International Journal of Computer Applications (0975 - 8887)
Volume 147 - No.14, August 2016

Wavelet Approximations using $(\Lambda \cdot C_1)$ Matrix-Cesàro Summability Method of Jacobi Series

Shyam Lal
Banaras Hindu University
Institute of Science
Department of Mathematics
Varanasi-221005, India

Manoj Kumar
Banaras Hindu University
Institute of Science
Department of Mathematics
Varanasi-221005, India

ABSTRACT

In this paper, an application to the approximation by wavelets has been obtained by using matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials. The rapid rate of convergence of matrix-Cesàro method of Jacobi polynomials are estimated. The result of Theorem (6.1) of this research paper is applicable for avoiding the Gibbs phenomenon in intermediate levels of wavelet approximations. There are major roles of wavelet approximations (obtained in this paper) in computer applications. The matrix-Cesàro $(\Lambda \cdot C_1)$ method includes $(N, p_n) \cdot C_1$ method as a particular case. The comparison between the numerical results obtained by the $(N, p_n) \cdot C_1$ and matrix-Cesàro $(\Lambda \cdot C_1)$ summability method reveals a slight improvement concerning the reduction of the excessive oscillations by using the approach of present paper.

General Terms

Summability methods, Jacobi polynomials, wavelet expansions, wavelet approximation, projection, the Gibbs phenomenon in wavelet analysis.

Keywords

Jacobi orthogonal polynomials, matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials, $(N, p_n) \cdot C_1$ method, multiresolution analysis, orthogonal projection, the Gibbs phenomenon in wavelet analysis.

1. INTRODUCTION

Approximation of Fourier series has been studied by several researchers like Osilenker [1], Szegö [2], Zygmund [3] and Móricz ([4], [5]). Recently these results has been generalized for wavelet expansions by researchers Lal and Sharma [6], Kelly ([7], [8]), Mallat [9]. It is important to note that wavelet expansion exhibits the same oscillatory behaviour as Fourier expansion and classical summability methods can not be applied straight way in wavelet expansions because the approximation is obtained by an infinite partial sums. In this paper, a new application to the approximation by wavelets based on the matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials has been obtained. The matrix-Cesàro $(\Lambda \cdot C_1)$ method is linear and also a generalization of the $(N, p_n) \cdot C_1$ method. It depends on two parameters $\theta \in [0, \pi]$ and $r \in (0, 1)$ and, this additional degree of freedom makes possible to improve the reduction of the Gibbs phenomenon in comparison with the single parametric approach based on Cesàro summability and Abel summability method (Walter and Sen [10]). The rapid rate of convergence of the introduced method and the effect of the matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials on the wavelet approximating expansions have been discussed.
This paper is mainly concerned with the following three investigations:

(1) an application to the approximation by wavelets based on the matrix - Cesàro $(\Lambda \cdot C_1)$ summability method of the Jacobi polynomials,
(2) the rapid rate of convergence by the matrix - Cesàro $(\Lambda \cdot C_1)$ summability method and
(3) the effect of the matrix - Cesàro $(\Lambda \cdot C_1)$ summability method of the Jacobi polynomials on the wavelet approximating expansions.

Oscillatory behaviour in the neighbourhood of jump discontinuities of a function which is approximated by using these classical expansions. The excessive oscillations near the jump are called Gibbs phenomena.

2. DEFINITIONS AND PRELIMINARIES

The trigonometric Fourier series $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ is associated with a periodic real function f of coefficients

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx.$$

In this case, we consider the trigonometric polynomials

$$(s_n f)(x) = \sum_{k=-n}^{n} c_k e^{ikx}, \tag{1}$$

where n is a non negative integer. These are called "partial sums" of the Fourier series f. The $(C, 1)$ means of the partial sums $(s_n f)_{n=o}^{\infty}$ is given by

$$(\sigma_m f)(x) = \frac{1}{m+1} \sum_{k=0}^{m} (s_k f)(x)$$

$$= \sum_{k=-m}^{m} \left(1 - \frac{|k|}{m+1}\right) c_k e^{ikx}. \quad (2)$$

The Gibbs phenomenon can be reduced by replacing the corresponding partial sums $\{s_k\}_{k=0}^{m}$ by their arithmetic means $\{\sigma_m\}_{m=0}^{\infty}$. The Fourier coefficients are affected by a term, which reduces their sizes and, as a consequence, the excessive oscillations are avoided.

Matrix summability method

Consider an infinite lower triangular matrix

$$\Lambda = (a_{n,k}),\ n = 0, 1, 2, \cdots,\ k = 0, 1, 2, \cdots,$$

where $a_{n,k} = 0$ for $k > n$. The conditions of regularity of infinite lower triangular matrix Λ are $\sum_{k=0}^{n} a_{n,k} \to 1$ as $n \to \infty$, $a_{n,k} = 0$ for $k > n$ and $\sum_{k=0}^{n} |a_{n,k}| \leq M$, a finite constant (Silverman-Toeplitz [11]).

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having n^{th} partial sum $s_n = \sum_{\nu=0}^{n} u_\nu$.

The sequence to sequence transformation

$$t_n^{\Lambda} := \sum_{k=0}^{\infty} a_{n-k} s_{n-k}$$

defines the sequence $\{t_n^{\Lambda}\}$ as matrix means of the sequence $\{s_n\}$, generated by the sequence of coefficients $(a_{n,k})$. The series $\sum_{n=0}^{\infty} u_n$ is said to be summable to the sum s by matrix method if $\lim_{n\to\infty} \{t_n^{\Lambda}\}$ exists and is equal to s (Zygmund [3], p. 74).

Cesàro means of order 1 or (C_1) summability

If $\sigma_n = \frac{1}{n+1} \sum_{\nu=0}^{n} s_\nu \to s$ as $n \to \infty$ then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by Cesàro means of order 1 and write

$$\sum_{n=0}^{\infty} u_n = s(C_1).$$

Matrix-Cesàro $(\Lambda \cdot C_1)$ summability method

$(\Lambda \cdot C_1)$-means, $t_n^{\Lambda \cdot C_1}$ is given by

$$t_n^{\Lambda \cdot C_1} := \sum_{k=0}^{n} a_{n,k} \sigma_k = \sum_{k=0}^{n} a_{n,k} \frac{1}{k+1} \sum_{\nu=0}^{k} s_\nu \quad (3)$$

If $t_n^{\Lambda \cdot C_1} \to s$ as $n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable to the sum s by $(\Lambda \cdot C_1)$ -method. It is written as

$$s_n \to s(\Lambda \cdot C_1)$$

or

$$\sum_{n=0}^{\infty} u_n = s(\Lambda \cdot C_1),$$

(Dhakal [12]).

If the triangular matrix Λ-summability method is superimposed on the Cesàro means of order 1, C_1, another method of summability $(\Lambda \cdot C_1)$ i.e., Matrix-Cesàro summability method is obtained. The triangular matrix Λ-summability method includes several summability methods like

$$(C, 1) \cdot C_1, (C, \delta) \cdot C_1, (N, p_n) \cdot C_1, (N, p, q) \cdot C_1, (H, p) \cdot C_1$$

as particular cases.

(1) Harmonic C_1 means when $a_{n,k} = \frac{1}{(n-k+1)\log n}$.

(2) $(H, p) \cdot C_1$ means when

$$a_{n,k} = \frac{1}{(\log)^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^q(k+1).$$

(3) $(N, p_n) \cdot C_1$ means (Nörlund [13]) if $a_{n,k} = \frac{p_{n-k}}{P_n}$ where $P_n = \sum_{k=0}^{n} p_k \neq 0$.

(4) $(N, p, q) \cdot C_1$ means (Borwein [14]) if $a_{n,k} = \frac{p_{n-k} q_k}{R_n}$ where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$.

(5) $(C, \delta) \cdot C_1$ means if $a_{n,k} = \frac{\binom{n-k+\delta-1}{\delta-1}}{\binom{n+\delta}{\delta}}$.

Following Askey [15], the normalized Jacobi polynomials defined as

$$R_n^{(\alpha,\beta)}(cos\theta) = \frac{P_n^{(\alpha,\beta)}(cos\theta)}{P_n^{\alpha,\beta}(1)},$$

where $P_n^{\alpha,\beta}(1) = \binom{n+\alpha}{n} \neq 0, \alpha, \beta > -1$, $w(x) = (1-x)^\alpha (1+x)^\beta$, form a complete orthogonal system in $L^2([0, \pi]; w)$ and, for each $n \geq 0$ and $\alpha \geq -\frac{1}{2}$, and

$$\left|R_n^{(\alpha,\beta)}(cos\theta)\right| \leq 1.$$

Askey [15] proved the following theorem:

Theorem If $\sum_{n=0}^{\infty} a_n$ converges to s, then

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n R_n^{(\alpha,\beta)}(cos\theta) r^n$$

tends to s for $r \to 1, \theta = O(1-r)$. If $\alpha > \frac{1}{2}$ then $u(r, \theta)$ tends to s for $r \to 1, \theta \to 0$, without the restriction $\theta = O(1-r)$.

Let $\sum_{m=0}^{\infty} a_m$ be an infinite series having m^{th} partial sums

$$s_m = \sum_{\nu=0}^{m} a_\nu \ \forall\ m \geq 0.$$

If

$$a_{m,k} = \begin{cases} R_k^{(\alpha,\beta)}(cos\theta) r^k - R_{k+1}^{(\alpha,\beta)}(cos\theta) r^{k+1}, & k = 0, \cdots, m-1 \\ R_m^{(\alpha,\beta)}(cos\theta) r^m, & k = m \\ 0, & k \geq m+1 \end{cases} \quad (4)$$

then this theorem shows that when $\sigma_m = \frac{1}{m+1}\sum_{\nu=0}^{m} s_\nu \to s$ as $m \to \infty, t_m^{\Lambda\cdot C_1} \to s$ as $m \to \infty$. The method $t_m^{\Lambda} := \sum_{k=0}^{\infty} a_{m-k}s_{m-k}$ is regular. By letting,

$$\tilde{\Lambda} = \tilde{a}_{m,k} = a_{m,m-k},$$

$(\tilde{a}_{m,k})$ is also regular. Consequently, the matrix $\tilde{\Lambda}$ defines a regular summability method of $\{\sigma_m\}$, given by

$$\begin{aligned} t_m^{\Lambda\cdot C_1}(s_m) \;&=\; R_m^{(\alpha,\beta)}(cos\theta)r^m s_0 \\ &+\sum_{k=1}^{m}\left(R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right) \\ &\frac{1}{k+1}\sum_{\nu=0}^{k} s_\nu \\ = \;& R_m^{(\alpha,\beta)}(cos\theta)r^m\sigma_0 \\ &+\sum_{k=1}^{m}\left(R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right)\sigma_k \end{aligned} \tag{5}$$

Nörlund summability method

If $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex. And let us write

$$P_n = p_1 + p_2 + p_3 + \cdots + p_n.$$

The sequence to sequence transformation, viz.,

$$t_n^N = \frac{1}{P_n}\sum_{\nu=0}^{n} p_{n-\nu}s_\nu = \frac{1}{P_n}\sum_{\nu=0}^{n} p_n s_{n-\nu}, \quad (P_n \neq 0), \tag{6}$$

define the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of constants $\{s_n\}$. The series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable by Nörlund means or summable (N, p_n) to s, if $\lim_{n\to\infty} t_n^N$ exists and equals s.
The condition of regularity of the method of the summability (N, p_n) defined by (6) are

$$\lim_{n\to\infty}\frac{p_n}{P_n} = 0 \tag{7}$$

and

$$\sum_{k=0}^{n} |p_k| = 0, \qquad as \quad n \to \infty. \tag{8}$$

If $\{p_n\}$ is real and non-negative, (8) is automatically satisfied and then (7) is the necessary and sufficient condition for the regularity of the method of summation (N, p_n).

If $\sigma_n = \dfrac{1}{n+1}\sum_{\nu=0}^{n} s_\nu$ tends to s as $n \to \infty$ then $\sum_{n=0}^{\infty} u_n$ or $\{s_n\}$ is said to summable to s by Cesàro's means of order 1, i.e. $(C, 1)$ method.
The product of (N, p_n) summability with $(C, 1)$ summability defines $(N, p_n)\cdot C_1$ summability. Thus the $(N, p_n)\cdot C_1$ means is given by

$$t_n^{NC} = \frac{1}{P_n}\sum_{k=0}^{n} p_k\sigma_{n-k} = \frac{1}{P_n}\sum_{k=0}^{n} p_k\frac{1}{(n-k+1)}\sum_{\nu=0}^{n-k} s_k. \tag{9}$$

If $t_n^{NC} \to s$ as $n \to \infty$ then the series $\sum_{n=0}^{\infty} u_n$ or the sequence $\{s_n\}$ is said to be summable to the sum s by $(N, p_n)\cdot C_1$ method.

2.1 The Gibbs phenomenon in wavelet expansions

Wavelets have wide applications in the subject of orthogonal series. It has effective applications in non-stationary signals due to orthogonal and non-orthogonal sequences of wavelets.
Let the approximation space at level j be V_j and the collection $\{V_j : j \in Z\}$ be a multiresolution analysis for the space $L^2(R)$. A scale relation between two consecutive subspaces is satisfied as

$$f(\cdot) \in V_j \Rightarrow f(2\cdot) \in V_{j+1}.$$

There exists a scaling function $\phi \in L^2(R)$ such that

$$\{\phi_{j,k}(\cdot) = 2^{j/2}\phi(2^j\cdot - k), k \in Z\}$$

is an orthonormal basis of V_j. Let W_j are be the orthogonal complement of V_j in V_{j+1}, given by

$$V_j \oplus W_j = V_{j+1}. \tag{10}$$

The spaces W_j are usually called the detail spaces at level j. Under these conditions, there exists a wavelet function $\psi \in L^2(R)$ (Cohen [16], Daubechies [17], Keinert [18] and Mallat [9]), such that $\{\psi_{j,k}(\cdot) = 2^{j/2}\psi(2^j\cdot - k), k \in Z\}$ is an orthonormal basis of W_j. Since $\{V_j\}$ is a multiresolution analysis, therefore

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \cdots \subset V_j \subset V_{j+1} \subset \cdots \tag{11}$$

and for a fixed level j, it follows that

$$V_j = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{j-1}$$

and

$$L^2(R) = V_0 \oplus_{j\geq 0} W_j.$$

Let P_j be the orthogonal projection of $L^2(R)$ on to V_j. If $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $L^2(R)$ and $f \in L^2(R)$, then by equation (10),

$$P_{j+1}f = P_jf + \sum_{k\in Z} d_{j,k}\psi_{j,k}, \tag{12}$$

where $d_{j,k} = \langle f, \psi_{j,k}\rangle$ are the wavelet or detail coefficients. For each approximation level j, a sequence of $j+1$ projections

$$\{P_0f, P_1f, P_2f, \cdots, P_{j-1}f, P_jf\} \tag{13}$$

can be defined by using equation (11). Kelly [7] studied the existence of the Gibbs phenomenon in approximations by using equation (12) with some compactly supported wavelets and a bounded variation function with a jump discontinuity. Working in the same direction, Shim and Volkmer [19] exihibited the same phenomenon in wavelet expansion under non-restricted conditions on the scaling function.

3. ANALYSIS OF MATRIX-CESÀRO $(\Lambda \cdot C_1)$ SUMMABILITY METHOD OF JACOBI POLYNOMIALS

3.1 Rate of convergence

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having its n^{th} partial sums $s_n = \sum_{\nu=0}^{n} u_\nu, \forall\ n \geq 0$ such that $\sigma_1, \sigma_2, \cdots, \sigma_m, \cdots$ converges to s, where $\sigma_m = \frac{1}{m+1}\sum_{\nu=0}^{m} s_\nu$. In this paper, two new theorems have been established to analyze the rate of convergence of $t_m^{\Lambda \cdot C_1}(s_m)$ to s in the following forms:

4. THEOREMS

4.1 Theorem

Let $\alpha \in (0,1)$, such that

$$\|\sigma_m - s\|_\infty = \left\| \frac{1}{m+1}\sum_{k=0}^{m}(s_k - s)\right\|_\infty = O(\alpha^m).$$

(i) If $\alpha = r$ then there exists a positive constant K_1 such that

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty \leq K_1(1+m)r^m.$$

(ii) If $r < \alpha$ then there exists $K_2 > 0$ such that

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty \leq K_2\alpha^m.$$

(iii) If $\alpha < r$ then there exists $K_3 > 0$ such that

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty \leq K_3 r^m.$$

4.2 Theorem

Let $\alpha \in (0,1)$, such that $\|\sigma_m - s\|_\infty = O(\alpha^m)$.

(i) If $\alpha = r$ then

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty = O\left(\frac{1-r^m}{1-r}\right).$$

(ii) If $r < \alpha$ then

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty = O\left(\frac{1-\alpha^m}{1-\alpha}\right).$$

(iii) If $\alpha < r$ then

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty = O\left(\frac{1-r^m}{1-r}\right).$$

5. PROOFS

5.1 Proof of Theorem 4.1

(i) From eq.(5) it follows that

$$\begin{aligned}
\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty &\leq \left|R_m^{(\alpha,\beta)}(cos\theta)r^m s_0\right| \\
&+ \sum_{k=1}^{m}\left|R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right| \\
&\left\|\frac{1}{k+1}\sum_{\nu=0}^{k}(s_\nu - s)\right\|_\infty \\
&\leq \left|R_m^{(\alpha,\beta)}(cos\theta)r^m s_0\right| \\
&+ \sum_{k=1}^{m}\left|R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right| \\
&C_1 r^k, \quad C_1 > 0 \\
&\leq |s_0|\, r^m + C_1\sum_{k=1}^{m}\left(r^{m-k} + r^{m-k+1}\right) r^k \\
&= |s_0|\, r^m + C_1\sum_{k=1}^{m} r^m(1+r) \\
&= |s_0|\, r^m + C_1 m r^m(1+r) \\
&= \left(|s_0| + C_1 m(1+r)\right) r^m \\
&\leq \left(|s_0| + C_1(m+1)(1+r)\right) r^m \\
&\leq \left((m+1)\,|s_0| + C_1(m+1)(1+r)\right) r^m \\
&= (m+1)\left(|s_0| + C_1(1+r)\right) r^m \\
&= K_1(m+1)r^m, \quad where\ |s_0| + C_1(r+1) = K_1
\end{aligned}$$

(ii) From eq.(5) it follows that

$$\begin{aligned}
\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty &\leq \left|R_m^{(\alpha,\beta)}(cos\theta)r^m s_0\right| \\
&+ \sum_{k=1}^{m}\left|R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right| \\
&\left\|\frac{1}{k+1}\sum_{\nu=0}^{k}(s_\nu - s)\right\|_\infty \\
&\leq \left|R_m^{(\alpha,\beta)}(cos\theta)\right| |s_0|\, r^m + \sum_{k=1}^{m}\left(\left|R_{m-k}^{(\alpha,\beta)}(cos\theta)\right| r^{m-k}\right. \\
&\left. + \left|R_{m-k+1}^{(\alpha,\beta)}(cos\theta)\right| r^{m-k+1}\right) C_2\alpha^k, \quad C_2 > 0 \\
&\leq \alpha^m\,|s_0| + C_2\sum_{k=1}^{m} r^{m-k}(1+r)\alpha^k \\
&\leq \alpha^m\,|s_0| + C_2\sum_{k=1}^{m}\left(\frac{r}{\alpha}\right)^{m-k}\alpha^m(1+\alpha) \\
&= \left(|s_0| + C_2\left(\frac{1-\left(\frac{r}{\alpha}\right)^m(1+\alpha)}{1-\frac{r}{\alpha}}\right)\right)\alpha^m \\
&\leq \left(|s_0| + C_2\frac{\alpha(1+\alpha)}{\alpha - r}\right)\alpha^m \\
&= K_2\alpha^m, \quad where\ K_2 = |s_0| + C_2\frac{\alpha(1+\alpha)}{\alpha - r}.
\end{aligned}$$

(iii) From eq.(5) it follows that

$$\left\|t_m^{\Lambda \cdot C_1}(s_m) - s\right\|_\infty$$

$$\leq \left| R_m^{(\alpha,\beta)}(cos\theta) r^m s_0 + \sum_{k=1}^{m} R_{m-k}^{(\alpha,\beta)}(cos\theta) r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta) r^{m-k+1} \right| \left\| \frac{1}{k+1} \sum_{\nu=0}^{k} (s_\nu - s) \right\|_\infty$$

$$\leq \left| R_m^{(\alpha,\beta)}(cos\theta) r^m s_0 + \sum_{k=1}^{m} R_{m-k}^{(\alpha,\beta)}(cos\theta) r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta) r^{m-k+1} \right| C_3 \alpha^k, \quad C_3 > 0$$

$$\leq C_4 \left| R_m^{(\alpha,\beta)}(cos\theta) r^m + \sum_{k=1}^{m} R_{m-k}^{(\alpha,\beta)}(cos\theta) r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta) r^{m-k+1} \alpha^k \right|, C_4 = \max\{|s_0|, C_3\}$$

$$= C_4 \left| \alpha^m + \sum_{k=0}^{m-1} R_{m-k}^{(\alpha,\beta)}(cos\theta)(\alpha^k - \alpha^{k+1}) r^{m-k} \right|$$

$$\leq C_4 \left| \alpha^m + r^m \sum_{k=0}^{m-1} \left(\frac{\alpha}{r}\right)^k (1-\alpha) \right|$$

$$= C_4 \left(\alpha^m + r^m \frac{\left(1 - \left(\frac{\alpha}{r}\right)^m\right)}{\left(1 - \frac{\alpha}{r}\right)} (1-\alpha) \right)$$

$$\leq C_4 \left(r^m + r^m \frac{r(1-\alpha)}{(r-\alpha)} \right)$$

$$= C_4 \left(1 + \frac{r(1-\alpha)}{(r-\alpha)} \right) r^m$$

$$= K_3 r^m, \quad where\ K_3 = C_4 \left(1 + \frac{r(1-\alpha)}{(r-\alpha)} \right).$$

5.2 Proof of Theorem 4.2

(i) From eq.(5) it follows that

$$\left\| t_m^{\Lambda \cdot C_1}(s_m) - s \right\|_\infty \leq \left| R_m^{(\alpha,\beta)}(cos\theta) r^m s_0 \right| + \left| \sum_{k=1}^{m} R_{m-k}^{(\alpha,\beta)}(cos\theta) r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta) r^{m-k+1} \right| \left\| \frac{1}{k+1} \sum_{\nu=0}^{k} (s_\nu - s) \right\|_\infty$$

$$\leq \left| R_m^{(\alpha,\beta)}(cos\theta) \right| r^m |s_0| + \sum_{k=1}^{m} \left| R_{m-k}^{(\alpha,\beta)}(cos\theta) r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta) r^{m-k+1} \right| \left(\frac{1}{k+1} \sum_{\nu=0}^{k} \|s_\nu - s\|_\infty \right)$$

$$\leq \left| R_m^{(\alpha,\beta)}(cos\theta) \right| r^m |s_0| + \sum_{k=1}^{m} \left| R_{m-k}^{(\alpha,\beta)}(cos\theta) r^{m-k} - R_{m-k+1}^{(\alpha,\beta)}(cos\theta) r^{m-k+1} \right| \left(\frac{1}{k+1} \sum_{\nu=0}^{k} C_1 r^\nu \right), \quad C_1 > 0$$

$$\leq r^m |s_0| + C_1 \sum_{k=1}^{m} \left(r^{m-k} + r^{m-k+1} \right) \left(\frac{1 + r + r^2 + \cdots + r^k}{k+1} \right)$$

$$= r^m |s_0| + C_1 \sum_{k=1}^{m} r^{m-k}(1+r) \left(\frac{1 + r + r^2 + \cdots + r^k}{k+1} \right)$$

$$= r^m |s_0| + C_1 (1+r) \sum_{k=1}^{m} \frac{1}{k+1} \left(r^{m-k} + r^{m-k+1} + \cdots + r^m \right)$$

$$\leq r^m |s_0| + C_1 (1+r) \sum_{k=1}^{m} \left(\frac{r^{m-k} + r^{m-k} + \cdots + r^{m-k}}{k+1} \right)$$

$$= r^m |s_0| + C_1 (1+r) \sum_{k=1}^{m} \left(\frac{(k+1) r^{m-k}}{k+1} \right)$$

$$\leq r^{m-k} |s_0| + C_1 (1+r) \sum_{k=1}^{m} r^{m-k}$$

$$\leq (|s_0| + C_1(1+r)) \sum_{k=1}^{m} r^{m-k}$$

$$\leq (|s_0| + C_1(1+r)) \left(\frac{1 - r^m}{1 - r} \right)$$

$$= O\left(\frac{1 - r^m}{1 - r} \right).$$

(ii) From eq.(5) it follows that

$$\left\| t_m^{\Lambda \cdot C_1}(s_m) - s \right\|_\infty$$

$$\leq |R_m^{(\alpha,\beta)}(cos\theta)| r^m |s_0| + \sum_{k=1}^{m} \left(|R_{m-k}^{(\alpha,\beta)}(cos\theta)| r^{m-k} + |R_{m-k+1}^{(\alpha,\beta)}(cos\theta)| r^{m-k+1} \right) \left\| \frac{1}{k+1} \sum_{\nu=0}^{k} (s_\nu - s) \right\|_\infty$$

$$\leq |R_m^{(\alpha,\beta)}(cos\theta)| r^m |s_0| + \sum_{k=1}^{m} \left(|R_{m-k}^{(\alpha,\beta)}(cos\theta)| r^{m-k} + |R_{m-k+1}^{(\alpha,\beta)}(cos\theta)| r^{m-k+1} \right) \frac{1}{k+1} \sum_{\nu=0}^{k} \|s_\nu - s\|_\infty$$

$$\leq |R_m^{(\alpha,\beta)}(cos\theta)| r^m |s_0| + \sum_{k=1}^{m} \left(|R_{m-k}^{(\alpha,\beta)}(cos\theta)| r^{m-k} + |R_{m-k+1}^{(\alpha,\beta)}(cos\theta)| r^{m-k+1} \right) \left(\frac{1}{k+1} \sum_{\nu=0}^{k} C_2 \alpha^\nu \right), \ C_2 > 0$$

$$\leq |s_0| r^m + C_2 \sum_{k=1}^{m} (r^{m-k} + r^{m-k+1}) \left(\frac{1 + \alpha + \cdots + \alpha^k}{k+1} \right)$$

$$= |s_0| r^m + C_2 \sum_{k=1}^{m} (1+r) r^{m-k} \left(\frac{1 + \alpha + \cdots + \alpha^k}{k+1} \right)$$

$$\leq |s_0| \alpha^m + C_2 (1+r) \sum_{k=1}^{m} \alpha^{m-k} \left(\frac{1 + \alpha + \cdots + \alpha^k}{k+1} \right)$$

$$= |s_0|\alpha^m + C_2(1+r)\sum_{k=1}^{m}\left(\frac{\alpha^{m-k}+\alpha^{m-k+1}+\cdots+\alpha^m}{k+1}\right)$$
$$\leq |s_0|\alpha^m + C_2(1+r)\sum_{k=1}^{m}\left(\frac{\alpha^{m-k}+\alpha^{m-k}+\cdots+\alpha^{m-k}}{k+1}\right)$$
$$= |s_0|\alpha^m + C_2(1+r)\sum_{k=1}^{m}\frac{(k+1)\alpha^{m-k}}{(k+1)}$$
$$\leq (|s_0| + C_2(1+r))\sum_{k=1}^{m}\alpha^{m-k}$$
$$= (|s_0| + C_2(1+r))\left(\frac{1-\alpha^m}{1-\alpha}\right)$$
$$= O\left(\frac{1-\alpha^m}{1-\alpha}\right).$$

(iii) From eq.(5) it follows that

$$\left\|t_m^{\Lambda\cdot C_1}(s_m) - s\right\|_\infty$$

$$\leq \left|R_m^{(\alpha,\beta)}(cos\theta)r^m s_0 + \sum_{k=1}^{m}\left(R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k}\right.\right.$$
$$\left.\left.-R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right)\right|\left\|\frac{1}{k+1}\sum_{\nu=0}^{k}(s_\nu - s)\right\|_\infty$$
$$\leq \left|R_m^{(\alpha,\beta)}(cos\theta)r^m s_0 + \sum_{k=1}^{m}\left(R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k}\right.\right.$$
$$\left.\left.-R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right)\right|\left(\frac{1}{k+1}\sum_{\nu=0}^{k}\|s_\nu - s\|_\infty\right)$$
$$\leq \left|R_m^{(\alpha,\beta)}(cos\theta)r^m s_0 + \sum_{k=1}^{m}\left(R_{m-k}^{(\alpha,\beta)}(cos\theta)r^{m-k}\right.\right.$$
$$\left.\left.-R_{m-k+1}^{(\alpha,\beta)}(cos\theta)r^{m-k+1}\right)\right|\left(\frac{1}{k+1}\sum_{\nu=0}^{k}C_3\alpha^\nu\right),\ C_3 > 0$$
$$\leq |R_m^{(\alpha,\beta)}(cos\theta)|r^m|s_0| + C_3\sum_{k=1}^{m}(|R_{m-k}^{(\alpha,\beta)}(cos\theta)|r^{m-k}$$
$$+|R_{m-k+1}^{(\alpha,\beta)}(cos\theta)|r^{m-k+1})\left(\frac{1}{k+1}\sum_{\nu=0}^{k}\alpha^\nu\right)$$
$$\leq r^m|s_0| + C_3\sum_{k=1}^{m}(r^{m-k}+r^{m-k+1})\left(\frac{1}{k+1}\sum_{\nu=0}^{k}r^\nu\right)$$
$$= r^m|s_0| + C_3\sum_{k=1}^{m}(1+r)r^{m-k}\left(\frac{1+r+r^2+\cdots+r^k}{k+1}\right)$$
$$= r^m|s_0| + C_3(1+r)\sum_{k=1}^{m}\left(\frac{r^{m-k}+r^{m-k+1}+\cdots+r^m}{k+1}\right)$$
$$\leq r^m|s_0| + C_3(1+r)\sum_{k=1}^{m}\left(\frac{r^{m-k}+r^{m-k}+\cdots+r^{m-k}}{k+1}\right)$$
$$= r^m|s_0| + C_3(1+r)\sum_{k=1}^{m}\frac{(k+1)r^{m-k}}{(k+1)}$$
$$\leq (|s_0| + C_3(1+r))\sum_{k=1}^{m}r^{m-k}$$
$$= (|s_0| + C_3(1+r))\left(\frac{1-r^m}{1-r}\right)$$
$$= O\left(\frac{1-r^m}{1-r}\right).$$

6. EFFECT OF THE MATRIX-CESÀRO (Λ, C_1) METHOD OF JACOBI POLYNOMIALS ON THE WAVELET EXPANSIONS

A system of orthogonal wavelet functions is considered in this study fixing approximation level j. A sequence of $j+1$ associated projections is defined by equation (13). This plays the role of partial sums in the mathematical analysis.

6.1 Theorem

Under the previous assumptions, it follows that

$$t_j^{\Lambda\cdot C_1}(P_j f) = \frac{P_0 f}{k+1} + \sum_{k=0}^{j-1}\sum_{\nu=0}^{k}\sum_{n\in Z}\left(\frac{1-R_{j-k}^{(\alpha,\beta)}(\cos\theta)r^{j-k}}{k+1}\right)d_{\nu,n}\psi_{\nu,n}. \tag{14}$$

6.2 Proof of Theorem 6.1

From equation(5) it follows that

$$t_j^{\Lambda\cdot C_1}(P_j f) = R_j^{(\alpha,\beta)}(cos\theta)r^j\frac{P_0 f}{k+1}$$
$$+\sum_{k=1}^{j}\left(R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k} - R_{j-k+1}^{(\alpha,\beta)}(cos\theta)r^{j-k+1}\right)$$
$$\left(\frac{1}{k+1}\sum_{\nu=1}^{k}P_\nu f\right)$$
$$= R_j^{(\alpha,\beta)}(cos\theta)r^j\frac{P_0 f}{k+1} + \sum_{k=1}^{j}R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}\left(\frac{1}{k+1}\sum_{\nu=1}^{k}P_\nu f\right)$$
$$-\sum_{k=1}^{j}R_{j-k+1}^{(\alpha,\beta)}(cos\theta)r^{j-k+1}\left(\frac{1}{k+1}\sum_{\nu=1}^{k}P_\nu f\right)$$
$$= \sum_{k=0}^{j}R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}\left(\frac{1}{k+1}\sum_{\nu=0}^{k}P_\nu f\right)$$
$$-\sum_{k=0}^{j-1}R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}\left(\frac{1}{k+1}\sum_{\nu=0}^{k}P_{\nu+1}f\right)$$
$$= \frac{1}{j+1}\sum_{\nu=0}^{j}P_\nu f + \sum_{k=0}^{j-1}R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}\left(\frac{1}{k+1}\sum_{\nu=0}^{k}P_\nu f\right)$$
$$-\sum_{k=0}^{j-1}R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}\left(\frac{1}{k+1}\sum_{\nu=0}^{k}P_{\nu+1}f\right)$$

$$= \frac{1}{j+1}\sum_{\nu=0}^{j} P_\nu f + \sum_{k=0}^{j-1} R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k} \left(\frac{1}{k+1}\sum_{\nu=0}^{k} P_\nu f - \frac{1}{k+1}\sum_{\nu=0}^{k} P_{\nu+1} f\right) \quad (15)$$

Since $P_{j+1}f = P_j f + \sum_{k\in Z} d_{j,k}\psi_{j,k}$. Therefore

$$\frac{1}{j+1}\sum_{\nu=0}^{j} P_{\nu+1} f = \frac{1}{j+1}\sum_{\nu=0}^{j} P_\nu f + \frac{1}{j+1}\sum_{\nu=0}^{j}\sum_{k\in Z} d_{\nu,k}\psi_{\nu,k}.$$

This can be written as

$$\frac{1}{k+1}\sum_{\nu=0}^{k} P_{\nu+1} f = \frac{1}{k+1}\sum_{\nu=0}^{k} P_\nu f + \frac{1}{k+1}\sum_{\nu=0}^{k}\sum_{n\in Z} d_{\nu,n}\psi_{\nu,n}, \quad (16)$$

$\Rightarrow$

$$\frac{1}{k+1}\sum_{\nu=0}^{k} P_{\nu+1} f - \frac{1}{k+1}\sum_{\nu=0}^{k} P_\nu f = \sum_{n\in Z}\sum_{\nu=0}^{k} \frac{d_{\nu,n}\psi_{\nu,n}}{k+1}. \quad (17)$$

Also, from eq.(16),

$$\frac{1}{j+1}\sum_{\nu=0}^{j} P_\nu f = \frac{P_0 f}{k+1} + \sum_{k=0}^{j-1}\sum_{n\in Z}\sum_{\nu=0}^{k} \frac{d_{\nu,n}\psi_{\nu,n}}{k+1}. \quad (18)$$

Substituting eqs.(17) and (18) in eq.(15),

$$\begin{aligned} t_j^{\Lambda\cdot C_1}(P_j f) &= \frac{P_0 f}{k+1} + \sum_{k=0}^{j-1}\sum_{n\in Z}\sum_{\nu=0}^{k} \frac{d_{\nu,n}\psi_{\nu,n}}{k+1} \\ &\quad - \sum_{k=0}^{j-1} R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}\left(\sum_{n\in Z}\sum_{\nu=0}^{k} \frac{d_{\nu,n}\psi_{\nu,n}}{k+1}\right) \\ &= \frac{P_0 f}{k+1} + \sum_{k=0}^{j-1}\sum_{n\in Z}\sum_{\nu=0}^{k} \frac{d_{\nu,n}\psi_{\nu,n}}{k+1} \\ &\quad - \sum_{k=0}^{j-1}\sum_{n\in Z}\sum_{\nu=0}^{k} \frac{R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}}{k+1} d_{\nu,n}\psi_{\nu,n} \\ &= \frac{P_0 f}{k+1} + \sum_{k=0}^{j-1}\sum_{n\in Z}\sum_{\nu=0}^{k}\left(\frac{1 - R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k}}{k+1}\right) d_{\nu,n}\psi_{\nu,n}. \end{aligned} \quad (19)$$

Thus the proof of the theorem 6.1 is complete.

7. REMARKS

(1) If $\lambda_{j,k}(r,\theta) = 1 - R_{j-k}^{(\alpha,\beta)}(cos\theta)r^{j-k} = 1$, then

$$t_j^{\Lambda\cdot C_1}(P_j f) = \sigma_j(P_j f)$$

and if $P_0 f$ exihibits the Gibbs phenomenon, the excessive oscillations are not avoidable when $\lambda_{j,k}(r,\theta) \approx 0$. The oscillations are reduced under the condition $\lambda_{j,k}(r,\theta) \in (0,1)$ for suitable values of r and θ.

(2) For a collection of $j+1$ projections of the form $\{P_j f, P_{j+1} f, \cdots, P_{2j} f\}$, Theorem 6.1 can be established in the following form

$$t_{2j}^{\Lambda\cdot C_1}(P_{2j} f) = \frac{P_j f}{k+1} + \sum_{k=j}^{2j-1}\sum_{\nu=k}^{2k-1}\sum_{n\in Z}\left(\frac{1 - R_{2j-k}^{(\alpha,\beta)}(cos\theta)r^{2j-k}}{k+1}\right) d_{\nu,n}\psi_{\nu,n}. \quad (20)$$

8. APPLICATIONS

(1) Considering the function

$$f(x) = \begin{cases} \frac{(x-2)^2}{35}, & -1 < x < 0; \\ (x - \frac{1}{3})^3, & 0 \le x < 1, \end{cases} \quad (21)$$

with the size of the jump $J_f = |f(0^-) - f(0^+)| = 0.15132$.
The projection $P_8 f$ obtained by the symlet system sym4 (Mallat [9], p. 253) exihibits the Gibbs phenomenon at 0, (Figure 1) where

$$|J_{P_8 f} - J_f| = 0.0290\ (approx.).$$

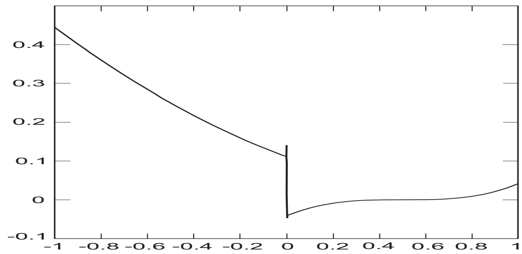

Fig. 1. Projection $P_8 f$ performed by using the wavelet $sym4$

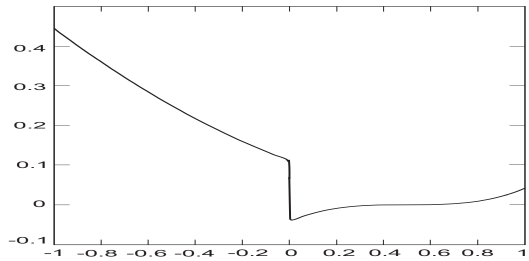

Fig. 2. $t_8^{\Lambda\cdot C_1}(P_8 f)$: Application of the matrix-Cesàro summability method of Jacobi polynomials on the projection.

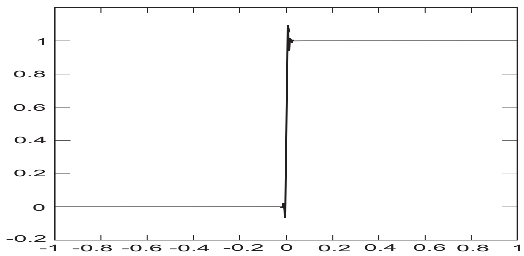

Fig. 3. Projection P_8u performed by using the wavelet $db4$.

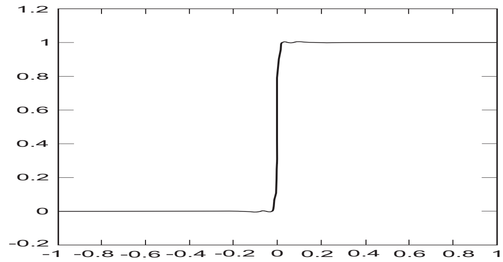

Fig. 4. $t_8^{[J]}(P_8u)$: Application of the matrix-Cesàro $(\Lambda \cdot C_1)$ summability method of Jacobi polynomials on the projection.

The matrix-Cesàro (Λ, C_1) method of Jacobi polynamials has been applied on the projections $\{P_4f, P_5f, P_6f, P_7f, P_8f\}$ with

$$\{\lambda_{8,k}(0.4580, 0.135)\}_{k=4}^{7} = \{0.9466, 0.8918, 0.7750, 0.5260\}$$

and $\alpha = \beta = 1, r = 0.4580$, $\theta = 0.135$. In this case,

$$|J_{t_8^{\Lambda \cdot C_1}(P_8f)} - J_f| = 4.6957e - 0.05\ (approx.)$$

is minimum and the values r and θ are optimal. The Gibbs phenomenon has been reduced by using the matrix-Cesàro (Λ, C_1) method of Jacobi polynamials performed on the projection $P_8f(x)$ which is shown in Figure 2.
In particular, by taking

$$a_{n,k} = \frac{p_{n-k}}{P_n}, P_n = \sum_{k=0}^{n} p_k \neq 0$$

in considered summability method (Λ, C_1), it reduces to $(N, p_n) \cdot C_1$ method t_8^{NC} has been applied on the projections $\{P_4f, P_5f, P_6f, P_7f, P_8f\}$ and it is observed that

$$|J_{t_8^{NC}(P_8f)} - J_f| = 5.2841e - 0.05\ (approx.)$$

is minimum with the optimal value $r = 0.4580$.
(2) Considering the unitary step function, u, defined by

$$u(x) = \begin{cases} 0, & x < 0; \\ 1, & x \geq 0, \end{cases} \tag{22}$$

with a jump discontinuity at 0. The Daubechies wavelet system db4, (Daubechies [17], p.195), has been used to compute the projections $\{P_4u, P_5u, P_6u, P_7u, P_8u\}$ (Figure 3). The application of the $(N, p_n) \cdot C_1$ method for $r = 0.450$ gives

$$|J_{t_8^{NC}}(P_8u) - J_u| = 1.0877e - 004\ (approx.).$$

For the computation of equation (20), the values

$$\{\lambda_{8,k}(0.4552, 0.015)\}_{k=4}^{7} = \{0.9573, 0.9058, 0.7931, 0.5452\}$$

are obtained by applying the matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials with $\alpha = \beta = \frac{1}{3}, r = 0.4452$ and $\theta = 0.015$. The effect of the matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials on the projection P_8u is more clear and effective as shown in Figure 4. Thus

$$|J_{t_8^{[J]}(P_8u)} - J_u| = 7.1724e - 0.045\ (approx.).$$

Consequently, in this paper, a better approximation to the Jump J_u is obtained by using the optimal values of r and θ in the matrix-Cesàro $(\Lambda \cdot C_1)$ method of Jacobi polynomials.

9. CONCLUSION

In this paper, the matrix-Cesàro $(\Lambda \cdot C_1)$ summability method of Jacobi polynomials is studied and it is applied to reduce the Gibbs phenomenon in wavelet analysis. The suitable estimators for the wavelet approximation of the functions belonging to generalized Lipschitz class are to be obtained using the idea of this paper.

10. ACKNOWLEDGEMENTS

Shyam Lal, one of the authors, is thankful to Prof. L. M. Tripathi, ExHead, Department of Mathematics, Banaras Hindu University, Varanasi and DST-CIMS for encouragement to this work.
Manoj Kumar is grateful to CSIR, India for providing financial assistance in the form of Junior Research Fellowship vide Reference no. 17-06/2012 (i)EU-V dated 28-09-2012 for this research work.
Authors are grateful to the referee for his valuable suggestions and comments which improve the presentation of this research paper.

11. REFERENCES

[1] Osilenker, B. (1999), "Fourier Series in Orthogonal Polynomials", World Scientific, Singapore.

[2] Szegö, G. (1975), "Orthogonal Polynomials", Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI.

[3] Zygmund, A. (1959), "Trigonometric Series", Vols. I & II, Cambridge University Press, London.

[4] Móricz, F. (2013), "Statistical Convergence of Sequences and Series of Complex Numbers with Applications in Fourier Analysis and Summability", Analysis Mathematica, Vol. 39, pp. 271-285.

[5] Móricz, F. (2004), "Ordinary convergence follows from statistical summability (C, 1) in the case of slowly decreasing or oscillating sequences", Colloq. Math., Vol. 99, pp. 207219.

[6] Lal, Shyam and Sharma, Vivek Kumar (2016), "On the Approximation of a Continuous Function $f(x, y)$ by its Two Dimensional Legendre Wavelet Expansion", International Journal of Computer Applications, Vol. 143 - No. 6, pp. 1-9.

[7] Kelly, S. E. (1996), "Gibbs phenomenon for wavelets", Appl. Comp. Harmon. Anal., Vol. 3, pp. 7281.

[8] Kelly, S. E., Kon, M.A. and Raphael, L.A. (1994), Local Convergence for Wavelet Expansions, J. Funct. Anal., 126, pp. 102138.

[9] Mallat, S. (1999), “A Wavelet Tour of Signal Processing”, Cambridge University Press, London.

[10] Walter, G. and Shen, X. (1998), “Positive estimation with wavelets, in Wavelets, Multiwavelets and their Applications”, Contemporary Mathematics, Aldroubi and Lin, eds., Vol. 216, AMS, Providence RI, pp. 6379.

[11] Toeplitz, O. (1911), “ $\ddot{u}$ber all gemeine lineare Mittelbuildungen”, Press Mathematyezno Fizyezne, 22, 113-119.

[12] Dhakal, B. P. (2010), “Approximation of Functions Belonging to the Lip α Class by Matrix-Ces$\grave{a}$ro Summability Method”, International Mathematical Forum, Vol. 5, no. 35, pp. 1729-1735.

[13] N$\ddot{o}$rlund, N. E. (1919), “Surune application des functions permutables”, Lund. Universitets Arsskrift, 16, 1-10.

[14] Borwein, D. (1958), “On Product of Sequences”, Jour. London Math. Soc., 33, 352-357.

[15] Askey, R. (1972), “Jacobi summability”, J. Approx. Theory, 5, pp. 387-392.

[16] Cohen, A. (2003), “Numerical Analysis of Wavelets Methods”, Studies in Mathematics and its Applications, Vol. 32, North-Holland, Elsevier, Amsterdam.

[17] Daubechies, I. (1992), “Ten Lectures on Wavelets”, SIAM, Philadelphia.

[18] Keinert, F. (2004), “Wavelets and Multiwavelets”, Chapman & Hall/CRC, Florida.

[19] Shim, H. T. and Volkmer, H. (1996), “On the Gibbs Phenomenon for Wavelet Expansions”, J. Approx. Theory 84, pp. 74-95.