Characters for the Permutation Group of Degree n using Specht Module and Semi Standard Young Tableaux

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ABSTRACT
For any partition, the corresponding Specht module is the sub module of permutation module spanned by the poly-tabloids. The Specht modules for a partition of a positive integer n form a complete list of irreducible representations of permutation group of degree n. The Semi-Standard Young Tableau (SSYT), on *n* -symbols are one type of combinatorial objects occur naturally in many computational problems in Science, Engineering and Technology, which have one-to-one correspondence with Gelfand–Tzetlin bases set of the Unitary group *U (n)*.In this paper, we propose a method to construct character table of permutation group of degree n using Specht module and Semi Standard Young Tableaux. This method is illustrated with an example using a partition of degree 5 in permutation group S_5.

Keywords
Permutation group ; Partition ; Young Tableaux; Semi Standard Young Tableaux ; Permutation Module; Specht Module

1. INTRODUCTION
Specht[1]introduced set of polynomials using standard tableau to generate rational representation of permutation group of degree n. The polynomials and modules are Specht Polynomials and Specht modules corresponding to the partition respectively. The set of polynomials forms a basis (i.e., they are linearly independent and spans the vector space). He proved the irreducibility and in equivalence of the modules without the use of characters of the permutation group representation. In the paper, [2] Garnir constructed irreducible integral matrix representations of the permutation groups and derived relations between the Specht polynomials and also showed that Specht polynomial is an integral linear combination of standard Specht polynomials.

One of the essential objectives of the representation theory of finite groups is in computation of characters of irreducible representations. For permutation group, the irreducible characters can be computed using either Frobenius formula, or the determinantal formula, or the Murnaghan-Nakayama rule [7,8,9,10].

This procedure is to explicate on how to construct character tables of symmetric groups.

2. PRELIMINARIES
2.1 Young diagram or Ferrer's diagram
Since the number of nonequivalent irreducible representation of a group is equal to its classes, the non-equivalent irreducible representations of Sn are defined, as are the classes, by the different partitions of the number n into the positive integral components. The partitions are usually denoted by [λ] = [λ1 λ2 λ3. . . . . λk] where λ1+ λ2+λ3+. . . . .+ λk = n is λ1≥ λ2≥λ3≥. . . . . ≥ λk≥0 and of the λi's may coincide. It is clear that k cannot exceed n. These partitions can be represented graphically by means of a diagram known as Young diagram. In which λi is represented by a row of λi cells. Young diagrams can be represented by a symbol

[λ] = [λ1 λ2 λ3. . . . . λk]

2.2 Self-conjugate diagram
For each [λ] = [λ1 λ2 λ3. . . . . λk] in the diagram, interchange the rows into the columns and the columns into the rows then the resultant diagram is known as conjugate to the given diagram and denoted by [$\tilde{\lambda}$] .A Young diagram [λ] is said to be self-conjugate diagram if [λ] = [$\tilde{\lambda}$],

2.3 Young Tableau
Let n be a positive number. A partition of n is a vector λ = [λ1 λ2 λ3. . . . . λk] of positive integers, where λ1 ≥ λ2 ≥ λ3 ≥. . . . . ≥ λk ≥ 0 and λ1+λ2+λ3+. . . . .+ λk = n. A Young diagram of shape λ is a subset of a rectangular table such that the ith row contains λi cells, with all rows starting at the first column. A Young tableau of shape λ is obtained by inserting the integers 1, 2, 3. . . n as entries in the cells of the Young diagram of shape λ without repetitions. The Young tableau is called standard (SYT) if its entries increase along all rows and columns.

2.4 Method to find dimension of a representation of SYT by using hook lengths
The dimension of a representation is equal to the number of different Young tableau that can be obtained from the diagram of the representation this number can be calculated by hook-length formula. By the hook length formula, the dimension of an irreducible representation is n! divided by the product of the hook lengths of all the boxes in the diagram of the representation.

The dimension f λ = $\dfrac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$

The hook length of cell (i, j) is the number hi, j = λi+ λj-i-j+1, in this formula λi is the length of the ith row and λj denotes the length of the jth column.

3 TABLOIDS & PERMUTATION MODULE
In this section, tabloids and permutation module are discussed.

3.1 Tabloids

Consider certain permutation representations of Sn on the elements{1,2,…, n} , which extends to the defining representation. In this merits, construct other representation of Sn using equivalence classes of tableaux, known as tabloids. Tabloids are used and used to construct a representation of Sn known as the permutation module Mλ, however, permutation modules are generally reducible.

Two λ- tableaux t1 and t2 are row-equivalent denoted t1 ~ t2. If the corresponding rows of the tableaux contain the same elements.

A tabloid of shape λ,or λ-tabloid is such an equivalence class, denoted by {t} ={t1/t1~ t} where t is a λ- tabloid. The tabloid {t} is drawn as the tableaux t without vertical bars separating the entries within each row.

For instance, if

t =

1	2
3	

Then {t} is the tabloid drawn as

{t} =

1 2
3

This represents the equivalence class containing the following two tableaux

1	2
3	

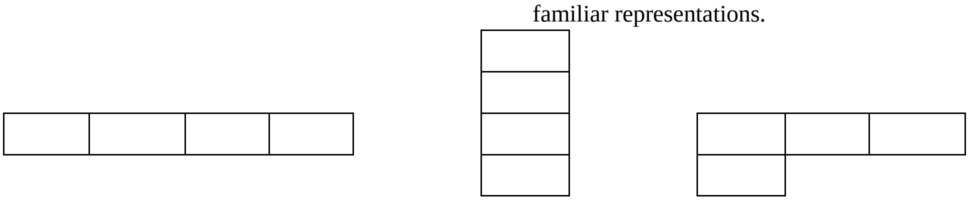
2	1
3	

The notation is suggestive as it emphasizes that the order of the entries within each row is irrelevant, so that each row may be shuffled arbitrary.

3.2 Permutation module

Suppose λ ⊢ n. Let M λ denote the vector space whose basis is the set of

λ- Tabloids. Then M λ is a representation of Sn known as the permutation module corresponding to λ. The M λ corresponding to the Young diagrams(Fig.1) are in fact familiar representations.



Trivial Sign Standard

Fig.1 Familiar Representations

Case 1

In λ = [n], M λ is the vector space generated by the single tabloid

1 2 . . . . n

Since this tabloid is fixed by Sn, M[n] is one-dimensional trivial representation

Case 2

For λ = [1n]=[1,1,……] then a λ- tabloid is simply a permutation of {1,2,3,….n} into n rows it follows that

$M[1^n]$ is isomorphic to the regular representation C[Sn].

Case 3

Consider λ= [n-1, 1].

Let {ti} be the λ-tabloid with i on the second row. Then M λ has basis {t1 }, { t2 },…{ tn }. Also, note that the action of π ∈ Sn sends ti to tπ(i) . M[n-1,1] is isomorphic to the representation C{1,2,……..n}.

For example for n=5, the representation $M^{(4,1)}$ has the following basis

t_1 =

2 3 4 5
1

t_2 =

1 3 4 5
2

t_3 =

1 2 4 5
3

t_4=

1 2 3 5
4

t_5=

1 2 3 4
5

3.3 Dimension of the permutation Module M λ

If $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ then

$$\dim M^{\lambda} = \frac{k!}{\lambda_1 ! \lambda_2 ! \dots \lambda_k !}$$

Example : The dimension of $M^{[2,2]}$ at $e \in S_4$ is $\dim M^{[2,2]} = \frac{4!}{2!2!} = 6$

3.4 Characters of the permutation module Mλ

Suppose λ=[λ1,, λ2,…………….. λk] and

μ= [μ1, μ2,…………. μn] are partition of n.

The characters of Mλ evaluated at an element of Sn with cycle type μ is equal to the coefficient of

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_k^{\lambda_k} \text{ in } \prod_{i=1}^{m} (x_1^{\mu_1} + x_2^{\mu_2} + \dots + x_i^{\mu_i})$$

the permutation module Mλ can be realized as a permutation representation on the λ-tabloids, its character at an element π ∈ Sn is equal to the number of tabloids fixed by π.

The dimension of a representation is simply the value of the character at the identity element, which has cycle type μ=[1n].

So above equation tells us that dimension of M^{λ} is the coefficient of

$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_k^{\lambda_k}$ in (x1+x2+……..xn)n. Which is equal to Dim $M^{\lambda} = \frac{n!}{\lambda_1!\lambda_2!\ldots\ldots\lambda_k!}$

Other characters can be similarly computed, and the result is shown in the Table 1.

Let us compute the full list of the characters of the permutation modules for S5 ,the character at the identity element is same to the dimension ,and it can found through [3,3], For instance ,

the character of M(2,2,1) = $\frac{5!}{2!2!1!} = 30.$

To compute the character of $M^{\lambda=(3,2)}$ at the permutation $\mu_{(2,2,1)}$ which has cycle type (2, 2, 1), by using [3 ,4], we see that the character is equal to the coefficient of $(x_1^3 x_2^2)$ in: $(x_1^2 + x_2^2 + x_3^2)^2(x_1 + x_2 + x_3)$

Table 1 : Characters of the representation

	$\mu_{(1^5)}$	$\mu_{(2,1^3)}$	$\mu_{(3,1^2)}$	$\mu_{(4,1)}$	$\mu_{(2^2,1)}$	$\mu_{(3,2)}$	$\mu_{(5)}$
$M^{\lambda=(5)}$	1	1	1	1	1	1	1
$M^{\lambda=(4,1)}$	5	3	2	1	1	0	0
$M^{\lambda=(3,2)}$	10	4	1	0	2	1	0
$M^{\lambda=(3,1,1)}$	20	6	2	0	0	0	0
$M^{\lambda=(2,2,1)}$	30	6	0	0	2	0	0
$M^{\lambda=(2,1,1,1)}$	60	6	0	0	0	0	0
$M^{\lambda=(1,1,1,1,1)}$	120	0	0	0	0	0	0

Note that in the above example, It is not possible to construct the character table S5, as all the M^{λ} are in fact reducible with the exception of $M^{(5)}$. In the next, we take a step further and construct the irreducible representation of Sn.

The table which was constructed in the table 1 is done depending on the [3.4], and a compound character biodegradable to irreducible characters are constructed by using the method of subtraction. The establishment of such a table 1 to the higher degree of group of the clique is very complex and not easy to get him so the item will show another way to, calculating the irreducible characters.

Main procedure to compute characters

This method depended on the inner product formula [3, 4, 5, 6], to inference irreducible

Character indicator in the example below. The trivial representation [6], is already irreducible, so the top row is an Since we know the copies of χ_5 occur in the lower representations, we can subtract them of and construct a table 2

irreducible character; let it be $\chi_5 = M^{(5)}$. Find out many copies of χ_5 each of the lower characters contains by taking inner products.

$< \chi_5, M^{(4,1)} >$ =1

$< \chi_5, M^{(3,2)} >$ =1

$< \chi_5, M^{(3,1,1)} >$ =1

$< \chi_5, M^{(2,2,1)} >$ =1

$< \chi_5, M^{(2,1,1,1)} >$ =1

$< \chi_5, M^{(1,1,1,1,1)} >$ =1

Then it is to find the copies of χ_1 occur in the lower representation and subtract them and construct the table 3.

Table 2 : Characters

Cycle type	(1,1,1,1,1)	(2,1,1,1)	(3,1,1)	(4,1)	(2,2,1)	(3,2)	(5)
χ_5	1	1	1	1	1	1	1
$\chi_{(4,1)}$	4	2	1	0	0	-1	-1
$M^{\lambda=(3,2)'}$	9	3	0	-1	1	0	-1
$M^{\lambda=(3,1,1)'}$	19	5	1	-1	-1	-1	-1
$M^{\lambda=(2,2,1)'}$	29	5	-1	-1	1	-1	-1
$M^{\lambda=(2,1,1,1)'}$	59	5	-1	-1	-1	-1	-1
$M^{\lambda=(1,1,1,1,1)'}$	119	-1	-1	-1	-1	-1	-1

Now row 2 is an irreducible character $\chi_{(4,1)}$; you can see this by taking its inner product with itself. We can now repeat by taking the inner product of $\chi_{(4,1)}$ with the characters and subtracting them off.

$$< \chi_{(4,1)}, M^{(3,2)} > =1$$

$$< \chi_{(4,1)}, M^{(3,1,1)} > =2$$

$$< \chi_{(4,1)}, M^{(2,2,1)} > =2$$

$$< \chi_{(4,1)}, M^{(2,1,1,1)} > =3$$

$$< \chi_{(4,1)}, M^{(1,1,1,1,1)} > =4$$

Table 3 : Characters for different cycles -I

Cycle type	(1,1,1,1,1)	(2,1,1,1)	(3,1,1)	(4,1)	(2,2,1)	(3,2)	(5)
χ_5	1	1	1	1	1	1	1
$\chi_{(4,1)}$	4	2	1	0	0	-1	-1
$\chi_{(3,2)}$	5	1	-1	-1	1	1	0
$M^{\lambda=(3,1,1)''}$	11	1	-1	-1	-1	1	1
$M^{\lambda=(2,2,1)''}$	21	1	-3	-1	1	1	1
$M^{\lambda=(2,1,1,1)''}$	37	-1	-4	-1	-1	2	2
$M^{\lambda=(1,1,1,1,1)''}$	107	-7	-4	-1	-1	2	2

Row 3 is irreducible and let it be $\chi_{(3,1,1)}$, and subtract it off from the lower rows and represented in table 4.

$$< \chi_{(3,2)}, M^{(3,1,1)} > =1$$

$$< \chi_{(3,2)}, M^{(2,2,1)} > =2$$

$$< \chi_{(3,2)}, M^{(2,1,1,1)} > =1$$

$$< \chi_{(3,2)}, M^{(1,1,1,1,1)} > =1$$

Table 4 : Characters for different cycles II

Cycle type	(1,1,1,1,1)	(2,1,1,1)	(3,1,1)	(4,1)	(2,2,1)	(3,2)	(5)
χ_5	1	1	1	1	1	1	1
$\chi_{(4,1)}$	4	2	1	0	0	-1	-1
$\chi_{(3,2)}$	5	1	-1	-1	1	1	0
$M^{\lambda=(3,1,1)'''}$	6	0	0	0	-2	0	1
$M^{\lambda=(2,2,1)'''}$	11	-1	-1	1	-1	-1	1
$M^{\lambda=(2,1,1,1)'''}$							
$M^{\lambda=(1,1,1,1,1)'''}$							

If we proceed further we can construct the table 5. It gives character table for S_4

Table 5 : Characters for S_4

Cycle type	(1,1,1,1,1)	(2,1,1,1)	(3,1,1)	(4,1)	(2,2,1)	(3,2)	(5)
χ_5	1	1	1	1	1	1	1
$\chi_{(4,1)}$	4	2	1	0	0	-1	-1
$\chi_{(3,2)}$	5	1	-1	-1	1	1	0
$\chi_{(3,1,1)}$	6	0	0	0	-2	0	1
$\chi_{(2,2,1)}$	5	-1	-1	1	1	-1	0
$\chi_{(2,1,1,1)}$	4	-2	1	0	0	1	-1
$\chi_{(1,1,1,1,1)}$	1	-1	1	-1	1	-1	1

4 CONCLUSION

In this paper, Character table of permutation group of degree n using Specht module and Semi Standard Young Tableaux has been proposed. This method is explained with an example using a partition of degree 5 in permutation group S5.

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