Stability of Quartic Functional Equation in Random 2-Normed Space

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ABSTRACT
In this paper, we present the Hyers- Ulam- Rassias stability of quartic functional equation
$f(2x + y) + f(2x - y) = 4.f(x + y) + 4f(x - y) + 24f(x) - 6f(y)$
in Random 2- Normed space .

Keywords
Hyers-Ulam-Rassias stability, Quartic functional equation , Random 2- Normed space.

1. INTRODUCTION
In 1941, D.H. Hyers [2] has been studied the stability of function for a function from normed space to Banach space. He solved the problem given by Ulam [16] in 1940. He proved that for a function $f : X \to Y$, a function between normed space X and Banach space Y satisfying

$$\| f(x + y) - f(x) - f(y) \| \leq \delta$$

for each $x, y \in X$ and $\delta > 0$. Then there exists a unique additive function $T : X \to Y$ such that

$$\| f(x) - T(x) \| \leq \delta \text{ for each } x \in X.$$

Above result is generalized by Aoki [18] and Rassias [19] for additive mappings and linear mappings, respectively. A generalization of Rassias theorem was obtained by Gavruta [11] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach.

In 1990, Rassias asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [25] gave an affirmative solution to this question for $p > 1$. Gajda [25] as well Rassias and Semirl [20] investigated that one can not prove Rassias – type theorem when $p = 1$ (cf. the books of Czerwik [15], Hyers, Isac and Rassias [3]). In the similar way, using different methods, the stability problems for several functional equations have been extensively investigated by serval mathematicians([4-7], [12-14], [21-24]).

The functional equation

$$f(2x + y) + f(2x - y) = 4.f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \qquad (1.1)$$

is said to be quartic functional equation and every solution of quartic equation is said to quartic mapping. Karn Petapirak and Pasian Nakmahachalasint [9] proved the stability problem of quartic functional equation.

2. PRELIMINARIES
In this section, we recall some notations and basic definitions used in this article.

Definition 2.1 [1] : A distribution function is an element of Δ^+, where $\Delta^+ = \{f : R \to [0, 1]$; f is left-continuous, non decreasing, $f(0) = 0$ and $f(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set

$$D^+ = \{f \in \Delta^+; l^- f((+\infty) = 1\}.$$

Here $l^- f(+\infty)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in R$. For any $a \in R$, H_a is a distribution function defined by

$$H_a(x) = \begin{cases} 0 & \text{if} \quad x \leq a; \\ 1 & \text{if} \quad x > a. \end{cases}$$

The set Δ, as well as its subsets, can be partially ordered by the usual pointwise order : in this order, H_0 is the maximal element in Δ^+.

A triangle function is a binary operation on Δ^+, namely a function $\mu : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, non decreasing and which has ε_0 as unit, that is, for all f, g, h $\in \Delta^+$, we obtain :

i. $\mu(\mu(f, g), h) = \mu(f, \mu(g, h))$,
ii. $\mu(f, g) = \mu(g, f)$,
iii. $\mu(f, g) = \mu(g, f)$ whenever $f \leq g$,
iv. $\mu(f, H_0) = f$.

A t-norm is a continuous mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ such that ([0, 1], a) is abelian monoid with unit one and $c*d \geq a*b$ if $c \geq a$ and $d \geq b$ for all $a, b, c, d \in [0, 1]$.

The concept of 2-normed space was first introduced in [17].

Definition 2.2[10] : Let X be a linear space dimension greater than 1. Suppose $\| . , . \|$ is a real-valued function on $X \times X$ satisfying the following conditions :

i. $\| x, y \| = 0$ if and only if x, y are linearly dependent vectors,
ii. $\| x, y \| = \| y, x \|$ for all $x, y \in X$,
iii. $\| \lambda x, y \| = | \lambda | \| x, y \|$ for all $\lambda \in R$ and for all $x, y \in X$,
iv. $\| x + y, z \| \leq \| x, z \| + \| z, y \|$ for all $x, y, z \in X$.

Then $\| . , . \|$ is called a 2-norm on X and the pair $(X, \| . , . \|)$ is called 2-normed linear space. Some of the basic properties of 2-norm are that they are non-negative and $\|x, y + \lambda x \| = \| x, y \|$ for all $\lambda \in R$ and all $x, y \in X$.

Example 2.2[10] : Let X be a linear space with inner product $< . , . >$ and $\dim(X) \geq 2$. Then

$$\| x, y \| = \begin{vmatrix} < x, x > & < x, y > \\ < y, x > & < y, y > \end{vmatrix}^{\frac{1}{2}} \text{ is a 2-norm on X.}$$

Gelet [8] and Mursaleen introduced the notion of random 2-normed space (or in short RTN space).

Definition2.3[1] : Let X be a linear space of dimension greater than one, μ is a triangle function, and $F : X \times X \to \Delta^+$. Then F is called a probabilistic 2-norm on X and (X, F, μ) a probabilistic 2-normed space if the following conditions are fulfilled :

(i) $F_{x,y}(t) = H_0(t)$ if x and y are linearly dependent, where $F_{x,y}(t)$ denotes the value of $F_{x,y}$ at $t \in R$,

(ii) $F_{x,y} \neq H_0$ if x and y are linearly independent,

(iii) $F_{x,y} = F_{y,x}$ for every x, y in X,

(iv) $F_{\alpha x,y}(t) = F_{x,y}\left(\frac{t}{|\alpha|}\right)$ for every $t > 0$, $\alpha \neq 0$ and $x, y \in X$,

(v) $F_{x+y,z} \geq \mu(F_{x,z}, F_{y,z})$ whenever $x, y, z \in X$. If (v) is replaced by

(v′) $F_{x+y,z}(t_1 + t_2) \geq F_{x,z}(t_1 * F_{y,z}(t_2)$ for all $x, y, z \in X$ and $t_1, t_2 \in R_0^+$, then triple (X, F, *) is called a random 2-normed space.

Example 2.3[1] : Let (X, ‖ . , . ‖) be a 2-normed space with $\|x, z\| = \|x_1z_2 - x_2z_1\|$, $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for $a, b \in [0, 1]$. For all $x \in X$, $t > 0$ and non zero $z \in X$, Consider

$$F_{x,z}(t) = \begin{cases} \dfrac{t}{t+\|x,z\|} & \text{if} \quad t > 0 \\ 0 & \text{if} \quad t \leq 0; \end{cases}$$

Then (X, F, *) is a random 2-normed space.

Remark 2.3 : Every 2-normed space (X, ‖ . , . ‖) can be made a random 2-normed space in natural way, by setting $F_{x,y}(t) = H_0(t - \|x, y\|)$, for every $x, y \in X$, $t > 0$

and $a * b = \min\{a, b\}$, $a, b \in [0, 1]$.

Definition 2.4[1] : A sequence $x = (x_k)$ is convergent in (X, F, *) or simply F-convergent to l if for every $\epsilon > 0$ and $\theta \in (0, 1)$ there exists $k_0 \in N$ such that $F_{x_k - l, z}(\varepsilon) > 1 - \theta$ whenever $k \geq k_0$ and non zero $z \in X$. In this case, we write $F - \lim_{k\to\infty} x_k = l$ and l is called the F-limit of $x = (x_k)$.

Definition 2.5[1] : A sequence $x = (x_k)$ is said to be Cauchy sequence in (X, F, *) for every $\epsilon > 0$, $\theta > 0$ and non-zero $z \in X$ there exist a number $N = N(\epsilon, z)$ such that $\lim F_{x_n - x_m, z}(\varepsilon) > 1 - \theta$ for all $n, m \geq N$. RTN-space (X, F, *) is said to be complete if every F-Cauchy is F-convergent. In this case, (X, F, *) is called random 2-Banach space.

3. MAIN RESULT

In this section, we shall suppose that X and Y are linear spaces; (X, F, *) and (Z, F′, *) are random 2-normed spaces; and (Y, F, *) is a random 2-Banach space. Let ϕ be a function from X ×X to Z. A mapping $f : X \to Y$ is said to be ϕ-approximately quartic mapping if

$$F_{E_{x,y},z}(t) \geq F'_{\varphi(x,y),z}(t) \qquad (3.1)$$

Theorem 3.1 : Let us assume that a function $\phi : X \times X \to Z$ satisfies $\phi(2x, 2y) = \alpha\phi(x, y)$ for all $x, y \in X$ and $\alpha \neq 0$. Let $f : X \to Y$ be a ϕ-approximately quartic function. For some $0 < \alpha < 16$,

$$F'_{\psi(2x,2y),z}(t) \geq F'_{\alpha\psi(x,y),z}(t) \qquad (3.2)$$

and $$\lim_{n\to\infty} F'_{\psi(2^n x, 2^n y),z}(16^n t) = 1$$

for all $x, y \in X$, $t > 0$ and non-zero $z \in X$. Then there exists a unique quartic mapping $C : X \to Y$ such that

$$F_{C(x)-f(x),z}(t) \geq F'_{\psi(x,0),z}((16-\alpha)t) \qquad (3.3)$$

for all $x \in X$, $t > 0$ and non-zero $z \in X$.

Proof : Taking y = 0 in equation (3.1). Then for all $x \in X$, $t > 0$ and non-zero $z \in X$

$$F_{\frac{f(2x)}{16} - f(x),z}\left(\frac{t}{32}\right) \geq F'_{\psi(x,0)}(t) \qquad (3.4)$$

Replacing x by $2^n x$ in (3.4) and applying (3.2), we have

$$F_{\frac{f(2^{n+1}x)}{16^{n+1}} - \frac{f(2^n x)}{16^n},z}\left(\frac{t}{32.16^n}\right) \geq F'_{\psi(2^n x,0),z}(t) \geq F'_{\psi(x,0),z}\left(\frac{t}{\alpha^n}\right)$$

for all $x \in X$, $t > 0$ and non-zero $z \in X$; and for all $n \geq 0$. By replacing t by $\alpha^n t$, we obtain

$$F_{\frac{f(2^{n+1}x)}{16^{n+1}} - \frac{f(2^n x)}{16^n},z}\left(\frac{\alpha^n.t}{32.16^n}\right) \geq F'_{\psi(x,0),z}(t) \qquad (3.5)$$

It follows from $\frac{f(2^n x)}{16^n} - f(x) = \sum_{k=0}^{n-1}\left(\frac{f(2^{k+1}x}{16^{k+1}} - \frac{f(2^k x)}{16^k}\right)$ and (3.5) that

$$F_{\frac{f(2^n x)}{16^n} - f(x),z}\left(\sum_{k=0}^{n-1}\frac{\alpha^k t}{32(16^k)}\right) \geq \prod_{k=0}^{n-1} F_{\frac{f(2^{k+1}x)}{16^{k+1}} - \frac{f(2^k x)}{16^k},z}\left(\frac{\alpha^k t}{32(16^k)}\right) \geq F'_{\psi(x,0),z}(t)$$
(3.6)

for all $x \in X$, $t > 0$ and $n > 0$ where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \ldots . a_n$

By replacing x with $2^m x$, we obtain

$$F_{\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m},z}\left(\sum_{k=0}^{n-1}\frac{\alpha^k t}{32(16)^{k+m}}\right) \geq F'_{\psi(2^m x,0),z}(t) \geq F'_{\psi(x,0),z}\left(\frac{t}{\alpha^m}\right) \text{Then}$$

$$F_{\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m},z}\left(\sum_{k=m}^{n+m-1}\frac{\alpha^k t}{32(16)^k}\right) \geq F'_{\psi(x,0),z}(t)$$

for all $x \in X$, $t > 0$, $m > 0$, $n \geq 0$ and non-zero $z \in X$. Hence

$$F_{\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m},z}(t) \geq F'_{\psi(x,0),z}\left(\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{32(16)^k}}\right) \qquad (3.7)$$

for all $x \in X$, $t > 0$, $m > 0$, $n \geq 0$ and non-zero $z \in X$. Since $0 < \alpha < 16$ and $\sum_{k=0}^{\infty}\left(\frac{\alpha}{16}\right)^k < \infty$, the Cauchy criterion for convergence proves that $\left(\frac{f(2^n x)}{16^n}\right)$ is a Cauchy sequence in (Y, F, *). Since (Y, F, *) is complete, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.7) to obtain

$$F_{\frac{f(2^n x)}{16^n}-f(x),z}(t) \geq F'_{\psi(x,0),z}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{32(16)^k}}\right)$$

for all $t > 0$, $n > 0$ and non-zero $z \in X$. Thus we get

$$F_{C(x)-f(x),z}(t) \geq F_{C(x)-\frac{f(2^n x)}{16^n},z}\left(\frac{t}{2}\right) * F_{\frac{f(2^n x)}{16^n}-f(x),z}\left(\frac{t}{2}\right)$$
$$\geq F'_{\psi(x,0),z}\left(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{16.16^k}}\right)$$

for large n. Taking the limit as $n \to \infty$ and applying the definition of Random 2-Normed Space, we get

$$F_{C(x)-f(x),z}(t) \geq F'_{\psi(x,0),z}((16-\alpha)t)$$

Change x and y by $2^n x$ and $2^n y$, respectively, in (3.1), we obtain

$$F_{\frac{E2^n x, 2^n y}{16^n},z}(t) \geq F'_{\psi(2^n x, 2^n y),z}(16^n t)$$

for all $x, y \in X$, $t > 0$ and non-zero $z \in X$.

Since $\lim_{n\to\infty} F'_{\psi(2^n x, 2^n y),z}(16^n t) = 1$,

We conclude that C satisfy (1.1). To prove the uniqueness of the quartic function C, let us suppose that there exist a quartic function $Q : X \to Y$ which satisfy (3.3). For fix $x \in X$, clearly $C(2^n x) = 16^n C(x)$ and

$Q(2^n x) = 16^n Q(x)$ for all $n \in N$.

It follows from (3.3) that

$$F_{C(x)-Q(x),z}(t) = F_{\frac{C(2^n x)}{16^n}-\frac{Q(2^n x)}{16^n},z}(t)$$
$$\geq F_{\frac{C(2^n x)}{16^n}-\frac{f(2^n x)}{16^n},z}\left(\frac{t}{2}\right) * F_{\frac{f(2^n x)}{16^n}-\frac{Q(2^n x)}{16^n},z}\left(\frac{t}{2}\right)$$
$$\geq F'_{\psi(2^n x,0),z}\left(\frac{16^n(16-\alpha)t}{2}\right)$$
$$\geq F'_{\psi(x,0),z}\left(\frac{16^n(16-\alpha)t}{2\alpha^n}\right)$$

So, $$F'_{\psi(x,0),z}\left(\frac{16^n(16-\alpha)t}{2\alpha^n}\right) = 1$$

Thus, $F_{C(x)-Q(x),z}(t) = 1$ for all $x \in X$, $t > 0$ and non-zero $z \in X$. Hence $C(x) = Q(x)$.

4. REFERENCES

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