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On the Near-Common Neighborhood Graph of a Graph

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ABSTRACT

The near common-neighborhood graph of a graph G, denoted by $ncn(G)$, is the graph on the same vertices of G, two vertices being adjacent in $ncn(G)$ if there is at least one vertex in G not adjacent to both of them. A graph is called near-common neighborhood graph if it is the near-common neighborhood of some graph. In this paper we introduce the near-common neighborhood of a graph, the near common neighborhood graph, near-completeness number of a graph, basic properties of these new graphs are obtained and interesting results are established.

Keywords

Near-common neighborhood graph (of graph), common neighborhood graph (of graph), Near-common neighborhood graph, Near-completeness number (of graph)

1. INTRODUCTION

In this paper, we are concerned with simple graphs, that is graphs without multiple, weighted or directed edges, and without self–loops. The distance, $d(u, v)$, between the vertices u and v in G is the length of a shortest $u - v$ path in G. The diameter, $diam(G) = d$, of G is the largest of the distances between two vertices in G. Let further $e(v) = \max d(v, u) : u \in V(G)$ denote the eccentricity of the vertex v. The radius $r(G)$ and the diameter $diam(G)$ are the minimum and maximum eccentricity, respectively. The set of all vertices $v \in V(G)$ with minimum eccentricity is known as the center of G. The graph G is self-centered if all vertices have equal eccentricity.
Let G be such a graph with vertex set $\mathbf{V} = \mathbf{V}(G) = \{v_1, v_2, \dots, v_n\}$. Thus, the number of vertices G is n.
The adjacency matrix of the graph G is the symmetric square matrix $\mathbf{A} = \mathbf{A}(G) = ||a_{ij}||$ of order n whose (i, j)-entry is defined as

$$a_{ij} = \begin{cases} 1 & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise .} \end{cases} \tag{1}$$

A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . For more details about parameters of domination number, the reader refer to [7].

DEFINITION 1. *[3] Let G be a simple graph with vertex set $\{v_1, v_2, \dots, v_n\}$. The common neighborhood graph (con-graph) of G, denoted by $con(G)$, is the graph with vertex set $\{v_1, v_2, \dots, v_n\}$, in which two vertices are adjacent if and only if they have at least one common neighbor in the graph G.*

In politics relation between the countries will be stronger if they have the same another friend country or have the same enemy. Similarly, in the food webs. Let the vertices of a graph be species in an ecosystem. Include an edge from v to u if v preys on u or u preys on v. Consider a corresponding graph. The species in the ecosystem will be the vertices and any two vertices u and v are adjacent if there is one vertex not preys for any of them. In communication by considering the Vertices are transmitters and receivers and any two vertices v and u are adjacent if the message sent at u can be received at v or opposite, we can define a graph with transmitters and receivers as vertices and any two vertices v and u are adjacent if there is a receiver w such that message from u and v not received at w, in this case u and v are not interfering in w. Similarly for the influences opinion on the social network. These situations and many others motivated us to introduce the near common neighborhood graph.
In this paper, we introduce The near common neighborhood graph of G and near-completeness number of a graph, basic properties of near-common neighborhood graph $\mathbf{ncn}(G)$ and some interesting results are obtained.

2. NEAR-COMMON NEIGHBORHOOD GRAPH OF A GRAPH

Let G be a graph. The near common neighborhood graph (ncn-graph) of G denoted by $\mathbf{ncn}(G)$ is a graph with the same vertices as G in which two vertices u and v are adjacent if there is at least one vertex $w \in V(G)$ not adjacent to both of u and v.
For $i \neq j$, the non-common neighborhood set of the the vertices v_i and v_j, denoted by $\Gamma^{'}(v_i, v_j)$, is the set of vertices, different from v_i and v_j, that are not adjacent to both v_i and v_j .
We now introduce a symmetric square matrix $||a^{\prime}_{ij}||$ of order n, whose (i, j)-entry is defined as

$$a^{\prime}_{ij} = \begin{cases} 1 & \text{if } |\Gamma^{\prime}\{v_i, v_j\}| \geq 1 \text{ and } i \neq j; \\ 0 & \text{otherwise .} \end{cases} \tag{2}$$

Evidently, this matrix can be viewed as the adjacency matrix of the near common neighborhood graph.

Denote by $\overline{G}$ the complement of the graph G. As usual, P_n , C_n , W_n and K_n , are the n-vertex path, cycle,wheel and complete graph. In addition, $K_{a,b}$ is the complete bipartite graph on $a + b$ vertices. The following simple relations can easily be verified.

PROPOSITION 1. *For any complete graph K_n and totally disconnected graph $\overline{K_n}$, we have*

(*1*) $\mathbf{ncn}(K_n) = \overline{K_n}$.

(*2*) $\mathbf{ncn}(\overline{K_n}) = K_n, n \geq 3$.

PROPOSITION 2. *For any path P_n,*

$$\mathbf{ncn}(P_n) = \begin{cases} \overline{K_n}, & \text{if } n = 2, 3; \\ 2K_2, & \text{if } n = 4; \\ K_n, & \text{if } n \geq 7. \end{cases}$$

The near-common neighborhood of P_5 and P_6 in figures 1,2.

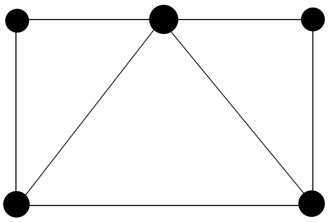

Fig. 1. Near-common neighbourhood graph of P_5

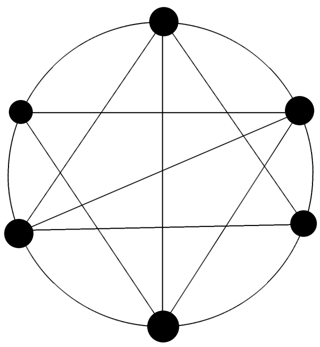

Fig. 2. Near-common neighbourhood graph of P_6

PROPOSITION 3. *For any complete bipartite graph $K_{a,b}$,*

$$\mathbf{ncn}(K_{a,b}) = \begin{cases} \overline{K_a \cup K_b}, & \text{if } a, b \geq 3; \\ \overline{K_{a+b}}, & \text{if } a, b < 3; \\ \overline{K_a} \cup K_b, & \text{if } a \in \{1, 2\} \text{ and } b \geq 3; \\ \overline{K_b} \cup K_a, & \text{if } b \in \{1, 2\} \text{ and } a \geq 3\ . \end{cases}$$

PROPOSITION 4. *For any wheel W_n, with $n \geq 8$ vertices,*

$$\mathbf{ncn}(W_n) = K_{n-1} \cup K_1\ .$$

PROPOSITION 5. *For any cycle C_n on n vertices*

$$\mathbf{ncn}(C_n) = \begin{cases} \overline{K_n} & \text{if } n = 3, \text{ or } 4; \\ C_5 & \text{if } n = 5; \\ K_n & \text{if } n \geq 7\ . \end{cases}$$

The near-common neighborhood of C_6 in figure 3.

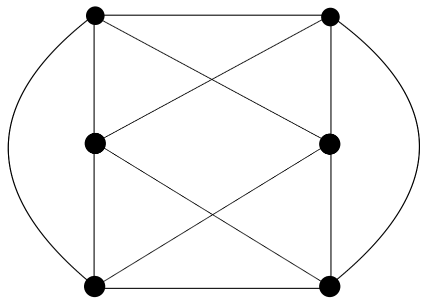

Fig. 3. Non-common neighbourhood graph of C_6

The following results are immediate.

PROPOSITION 6. $\mathbf{ncn}(K_2 \cup K_1) = K_2 \cup K_1$, $\mathbf{ncn}(2K_2) = 2K_2$, $\mathbf{ncn}(K_2) = \overline{K_2}$.

PROPOSITION 7. *Let G be a graph on $n \geq 3$ vertices and $\Delta(G) < n - r - 1$ for some positive integer r. If G has r isolated vertices, then $\mathbf{ncn}(G) \cong K_n$.*

PROPOSITION 8. *For any graph G on n vertices,*

$$\mathbf{ncn}(G \cup \overline{K_m}) \cong K_{n+m},$$

where $m \geq 2$.

THEOREM 2. *For any graph $G = (V, E)$, $\mathbf{ncn}(G) = \mathbf{con}(\overline{G})$.*

PROOF. Let uv be any edge in $\mathbf{ncn}(G)$. Then there exists at least one vertex w in V such that w is not adjacent to both u and v in G. Therefore w is adjacent to both u and v in $\overline{G}$. Thus uv is an edge in $\mathbf{con}(\overline{G})$.
Similarly, suppose xy be any edge in $\mathbf{con}(\overline{G})$, then there exist at least one vertex z in $\overline{G}$ which adjacent to both x and y in $\overline{G}$. Therefore z is not adjacent to both x and y in G. Thus xy is an edge in $\mathbf{ncn}(G)$. Hence $\mathbf{ncn}(G) = \mathbf{con}(\overline{G})$.
□

A graph which is isomorphic to its complement is said to be a self-complementary graph. These graphs have a high degree of structure, from Theorem immediately the following result.

PROPOSITION 9. *For any self complementary graph G, $\mathbf{ncn}(G) = \mathbf{con}(G)$.*

PROPOSITION 10. *For any graph G, $\mathbf{ncn}(G)$ is not tree.*

A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Let $\tau_2(G)$ be the number of dominating set of size two. Then the number of edges can be find out as in the following theorem.

THEOREM 3. *Let G be a graph and let q_{ncn} is the number of edges of the graph $\mathbf{ncn}(G)$. Then $q_{ncn} = \frac{n(n-1)}{2} - \tau_2(G)$.*

PROOF. In $\mathbf{ncn}(G)$ any two vertices u and v are not adjacent if and only if $V - (N_G[u] \cup N_G[v]) = \phi$. Then any two vertices u and v are not adjacent in $\mathbf{ncn}(G)$ if $\{u, v\}$ is dominating set in G. Therefore all the vertices are adjacent in $\mathbf{ncn}(G)$ except the pairs of vertices which form dominating set in G. Hence $q_{ncn} = \frac{n(n-1)}{2} - \tau_2(G)$. $\square$

PROPOSITION 11. *For any graph $G = (V, E)$ with n vertices and domination number $\gamma(G) \geq 3$, $\mathbf{ncn}(G) = K_n$.*

THEOREM 4. *Let G_1 and G_2 be any two graphs with n_1 and n_2 vertices respectively and without vertex of full degree. Then $\mathbf{ncn}(G_1 \cup G_2) = K_{n_1+n_2}$.*

PROOF. let u and v be any two vertices in $G_1 \cup G_2$. Then there are tow cases:
Case 1. If both of u and v belong either to G_1 or G_2, then if u and v belong to G_1 then every vertex in G_2 not adjacent to both of u and v. Similarly,if u and v belong to G_2. therefore u and v are adjacent in $\mathbf{ncn}(G_1 \cup G_2)$.
Case 2. If u in G_1 and v in G_2, since there is no vertex of full degree in G_1 and G_2 there is at least one vertex in G_1 and one vertex in G_2 not adjacent both of u and v.
Hence, any two vertices in $\mathbf{ncn}(G_1 \cup G_2)$ are adjacent. $\square$

Also the Theorem 4 can be generalize as following.

PROPOSITION 12. *There are many infinite graphs G with the property $\mathbf{ncn}(G) = G$.*

THEOREM 5. *Let $G_1, G_2, ..., G_m$ be any graphs with $m \geq 3$ and $n_1, n_2, ...n_m$ vertices respectively and let $G = \bigcup_{i=1}^{m} G_i$. Then*

$$\mathbf{ncn}(G) = K_{n_1+n_2+...+n_m}.$$

PROPOSITION 13. *For any graph $G = \bigcup_{i=1}^{r} G_i$ with n vertices and $r \geq 3$, $\mathbf{ncn}(G) = K_n$.*

For each vertex $v \in V(G)$, the *domination value* of v denoted by $DV(v)$ and defined as the number of minimum dominating sets which contains v. Here we denote by $DV_2(v)$ to the number of dominating sets of size two which contains v.

PROPOSITION 14. *Let $G = (V, E)$ be a graph on n vertices with $\gamma(G) \leq 2$ and $\mathbf{ncn}(G)$, is its near-common neighborhood graph and let $\delta(\mathbf{ncn}(G))$ and $\Delta(\mathbf{ncn}(G))$ be the minimum and maximum degree of $\mathbf{ncn}(G)$. Then*

(i) $\delta(\mathbf{ncn}(G)) = n - 1 - \max_{v \in V(G)} DV_2(v)$.
(ii) $\Delta(\mathbf{ncn}(G)) = n - 1 - \min_{v \in V(G)} DV_2(v)$.

PROOF. **(i)** Let u be any vertex in G. Since any two vertices x, y not adjacent in $\mathbf{ncn}(G)$ if and only if $\{x, y\}$ is dominating set in G, then $\mathbf{degre}(u) = n - 1 - DV_2(u)$. Therefore $\delta(\mathbf{ncn}(G)) = n - 1 - \mathbf{max}_{u \in V(G)} DV_2(u)$.
(ii) Similarly, as in (i) $\Delta(\mathbf{ncn}(G)) = n - 1 - \mathbf{min}_{v \in V(G)} DV_2(v)$ $\square$

PROPOSITION 15. *Let $G = (V_1, V_2, E)$ be bipartite graph which is not complete bipartite graph with $|V_1| \geq 3$ and $|V_2| \geq 3$. Then $\mathbf{ncn}(G)$ is connected.*

PROOF. Obviously from the definition of near-common neighborhood graph of a graph G, if $G = (V_1, V_2, E)$ be bipartite graph which is not complete bipartite graph with $|V_1| \geq 3$ and $|V_2| \geq 3$, then $\mathbf{ncn}(G)$ contains two cliques Q_1, Q_2 of orders $|V_1|, |V_2|$ and at least one edge between the two cliques. Hence $\mathbf{ncn}(G)$ is connected. $\square$

PROPOSITION 16. *For any bipartite graph G with $p \leq 3$ vertices, $\mathbf{ncn}(G)$ is disconnected and has exactly two components.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$. The sum $G_1 + G_2$ is defined as $G_1 \cup G_2$ with all edges joining vertices of V_1 to vertices of V_2.

PROPOSITION 17. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then $\mathbf{ncn}(G_1 + G_2) = \mathbf{ncn}(G_1) \cup \mathbf{ncn}(G_2)$.*

PROOF. Let uv be any edge in $\mathbf{ncn}(G_1 + G_2)$. Then there exist at least one vertex w in which not adjacent to both u and v and it is obviously from the definition of of $G_1 + G_2$, the all the vertices u, v and w together must belong to V_1 or V_2. Therefore the edge uv either in $\mathbf{ncn}(G_1)$ or $\mathbf{ncn}(G_2)$. Thus uv is an edge in $\mathbf{ncn}(G_1) \cup \mathbf{ncn}(G_2)$.
Similarly If uv be any edge in $\mathbf{ncn}(G_1) \cup \mathbf{ncn}(G_2)$, then uv belong to $\mathbf{ncn}(G_1)$ or $\mathbf{ncn}(G_2)$, if uv is an edge in $\mathbf{ncn}(G_1)$, then there exist at least one vertex w not adjacent to both u and v and u,v and w are vertices in V_1. Therefore uv is an edge in $\mathbf{ncn}(G_1 + G_2)$, similarly if uv belong to $\mathbf{ncn}(G_2)$, we get uv is an edge in $\mathbf{ncn}(G_1 + G_2)$.
Hence $\mathbf{ncn}(G_1 + G_2) = \mathbf{ncn}(G_1) \cup \mathbf{ncn}(G_2)$. $\square$

PROPOSITION 18. *Let G be a strongly regular graph with parameters (n, k, λ, μ). Then*

$$\mathbf{ncn}(G) = \begin{cases} \overline{K_n}, & \text{if } n = 2k - \lambda \text{ and } \mu = \lambda + 2; \\ K_n, & \text{if } n > 2k - \lambda \text{ and } \mu = \lambda + 2; \\ G, & \text{if } n = 2k - \mu + 2 \text{ and } \mu < \lambda + 2; \\ \overline{G}, & \text{if } n = 2k - \lambda \text{ and } \mu > \lambda + 2. \end{cases}$$

PROOF. In view of the definition of near-common neighborhood graph of a graph and the fact that for any strongly regular graph, we have

$$|\Gamma^{'}(u, v)| = \begin{cases} n - 2k + \lambda, & \text{if } u \text{ is adjacent with } v; \\ n - 2k + \mu - 2, & \text{if } u \text{ and } v \text{ are not adjacent.} \end{cases}$$

the proof is straightforward. $\square$

PROPOSITION 19. *Let G be the complement of strongly regular graph which is triangle free. Then $\mathbf{ncn}(G) = G$.*

THEOREM 6. *Let G be a graph with n vertices. Then $\mathbf{ncn}(G) = \overline{K_n}$ if and only if G is one of the following graphs:*

(i) $G \cong K_n$.
(ii) $G \cong \overline{K_2}$.

(iii) $G \cong K_n - \{e_1, ..., e_i\}$, *where* $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ *and* $\{e_1, ..., e_i\}$ *is matching in* K_n.

PROOF. If G is one of the following graphs K_n or $\overline{K_2}$ or $K_{s,t}$, where $s, t \leq 2$, then it is easy to check that $\mathbf{ncn}(G) = \overline{K_n}$. Similarly if $G \cong K_n - \{e_1, ..., e_i\}$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $\{e_1, ..., e_i\}$ is matching in K_n, then for any two vertices u and v in G there is no vertex in G not adjacent to both u and v since any vertex must be adjacent either to u or v.

Conversely, Suppose that G is a graph with n vertices such that $\mathbf{ncn}(G) = \overline{K_n}$, then we have two cases:
Case 1. If G is disconnected graph, and G should has at most two components because if it has more than two components $\mathbf{ncn}(G)$ will be complete graph by Proposition 13. Suppose that G has two component G_1 and G_2 and if G_1 or G_2 has more than one vertex then there is at least one edge in $G \cong G_1 \cup G_2$. Therefore $G \cong \overline{K_2}$.
Case 2. If G is connected graph. In this case the diameter of the graph should be less than or equal two, because if the diameter of G is three and u and v any two vertices such that $d(u, v) = 3$ and uw_1w_2v is the shortest path between them then u and w_1 are adjacent in $\mathbf{ncn}(G)$. Similarly if $diam(G) > 3$ we will get at least two vertices are adjacent in $\mathbf{ncn}(G)$. Therefore we have only two subcases:
Subcase 1. If $diam(G) = 1$, that means G is complete graph K_n.
Subcase 2. $diam(G) = 2$. In this subcase G must satisfied that for any two vertices in G there is no vertex not adjacent to both of them. That means for any two vertices u and v in G there is no vertex in G has distance two from u and distance two from v in another meaning any vertex w different from u and v must be adjacent to either u or v vertices. Therefor G is can be construct by deleted any matching from G and since the matching number of K_n is $\lfloor \frac{n}{2} \rfloor$. Hence, $G \cong K_n - \{e_1, ..., e_i\}$, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $\{e_1, ..., e_i\}$ is matching in K_n. □

PROPOSITION 20. *For any multipartite graph* $K_{n_1,n_2,...,n_r}$ *with* $n_1 \geq 3, n_2 \geq 3, ...n_r \geq 3$, $\mathbf{ncn}(K_{n_1,n_2,...,n_r}) = \bigcup_{i=1}^{r} K_{n_i}$.

DEFINITION 7. *For any positive integer* k *the* k*-iterated non-Common neighborhood graph* $\mathbf{ncn}^k(G)$ *is a graph with the same vertex set of* G *and any two distinct vertices in* $\mathbf{ncn}^k(G)$ *are adjacent if there exist a vertex* w *in* $\mathbf{ncn}^{k-1}(G)$ *not adjacent to both* u *and* v, *where* $\mathbf{ncn}^0(G) = G$

In almost graphs G with n vertices by taking k-iterated non-Common neighborhood graph for some positive integer k we will get the graph $\overline{K_n}$. This we give a new parameter for the graph as the following definition.

DEFINITION 8. *Let* $G = (V, E)$, *be a graph with* $p \geq 1$ *vertices. The near-Common neighborhood completeness number denoted by* $\mathbf{cncn}(G)$ *and defined as* $\mathbf{cncn}(G) = \min\{k \in \mathbf{N} | \mathbf{ncn}^k(G) \cong \overline{K_p}\}$.

DEFINITION 9. *A graph* G *is called* $\mathbf{ncn}$*-graph, if there exist at least one graph* H *such that* $\mathbf{ncn}(H) = G$.

PROPOSITION 21.

(i) *Any complete graph is* $\mathbf{ncn}$*-graph.*
(ii) *Any totally disconnected graph is* $\mathbf{ncn}$*-graph.*
(iii) *Let* G *be any connected graph which contains* t *cliques* $Q_1, Q_2, ..., Q_t$ *with sizes* $n_1 \geq 3, n_2 \geq 3, ..., n_t \geq 3$ *respectively. Then* G *is* $\mathbf{ncn}$*-graph.*
(iv) *Any tree is not* $\mathbf{ncn}$*-graph.*

3. CONCLUSION

In this paper, we have started to study the near common neighborhood graph of a graph and we defined the near neighborhood graph, near-completeness number of a graph, some basic properties, relations and results are obtained and still there are many problems can to study in this topic, we will state some open problems for future work.

(1) what is the sufficient and necessary conditions for a graph to be near common neighborhood graph.
(2) what is the sufficient and necessary conditions for a graph G such that $\mathbf{ncn}(G) = G$.
(3) which graphs have near-completeness number.
(4) what is the sufficient and necessary conditions for a graph to satisfy $\mathbf{ncn}(G) = \mathbf{con}(G)$.

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