Suppose it becomes necessary to search a word $A$ from among words in some set containing $A$. The maximum number of tests necessary to recognize $A$ is the height of the anti-arborescence. Similarly, searching for a letter of the alphabet can be done by an anti-arborescence where the height of the tree implies the number of tests required to locate the letter.

3. Consider a network of streams which does not contain cycles or loops. The topology of such a network can be modelled by an arborescence rooted at the ultimate stream source. Such models have been used by geographers to analyse the way in which stream systems evolve and together with their propensity to flood.

Now, few characterizations of arborescence are mentioned below.

**Proposition 3.5.1**

If $T$ is an arborescence with the root $r$, then $id_{ca}(v)=1$ for any vertex $v\neq r$ in $T$.

**Proof:**

Let $T$ be an arborescence with order $p$ and root at $r$.
Clearly, $T_{ca}$ is also an arborescence in its digraph setting.
So $\sum_{i=0}^{p} id(v_i)\leq p-1$ in $T_{ca}$.
Of the $p$ terms on the left hand side of this equation, only one is zero because of the root $r$ (since $id(r)=0$), while all others must be positive. Therefore, they all must be 1's.
Thus, $id(v)=1$ for any vertex $v\neq r$ in $T_{ca}$
This means that $id_{ca}(v)=1$ for any vertex $v\neq r$ in $T$. ■

**Proposition 3.5.2**

A disemigraph $D$ is an arborescence if and only if $D$ is a tree and there exists only one vertex $u$ in $D$ with $id(u)=0$.

**Proof:**

The proof follows immediately from the definition of arborescence. ■

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