

Variational Iteration Method for Solving Two Dimensional Volterra - Fredholm Nonlinear Integral Equations

M. H. Saleh
Mathematics Department,
Faculty of Science,
Zagazig University, Zagazig,
Egypt

D. Sh. Mohamed
Mathematics Department,
Faculty of Science,
Zagazig University, Zagazig,
Egypt

R. A. Taher
Mathematics Department,
Faculty of Science,
Zagazig University, Zagazig,
Egypt

ABSTRACT

In this paper we investigate the numerical solution of two dimensional Volterra - Fredholm integralequations by Variational iteration method. Two numerical examples are given to illustrate themethod.

Keywords

Variational iteration method ,Volterra-fredholm ,Lagrange multiplier ,Two dimensional equations.

1. INTRODUCTION

Many works have been focusing on the development of more advanced and efficient methods for the two dimensional nonlinear Volterra integral equations such as Collocation method , Legendre polynomials,Tau method , Iterative method ,Extrapolation methods and others see [1 - 10] .

In 1987 Inokuti in [17] proposed a general Lagrange multiplier method for solving nonlinear operator

forms as follows :

$$Lu + Nu = g(x) \quad (1)$$

where L is a linear operator , N is a nonlinear operator , g(x) is a known analytic function and u is an unknown function that to be determind. Also on the supposition that u₀ is the solution of LU = 0 , It can be found via Variational theorem in [12 ,17] .The Inokuti method is modified by he which it can be written

$$u_{n+1}(x_0) = u_n(x_0) + \int_0^{x_0} \lambda(Lu_n + Nu_n - g) ds \quad (2)$$

where u₀ is an initial approximation and u_n is are stricted variation , in other words $\delta u_n = 0$ in[12] , for arbitrary of x₀ ,author has converted Eq.(2) to the following equation [13 , 14].

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + Nu_n(s) - g(s)) ds \quad (3)$$

the above integral in Eq.(3) is called a correction function and index n denotes the nth approximation. Also Eq.(3)is called Variational iteration method (VIM). In [15 , 16] this method is used for solving nonlinear problems .The Variational iteration method is effective and easy for linear problem because

exact solution can be given by one iteration . In the above process Eq.(3) is written in following form

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(Lu_n(s) + Nu_n(s) - g(s)) ds \quad (4)$$

In Eq.(4) by using u₀(x) as an initial approximation , we obtain a sequence of approximations of exact solution of Eq.(1). for illustration of effectively , easily and accurately a large class of nonlinear problems with approximations which converge quickly , Exact solution of integral equations is approximated by (VIM) in [18] .Also in [11] ,(VIM) is applied for solving nonlinear system of ordinary differential equations . Thus , we can say Variational iteration method is a well known method to solve nonlinear equations . The convergence of (VIM) is discussed in [20]. In this work use (VIM) for solving nonlinear of mixed Volterra -Fredholm integral equations , for convenience we consider mixed Volterra – Fredholm integral equations as follows in [10] ,

$$u(x, y) = f(x, y) + \int_0^y \int_{\Omega} G(x, y, s, t, u(s, t)) ds dt \quad (5)$$

$$(x, y) \in \Omega \times [0, T]$$

where ,we suppose that Ω is colsed subset of R , f(x, y) and G(x, y, s, t, u(s, t)) are analytic on

$$D = \Omega \times [0, T] ,$$

\tilde{D} respectively such that :

$$\tilde{D} = \{(x, y, s, t) | 0 \leq t \leq y \leq T, (x, s) \in \Omega^2\} \times R$$

2. NONLINEAR OF MIXED VOLTERRA -FREDHOLM INTEGRAL

The nonlinear integral or differential equations are important in applied sciences and engineering. In [11] linear and nonlinear of ordinary equations are solved by using he is Variational iteration method and in [19] the exact solution of nonlinear of mixed Volterra -Fredholm integral equations , is approximated by Variational Iteration method .Now we consider this problem as follows :

$$u(x, y) = f(x, y) + \int_0^y \int_{\Omega} G(x, y, s, t, u(s, t)) ds dt \quad (6)$$

$$(x, y) \in \Omega \times [0, T]$$

we try obtain an effective method for solving nonlinear of mixed Volterra -Fredholm integral equations ,

so we suppose $\Omega=[0,1]$

and by differentiating form both sides of Eq.(6) respect to y , we have

$$\frac{\partial u}{\partial y} - \frac{\partial f}{\partial y} - \int_0^1 G(x, y, s, y, u(s, y)) ds - \int_0^y \int_0^1 \frac{\partial G}{\partial y} ds dt = 0 \quad (7)$$

now ,we use (VIM) for Eq.(7) , so we obtain following

$$u_{n+1}(x, y) = u_n(x, y) + \int_0^y \lambda \left[\frac{\partial u_n(x, \tau)}{\partial \tau} - \frac{\partial f(x, \tau)}{\partial \tau} - \int_0^1 G(x, \tau, s, \tau, u(s, \tau)) ds - \int_0^\tau \int_0^1 \frac{\partial G}{\partial \tau} ds dt \right] d\tau \quad (8)$$

$n=0,1,2,\dots$

by using the Variational theorem and effect δ in both sides of Eq.(10) and also with assume

$$\delta u_{n+1}(x, y) = 0,$$

and

$$\delta \left[- \int_0^1 G(x, y, s, y, u(s, y)) ds - \int_0^y \int_0^1 \frac{\partial G}{\partial y} ds dt \right] = 0$$

We have

$$u_{n+1}(x, y) = (1 + \lambda(y)) \delta u_n(x, y) - \int_0^1 \hat{\lambda}(\tau) \delta u_n(s, \tau) d\tau = 0$$

By considering

$$\delta u_{n+1}(x, y) = 0$$

optimal value of λ is given by the following differential equation :

$$1 + \lambda(y) = 0, \quad \hat{\lambda}(\tau)|_{\tau=y} = 0 \quad (9)$$

so , we get the Lagrange multiplier as follows :

$$\lambda = -1$$

by substituting $\lambda = -1$ in Eq.(8) , we obtain an algebraic for finding solution of Eq.(5).

3. NUMERICAL EXAMPLES

In this section ,we solve two examples of the nonlinear of mixed Volterra - Fredholm integral equations which have solved in [19] by using Collocation method numerical results show that our proposed method has a high accuracy and also VIM better than Collocation method.

Example 1.

Consider the following nonlinear of mixed Volterra - Fredholm integral equations:

$$u(x, y) = f(x, y) + \int_0^y \int_0^1 u^2(t, s) dt ds$$

where

$$f(x, y) = -\frac{1}{5}y^5 + \frac{1}{2}y^4 - \frac{4}{9}y^3 - y^2 + 2xy$$

with the exact solution

$$u(x, y) = -y^2 + 2xy$$

for solvin the above equation , when $\lambda = -1$ we find the iteration sequences as follows:

$$u_{n+1}(x, y) = u_n(x, y) - \int_0^y \left[\frac{\partial u_n(x, \tau)}{\partial \tau} - \frac{\partial \left(-\frac{1}{5}\tau^5 + \frac{1}{2}\tau^4 - \frac{4}{9}\tau^3 - \tau^2 + 2x\tau \right)}{\partial \tau} - \int_0^1 u_n^2(t, \tau) dt - \int_0^\tau \int_0^1 \frac{\partial}{\partial \tau} [u_n^2(t, s)] dt ds \right] d\tau$$

$n=0,1,2,\dots$

by choosing $u_0 = 0$ as an initial approximation .

$$u_1 = \int_0^y \frac{\partial}{\partial \tau} \left(-\frac{1}{5}\tau^5 + \frac{1}{2}\tau^4 - \frac{4}{9}\tau^3 - \tau^2 + 2x\tau \right) d\tau = -\frac{1}{5}y^5 + \frac{1}{2}y^4 - \frac{4}{9}y^3 - y^2 + 2xy$$

$$u_2 = -\frac{1}{5}y^5 + \frac{1}{2}y^4 - \frac{4}{9}y^3 - y^2 + 2xy - \int_0^y \left[\frac{\partial}{\partial \tau} \left(-\frac{1}{5}\tau^5 + \frac{1}{2}\tau^4 - \frac{4}{9}\tau^3 - \tau^2 + 2x\tau \right) - \frac{\partial}{\partial \tau} \left(-\frac{1}{5}\tau^5 + \frac{1}{2}\tau^4 - \frac{4}{9}\tau^3 - \tau^2 + 2x\tau \right) - \int_0^1 \left(-\frac{1}{5}\tau^5 + \frac{1}{2}\tau^4 - \frac{4}{9}\tau^3 - \tau^2 + 2t\tau \right)^2 dt - \int_0^\tau \int_0^1 \frac{\partial}{\partial \tau} \left(-\frac{1}{5}s^5 + \frac{1}{2}s^4 - \frac{4}{9}s^3 - s^2 + 2ts \right)^2 dt ds \right] d\tau$$

$$u_2 = -\frac{8}{45}y^5 - y^2 + 2xy + \frac{1}{275}y^{11} - \frac{1}{50}y^{10} + \frac{77}{1620}y^9 - \frac{1}{80}y^8 - \frac{487}{2835}y^7 + \frac{17}{54}y^6$$

Numerical results fpr example 1 , is shown in Table 1. for N = 4 and figures 1,2 and 3.

Table 1: Numerical results of example 1.N = 4

(x,t)	Exact sol	Approximate sol	Abs.error
(0,0)	0	0	0
(0.1,0.1)	0.01	0.0099999	1×10^{-11}
(0.2,0.2)	0.04	0.03999999702	2.98×10^{-9}
(0.3,0.3)	0.09	0.08999992059	7.941×10^{-8}
(0.4,0.4)	0.16	0.1599992906	7.094×10^{-7}
(0.5,0.5)	0.25	0.2499965804	3.4196×10^{-6}
(0.6,0.6)	0.36	0.3599890815	1.09183×10^{-5}

			5
(0.7,0.7)	0.49	0.4899743999	2.56001×10^{-5}
(0.8,0.8)	0.64	0.6399535465	4.64535×10^{-5}
(0.9,0.9)	0.81	0.8099333359	6.66641×10^{-5}
(1,1)	1.000	0.9999245235	7.54765×10^{-5}

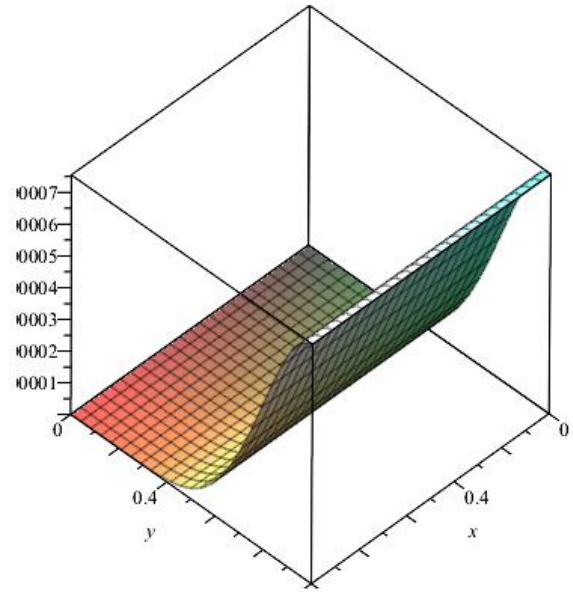


Figure 3: Absolute error of example 1

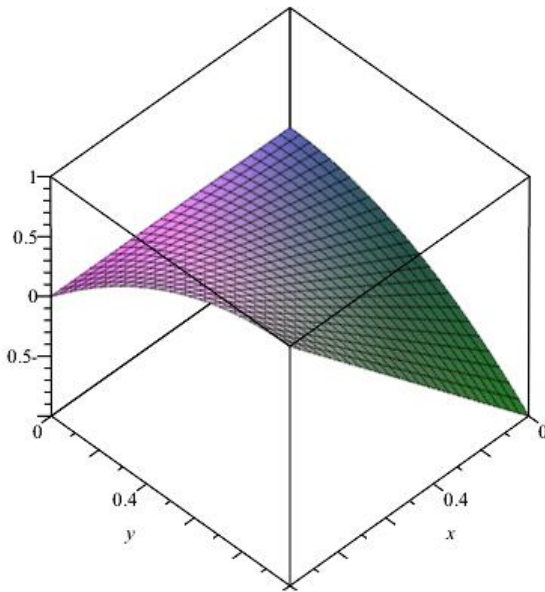


Figure 1: Approximate solution of example 1

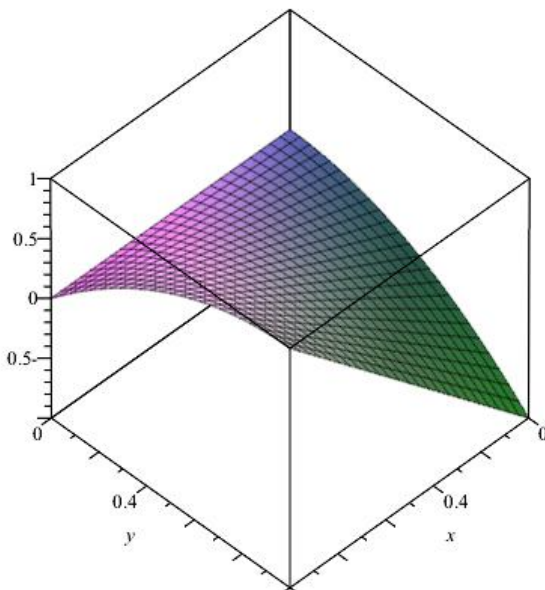


Figure 2: Exact solution of example 1

Example (2) Consider the following nonlinear of mixed Volterra -Fredholm integral equations:

$$u(x, y) = f(x, y) + (x - y^2) \int_0^y \int_0^1 u^2(t, s) dt ds$$

Where

$$f(x, y) = y^2 \sin(x) - 0.05453512866(x - y^2)y^5$$

with the exact solution

$$u(x, y) = y^2 \sin x$$

for solvin the above equation , when $\lambda = -1$ we find the iteration sequences as follows:

$$\begin{aligned} &= u_n(x, y) \\ &- \int_0^y \left[\frac{\partial u_n(x, \tau)}{\partial \tau} \right. \\ &\quad \left. - \frac{\partial(\tau^2 \sin(x) - 0.05453512866(x - \tau^2)\tau^5)}{\partial \tau} \right. \\ &\quad \left. - \int_0^1 (x - \tau^2)u_n^2(t, \tau) dt \right. \\ &\quad \left. - \int_0^\tau \int_0^1 \frac{\partial}{\partial \tau} [(x - \tau^2)u_n^2(t, s)] dt ds \right] d\tau \\ &n=0,1,2,\dots \end{aligned}$$

by choosing $u_0 = 0$ as an initial approximation

$$\begin{aligned} u_1 &= \int_0^y \frac{\partial}{\partial \tau} (\tau^2 \sin(x) - 0.05453512866(x - \tau^2)\tau^5) d\tau \\ &= 0.05453512866y^7 - 0.05453512866xy^5 + y^2 \sin(x) \end{aligned}$$

similarly u_2 .

Numerical results for example 2. is shown in Table 2. for $N = 2$ and figures 4,5 and 6

Table 2: Numerical results of example 2, $N = 2$

(x, t)	Exact Sol	Approximate sol	Abs.error
(0,0)	0	0	0
(0.1,0.1)	0.0009983341665	0.0009983341629	3.6×10^{-12}
(0.2,0.2)	0.007946771632	0.007946773232	1.6×10^{-9}
(0.3,0.3)	0.02659681860	0.02659676827	5.033×10^{-8}
(0.4,0.4)	0.06230693477	0.06230641571	5.1906×10^{-7}
(0.5,0.5)	0.1198563846	0.1198536043	2.7803×10^{-6}
(0.6,0.6)	0.2032712904	0.2032620438	9.2466×10^{-6}
(0.7,0.7)	0.3156666667	0.3156468072	1.98595×10^{-5}
(0.8,0.8)	0.4591078982	0.4590841580	2.37403×10^{-5}
(0.9,0.9)	0.6344947968	0.6344940476	7.492×10^{-7}
(1,1)	0.8414709848	0.8414709847	1×10^{-10}

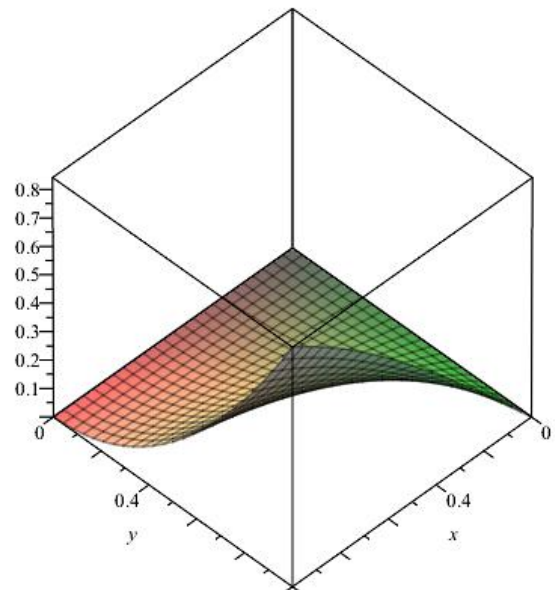


Figure 4: Approximate solution of example 2

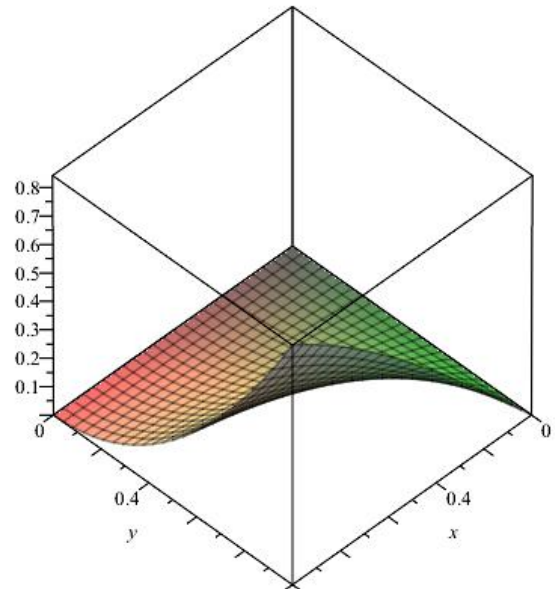


Figure 5: Exact solution of example 2

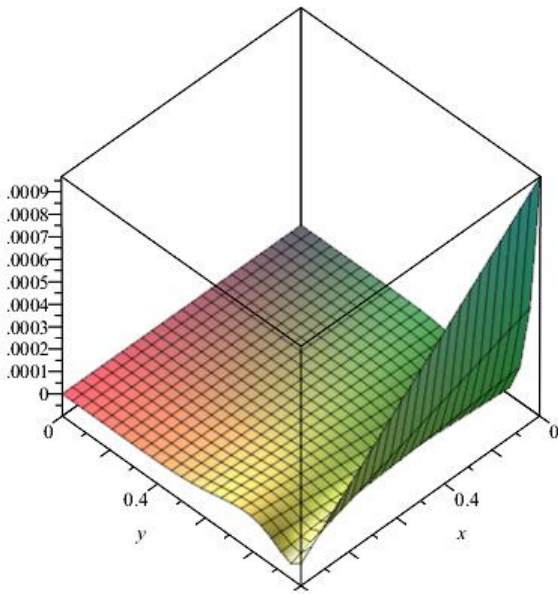


Figure 6: Absolute error of example 2

4. CONCLUSION

This paper concerns the numerical solutions of two dimensional volterra -Fredholm integral equations by using variational Iteration method, Examples 1 and 2 have been solve in [19] by collocation method, we show that our method is better than collocation method.

5. REFERENCES

[1] Darania . p and Ebadian . A ; Numerical solutions of the nonlinear two dimensional Voltterraintegral equations .Newzealand Journal of mathematics .vol36(2007), 163 – 174.

[2] Nemati .S and Ordokhanyi .Y ; Numerical solution of two dimensional nonlinear Voltterraintegral equations by the Legendre polynomials .J . SCI . Tarbiat Moalem University.vol11,No.2(2012), 195 – 210 .

[3] Saeedi . L , Tari . A and Masuleh . S . H . M ; Numerical solution of some nonlinearVolterra integral equations of the first kind .Applications and Applied Mathematics. vol8, (2013), 214 – 226

[4] Bazm .S and Lima .P ; Numerical solutions of the nonlinear second kind two dimensionalVolterra integral equations using Extrapolation method .Numerical Analysis and Applied Mathematics, International conference(2010), 1195 – 1198

[5] Gholami .A ;Solving system of two dimensional nonlinear Volterra -Fredholm integro -differential equations by hes Variational Iteration method .Researcher (2015), 81 – 85

[6] Erfanian .H .R and Mostahahsan .T ;Approximate solution of a class of nonlinear integralequations . The Journal of mathematics and computer science vol.3,No.3(2011), 278 – 286

[7] Bahzadi .Sh .S ;The use of Iterative methods to solve two dimensional nonlinear Volterra-Fredholm integro - differential equations .ISPACS , vol.(2012), 20pages

[8] Rabbani .M and Jamali .R ;solving nonlinear system of mixed Volterra -Fredholm integral equations by using Variational Iteration method . The Journal of mathematics and computerscience vol.5,No.4, (2012), 280 – 287

[9] Borzabadi .A .H and Hedari .M ; A successive iterative approach for two dimensional nonlinearVolterra - Fredholm integro -differential equations .Iranian Journal of Numerical Analysisand optimization .vol.4,No.1(2014), pp95 – 104

[10] Wazwaz . A ; Reliable Treatment for Mixed Volterra-Fredholm in Integral EquationsAppl.Math.comput.127(2002)405 – 414 .

[11] Biazar . J, Ghazvini . H ; He’s variational iteration method for solving linearand nonlinear systems of ordinary differential equations, Applied mathematics andcomputation,191(2007)287 – 297.

[12] Finlayson . B . A , The method of weighted residuals and variational principles , Academicpress, (1972) Newyork

[13] He . J . H ; A new approach to nonlinear partial differential equations, Communications inNonlinear science and Numerical simulation,V ol.2,No.4(1997)230 – 235.

[14] He . J . H ; Nonlinear oscillation with fractional derivative and it’s approximation, Int.conf.on vibration Engineering98,Dalian,China, (1998).

[15] He . J . H, Variational iteration method for nonlinear and it’s applications, Mechanics andpractice,20, (1)(1998)30 – 32(inchinese)

[16] He . J . H ; Variational Iteration method - a kind of nonlinear analytical technique:Someexamples, Int.Journal of Nonlinear Mechanics,34(1999)699 – 708.

[17] Inokuti . M ; General use of the Lagrange multiplier in in nonlinear mathematicalphysics,in: S. Nemat-nasser(Ed.), Variational Method in Mechanics of solids, Progamonpress, oxford, (1978)156 – 16210

[18] Xu. Lan ; Variational iteration method for solving integral equations, computers and Mathematicswith Applications,54(2007)1071 – 1078

[19] Abdollah Borhanifar. and Khadijeh Sadri, Numerical solution for systems of two dimensionalintegral equations by using Jacobi operational collocation method.Sohag J. Math. 1, No. 1,15-26 (2014).

[20] Tatari . M , Dehghan . M ; On the Convergence of He’s Variational Iteration Method, J.comput.Appl.Math,207(2007)121–128