Numerical Computation for Extracting Eigenvalues and Eigenvectors from a Large Order System of Equation - A Given Input Range

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ABSTRACT

Finite Element method is a numerical method for finding approximate solution to boundary value problems for partial differential equations. It uses subdivision of a whole problem domain into simpler domain, called finite elements, and variation methods from the calculus of variations to solve the problem by minimizing an associated error function.

Finite element method (FEM) is applied in engineering as a computational tool for performing engineering analysis. FEM form a global linear system of algebraic equations (homogenous/ non-homogenous) to find the solution of analysis problems.

In structural dynamic problems the algebraic equations will of the form $[K]\{u\}=\omega^2[M]\{u\}$, [K] and [M] are stiffness and mass matrices. Solving this large order system will generate natural frequencies and mode shape of the structure. Since most of the analysis is carried out for first few frequencies and mode shape of the structure, solving the entire system is required iterative method used to transform system into reduced system from where the eigenvalues are extracted in increasing order .(Lanczos/subspace).

In the proposed work, the Lanczos algorithm implementation is modified to extract the natural frequencies and mode shapes of the structure in a given input frequency range. The eigenvalues and eigenvectors are computed in multiple passes to Lanczos reduction algorithm. In order to get faster converges of previously calculated eigenvectors and used subsequent reorthogonalization vectors. This reduced the overall time required to extract frequencies and mode shapes .This is one of requirements of in-house developed FEM based structural analysis software FEAST^{SMT}.

General Terms

α^2	Small negative parameter for vibration problems.
ε	Tolerance for rejecting small elements [M].
$\lambda, \bar{\lambda}$	Exact and approximate values of inverted and shifted eigenvalues.
λ_a	Physical eigenvalue
λ_0	Shift in physical eigenvalue.
$\varphi, \bar{\varphi}$	Exact and approximate buckling eigenvalue.
ε_i	Bound on the absolute relative error in i^{th} physical eigenvalues.
ω_0	Center of frequency range of interest.
$\omega, \bar{\omega}$	Exact and approximate circular natural frequencies
a	General element of reduced eigenmatrix
c_i	Participation factor for i^{th} eigenvector
d	Off-diagonal element of reduced tridiagonal matrix
b	Vectors of scalar coefficients.
[A]	Tridiagonal, reduced eigenmatrix.
$[\bar{B}]$	Eigenmatrix for the inverse problem.
[B]	[D], [B], symmetric matrix.
[C]	Lower triangular choleskey factor of decomposed $[\bar{K}]$
[d-]	Diagonal factor of decomposed $[\bar{K}]$
[D]	Symmetric orthogonality matrix for the inverse problem.
[I]	Identity matrix
[K]	Stiffness Matrix.
$\bar{d}$	Normalization factor for v-vectors and approximate off-diagonal element of reduced tridiagonal matrix
f	Number of previously calculated modes.
K_{ij}	Element of $[K_{aa}]$ matrix
m	Size of reduced eigenproblem
m_{ij}	Element of mass[M] matrix
N	Size of the full, unreduced eigen problem
$\bar{n}$	Number of non-null columns and rows of [M] matrix
q	Total number of desired eigensolutions, including those previously computed.
$\bar{q}$	Total number of desired eigensolutions, including those previously computed.
r	Desired number of new eigensolutions
$\bar{r}$	Maximum possible size of reduced eigenproblem.
r_i	Weighted root-mean square residual.
T	Number of decimal digits carried by computer**.**
[M]	Equal to $[M_{aa}]$ or $[K_{aa}^d]$
$[M_{aa}]$	Mass matrix for analysis set.
[R]	Residual Vector.
[V]	Matrix of v-vectors assembled column -wise.
[v]	Trial Vector.
{w}	Pseudo -random vector.
{X}	Eigenvector of the symmetric inverse problem.
$\{\tilde{x}\}$	Previously generated eigenvector.
[y]	Eigenvector of the reduced problem.
[Z]	Exact Modal matrix.
$\emptyset, \bar{\emptyset}$	Exact and approximate physical eigenvector.
λ	Diagonal matrix of exact eigenvalues.
$\bar{K}$	Effective stiffness matrix after eigenvalue shift.
K_{aa}	Stiffness for analysis set.

Keywords

Lanczos algorithm, FEAST^{SMT} ,Finite element analysis.

1. INTRODUCTION

FEAST (**F**inite **E**lement **A**nalysis of **ST**ructures) is the Indian Space Research Organisation (ISRO) structural analysis solver software based on Finite Element Method

(FEM) realized by Structures group of Vikram Sarabhai Space Centre (VSSC).Numerical methods are utilized to solve large dimension problems that in most cases belong to elliptic, parabolic and hyperbolic class of partial differential equations encountered in the field of solid mechanics, which is the theoretical basis for structural engineering.[1]

Finite element method (FEM) is a very popular numerical method that has found use in finding practical solutions to field problems in solid mechanics, electro-magnetic thermal engineering etc.

Lanczos algorithms are very attractive because the multiplication by is the only large-scale linear operation. Since weighted-term text retrieval engines implement just this operation, the Lanczos algorithm can be applied efficiently to text**.**

Table I Comparison of Methods of Real Eigen value Extraction

Characteristic / Method	Tridiagonal Method	Lanczos Method
Matrix Pencil	**(A,I)** A = $\frac{\mathbf{K}}{\mathbf{M}}$ or A= $\frac{\mathbf{M}}{\mathbf{K}+\lambda\mathbf{M}}$	(M (K - $\sigma\, m)^{-1}$,M) Or K(K - $\sigma\, k_d)^{-1}$ K,K)
Restriction on matrix character	A real, symmetrical constant M ≠ Singular or K+λM ≠singular	M positive semi definite or K positive semi definite
Obtains eigenvalues in order	All At once	Several -nearest to the shift point
Takes advantage of bandwidth or sparsity	No	Yes
Number of calculations	$O(n^3)$	$O(nb^2E)$
Recommended	All modes	Few modes

2. LANCZOS ALGORITHM FOR SOLVING LARGE ORDER PROBLEMS

The basic Lanczos recurrence algorithm is a transformation process to a reduced tridiagonal form. The algorithm truncates the tridiagonalization process and provides approximations to the eigenpairs (eigenvalues and eigenvectors) of the original matrix.The tridiagonal reduction is an automatic matrix reduction scheme whereby the eigen solutions in the neighbourhood of a specified point in the eigen spectrum can be accurately extracted from a tridiagonal eigenvalue problem whose order is much lower than that of the full problem. Specifically, the order, **m** of the reduced problem is never greater than [4]

$$m = 2q + 10$$

Where; q is the desired number of accurately computed eigenvalues. Thus, the intrinsic power of the method lies in the fact that the size of the reduced eigenvalue problem is of the same order of magnitude as the number of desired roots, even though the discredited system model may possess thousands of degrees of freedom.

Tri-diagonal reduction method employs only a single initial shift of eigenvalues and hence usually requires only one matrix decomposition. The Lanczos tri-diagonal reduction method is implemented in FEAST^{SMT} for real eigenvalue analysis as typified by structural vibration and buckling problems.[2]

The usual first step in performing s dynamic analysis is determining the natural frequencies and mode shapes of the structure of the with damping neglected .these results characterize the basic dynamic behaviour of the structure and are an induction of how the structure will respond dynamic loading .

The natural frequencies of a structure are the frequencies at which the structure naturally tends to vibrate if it is subjected to a disturbance .for example, the strings of piano are each tuned to vibrate at a specific frequency .Some alternate terms for the natural frequency are characteristic, fundamental frequency, resonant frequency, and normal frequency.[2]

The deformed shape of the structure at a specific natural frequency of vibration is termed its normal mode of vibration .Some other terms used to describe the normal mode are mode shape ,characteristic shape ,and fundamental .each mode shape is associated with a specific natural frequency.

Natural frequencies and mode shapes are functions of the structural properties and boundary conditions. If the structural properties change, the natural frequencies change, but the mode shapes

The Lanczos algorithm is an iterative algorithm devised by Cornelius Lanczos that is an adaptation of power methods to find eigenvalues and eigenvectors of a square matrix or the singular value decomposition of a rectangular matrix. It is particularly useful for finding decompositions of very large sparse matrices. In latent semantic indexing, for instance, matrices relating millions of documents to hundreds of thousands of terms must be reduced to singular-value form.

The power method for finding the largest eigenvalue of a matrix A can be summarized by noting that if x_0 is a random vector and $x_{n+1=Ax_n}$, then in the large n limit, $x^n/||x^n$ approaches the normalized eigenvector corresponding to the largest eigen value in magnitude.

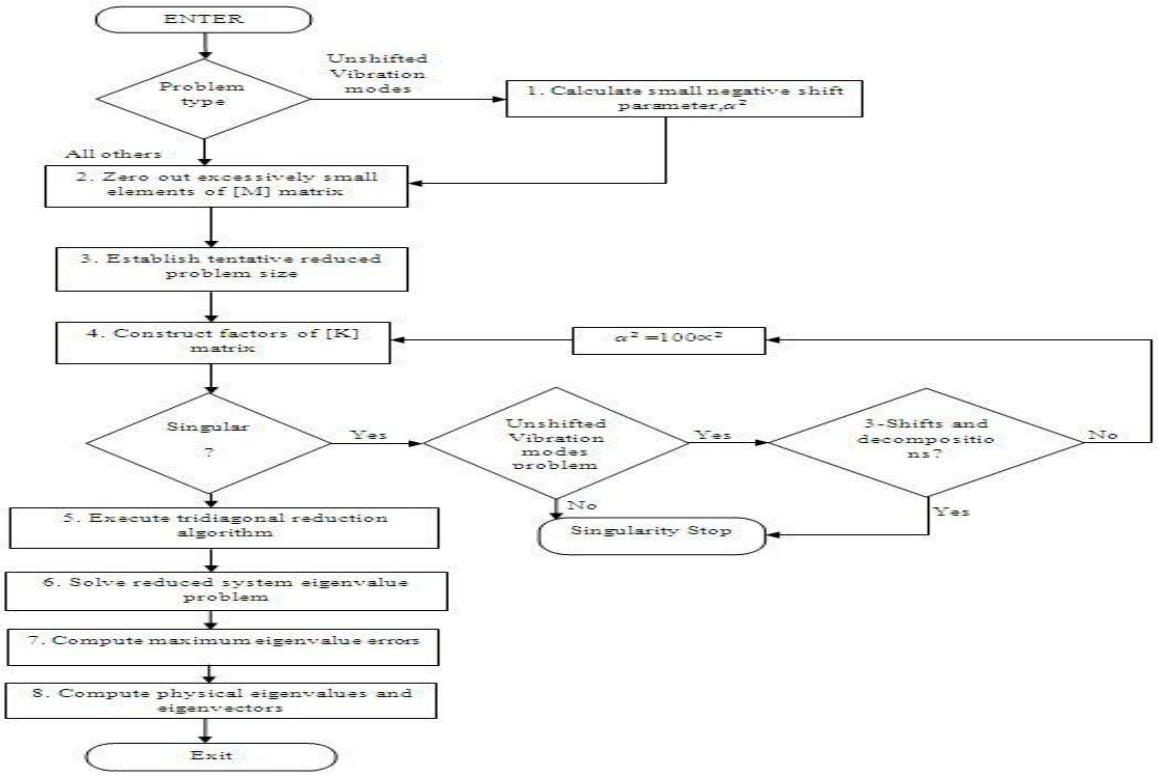

Fig 1 : Tridiagonal Reduction Method

3. MODIFIED LANCZOS ALGORITHM

Step **1**: Initialize converged modes and vector values

$(\sqrt{_i})$ = 0 and $(\emptyset)$ = null

Step 2: Declare the end frequency range.

E.g.: m_freq =500(Input)

Step 3: Find shift parameter (α^2) for removing possible singularities in the eigen system.

$$\alpha^2 = \text{MAX}\left(\alpha^2_{\min}, \alpha^2_0\right)$$

Where; $$\alpha^2_{\min} = n * 10^{2-t} * \left|\frac{K_{ii}}{M_{ii}}\right|_{\max} ; M_{ii} \neq 0$$

$$\alpha^2_0 = 10^{\frac{-t}{3}} * \left|\frac{K_{ii}}{M_{ii}}\right|_{\min}$$

K_{ii} and M_{ii} are the diagonal elements of $\lfloor K_{aa} \rfloor$ and $\lfloor M_{aa} \rfloor$. n is the number of $\{u_a\}$ degrees of freedom, and t is the decimal digit carried by the computer.

Step 4**:** Zero-out excessively small Elements of [M] matrix.

a. Compare the magnitudes of all off-diagonal elements of [M] with the corresponding diagonal elements and determine whether

$$\left|\frac{m_{ij}}{m_{ii}}\right| \leq 10^{-2t/3}; i = j, m_{ii} \neq 0$$

b. Set the m_{ij} =0 for every off-diagonal element for satisfying the condition.

Step5**:** Call the Lanczos algorithm, compute eigensolution for nr modes

Input: 1.Converged vector $(\sqrt{_i})$

2. Requested modes $(\bar{q})$.

Output**:** 1.Computed eigenvalues (λ_i)

2...Eigenvector $(\emptyset_i)$

Step6**:** Call extracts Eigen modes

Input: 1.Number of modes (nr),

2. Eigenvalues (λ)

3. Eigenvector $(\emptyset)$

Output: 1.converged modes $(\sqrt{_i})$ and computed frequencies $(\bar{f})$

2. The Termination flag

Step7: If the termination flag is False go to Step 6

Step8**:** Write converged frequencies (f) and converged modes$(\sqrt{}_i)$

A. Lanczos Algorithm[2]

Solve for eigen value and vectors(//K is assumed to be decomposed*)*

Input: nr, p, converged modes$(\sqrt{}_i)\ \bar{\boldsymbol{r}}$

Step1 **:** Establish the size of the reduced eigenvalue problem

a. Count the number of non-null columns or rows in the matrix [M]

$\bar{r} = \bar{n} - p$ Where; p is the number of previously computed eigensolutions

b. Calculate the size of reduced eigenvalue problem

$m = \min\left[2\bar{q} + 10, \bar{r}\right]$

Where, $\bar{q} = q - p$

q = Total number of accurate eigenvalues requested by the user

Step2**:** Construct factors of [K] Matrix

a. Set

i. $\left[\bar{K}\right] = \left[K_{aa} - \omega_0^2 M_{aa}\right]$

ii. $\left[\bar{K}\right] = \left[K_{aa} - \alpha^2 M_{aa}\right]$

iii. $\left[\bar{K}\right] = \left[K_{aa}\right]$

b. Perform non-square root decomposition

$$\left[\bar{K}\right] = \left[L\right]\left[d\right]\left[L\right]^T$$

Step3: Initialize recurrence algorithm

Initialize vector index, $i = 0$ and set $\{V_0\} = 0$

Where is an $n \times 1$ "null"vector.

Step 4: Generate a starting or restart vector

and set $d(i+1) = 0$

a. Construct an n-element vector $\{W\}$ using pseudo random number generator.

b. Solve for un-normalized trial vector from the equation

$$\{v_{i+1}\} = [\bar{B}]\{w\}$$

i. $\left[\bar{B}\right] = L^{-T}\left[d\right]\left[L\right]^{-1}$

ii. $\left[\bar{B}\right] = \left[L\right]^{-1}\left[M\right]\left[L\right]^{-T}$

iii. $\left[\bar{B}\right] = \left[L\right]^{-1}\left[K^{d}\right]\left[L\right]^{-T}$

Forward and backward passes are used to perform the above inverse operations

a. Normalize the above vector

$$\begin{Bmatrix} 0 \\ v_{i+1} \end{Bmatrix} = \left[\frac{1}{\{v_{i+1}\}^T [D] \{v_{i+1}\}} \right]^{1/2} \{v_{i+1}\}$$

Set $d(i+1) = 0$ and proceed to step 5.

Step 5**:** Create one approximate Trial vector and one diagonal coefficient.

The recurrence algorithm is:

$$a_{ii} = \{v_i\}^T [B] \{v_i\}$$

$$\{v_{(i+1)}\} = [\bar{B}]\{v_i\} - a_{ii}\{v_i\} - d_i\{v_{i-1}\}$$

$$d_{(i+1)} = \left[\frac{1}{\{V_{i+1}\}^T [D] \{v_{i+1}\}} \right]^{1/2}$$

$$\begin{Bmatrix} 0 \\ V_{i+1} \end{Bmatrix} = \left[\frac{1}{d(i+1)} v(i+1) \right]$$

Where; $[B] = [D][\bar{B}]$ and $\begin{Bmatrix} 0 \\ v_{i+1} \end{Bmatrix}$ is an approximation to new trial vector.

Step 6**:** Perform first normalization test.

The test is:

$$\{d_{i+1}\} > 10^{2-t} * |a_{ii}|$$

Pas**s:** Proceed directly to Step 5

Fail**:** Return back to Step 2. Generate a new restart vector for $\begin{Bmatrix} 0 \\ V_{i+1} \end{Bmatrix}$ and proceed to Step 5.

Step 7**:** Iterate to obtain orthogonalized vector

Designate $\{\tilde{X}_j\}$; j = 1 to p, as previously calculated and stored eigenvectors ($\emptyset$) and frequencies (f).

Perform the iterations:

$$\{V_{i+1}^{s+1}\} = \{V_{i+1}^{s}\} - \sum_{j=1}^{i} (V_j)^T [D] \{V_{i+1}^{s}\} \{V_j\} - \sum_{j=1}^{p} [\overline{X_j}]; s = 0,1,2.....$$

Until

$$\frac{MAX}{1 \le j \le i} \left| \{V_j\}^T [D] \{V_{i+1}^{(s)}\} \right| \le 10^{2-t}$$

and

$$\frac{MAX}{1 \le j \le i} \left| \{V_j\}^T [D] \{V_{i+1}^{(s)}\} \right| \le 10^{2-t}$$

If the orthogonality criterion is satisfied, proceed to Step 6. Otherwise, set problem size 'm = i' and proceed to exit.

Step 8: Normalize the orthogonalized trial vector. Compute

$$\{V_{i+1}\} = \{V_{i+1}^{s}\} \Big/ \left[\{V_{i+1}^{s+1}\} [D] \{V_{i+1}^{s+1}\} \right]^{1/2}$$

This is the new orthogonalized and normalized trial vector.

Step 9: Second normalization test and criteria of off-diagonal coefficient

a. Compute the next off-diagonal in the reduced tri-diagonal matrix form.

$$d_{i+1} = \{V_{i+1}\}^T [B] [V_i]$$

b. Verify whether the following test is met:

$$|d_{i+1}| > 10^{2-t} |a_{ii}|$$

If it has, set $i = i + 1$ and return to step 3 for the continuation of recurrence algorithm. If the test fails, set $n = i$ to reduce the problem size and proceed to exit and issue message to the user that only 'i' modes can be obtained. More than r-f modes may have been requested.

Step 10**:** Solve reduced eigenvalue problem.

a. Find the diagonal elements and off -diagonal elements of reduced tridiagonal matrix [A]

b. The m^{th} order eigenvalue problem

$$[A]\{y\} = \bar{\lambda}[y]$$

c. The reduced system eigenvectors are normalized.

$$\{y^T\} \{y_i\} = 1, I = 1, m$$

Step 11**:** Compute maximum eigenvalue errors.

a. The maximum absolute relative errors in the computed physical eigenvalues are

a.

$$\varepsilon_i = \left| \frac{(\overline{d}_{m+1})(y_{mi})}{(\lambda_i)(1 + \psi_0 \lambda_i)} \right|; i = 1, m$$

b. where ;

a. $(\overline{d}_{m+1})$ is the last off -diagonal term and (y_{mi}) is the last element in the vector $\{y_i\}$. If the physical eigenvalue $\frac{1}{\lambda_0} + \psi_0$ is corresponds to the rigid body mode, the above computation is invalid. A rigid body mode is assumed to occur $\left| \frac{1}{\lambda_0} + \psi_0 \right| \le 10^{-t/3}$ is denoted by setting the relative error ε_i equal to flat zero

b. The eigenvalues are processed in order of increasing distance from the center of range of interest, ψ_0 , to determine whether their associated ε_i values meet an acceptable relative error tolerance. (The default value is .001/n where n is the order of the stiffness matrix). The first Eigen value not meeting the tolerance test. As well as subsequent eigenvalues further removed from t the center of interest are considered to lack sufficient accuracy and are therefore rejected

c. Acceptances Eigen values are obtained in the above manner are reordered in terms of increasing physical values.

Step 12: Compute the physical Eigenvalues (λ_i) and Eigen vectors ($\emptyset_i$).

Output**:**

1. $\psi_i = \frac{1}{\lambda_i}$ (Buckling)

2. $\{\overline{\emptyset}_i\} = [c^{-1}]^T \{V\} \{y_i\}$(Buckling or Un-shifted vibration mode Problems

Where, ψ_i (shift in Physical eigenvalues)

i.e $\psi_i = \frac{1}{\lambda_i}$

$\emptyset_i$(Eigenvectors) and Natural frequencies (ω^2)

[V]= {v1},{v2}, ……{v_m}

Step1: \\Extract the modes\values.

Input: 1. λ_i , I = 1 to nr

2. Eigen values (λ_i)

3. Endfreq (m_freq)

4. nr (number of Modes)

bool termination = false

Step2**:** Frequency (f) calculation

$$f = \text{sqrt}\left(\frac{\lambda_i}{2\pi}\right)$$

For I = 0 to nr

begin

$$f = \text{sqrt}\left(\frac{\lambda_i}{2\pi}\right)$$

If (f ≥ m_freq)

termination = True;

break;

$\overline{f}$ .add (p);

$\sqrt{}_i$.add (Ø);

End

Step3: Return (termination flag,$\overline{\boldsymbol{f}}$,$\sqrt{}_{\boldsymbol{i}}$)

Save the values

Step 1: Save the values

Input: f, $\sqrt{}_i$

nmode = f. no ( );

For I =0 to nmode do

begin

Set V to dof and Save the Dof Values,

End

4. RESULTS

Model 1

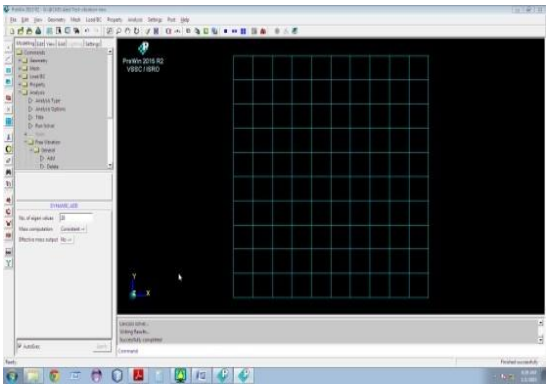

Figure 1 : Eigen values extraction usingLanczos solver

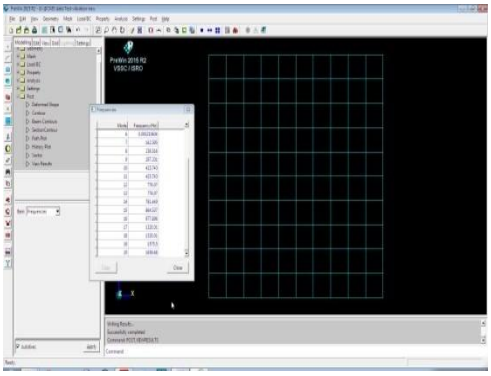

Figure.2 Extracting frequencies and modes of a structure using Lanczos algorithm

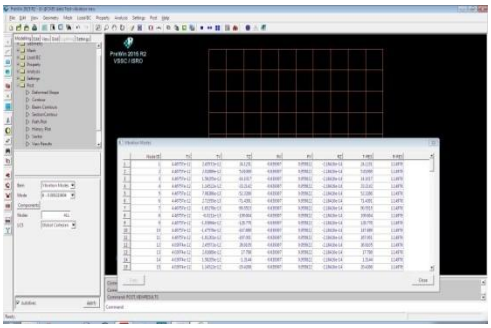

Figure.3 Vibration modes of a structure

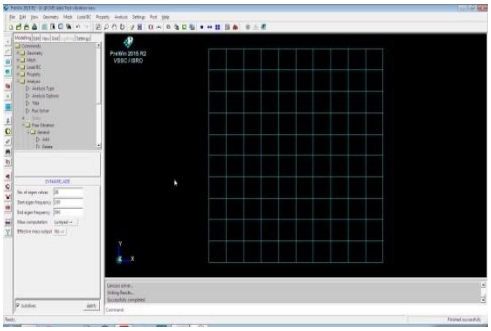

Figure.4 :Eigen values extraction using modified Lanczos solver

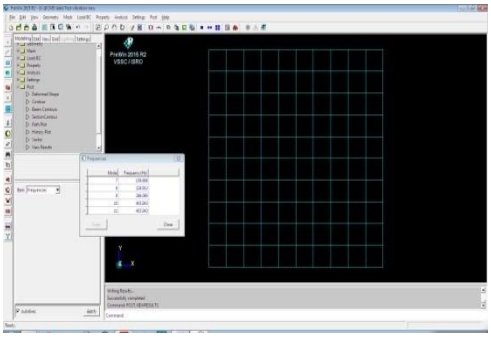

Figure.5.Frequencies and mode shape extraction using modified Lanczos Algorithm

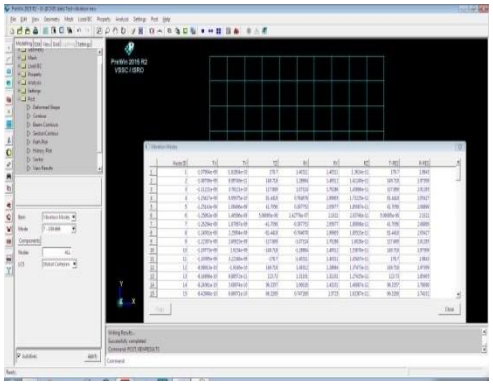

Figure.6 List of Vibration Modes within user input range

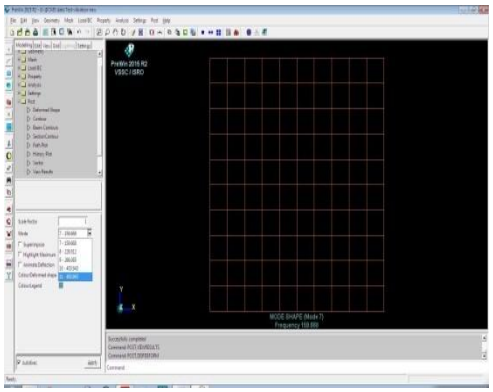

Figure.7 Modes and frequencies within user input range

5. CONCLUSION

Launch vehicle structural components are analyzed with vary large size FE Model. In this thesis, various aspects for the eigenvalues and eigenvectors extraction methods were familiarized. The details of Lanczos algorithm for solving large order system of homogeneous equations implemented in FEAST^{SMT} solver are studied. Since Lanczos method is a tridiagonal reduction method, with reduced size depends on the number of modes to be extracted, a modified version is implemented for users interested in structural frequencies in a given range. Lanczos algorithm is used multiple passes, which eliminates vector re-orthogonalization. Converged vectors of previous passes are retained and used for subsequent passes, ensuring speedy convergence of orthogonalized vectors. In order to enable users to feed frequency range, necessary changes are done in pre/post processor of FEAST^{SMT} to accept inputs.

The modified implementation is tested for various launch vehicle models and results are discussed in the thesis.

6. ACKNOWLEDGMENT

The authors duly acknowledge the help rendered by

Shri Deepak P, Scientist/Engineer's ' , SMSD/SATG/STR , VSSC , ISRO , Thiruvananthapuram , in understanding reordering algorithms for the implementation of Lanczos solution technique .

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