Lacunary Interpolation at Odd and Even Nodes

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ABSTRACT
Here a special (0, 2; 0, 3) lacunary interpolation scheme is considered where the data are prescribed unevenly at even and odd nodes of an arbitrarily defined partition of the unit interval I =[0,1].

The problem described as, we have the function values and second derivatives at odd nodes, whereas function values and the thirdderivatives ateven nodes are known, weproved that there existsa unique quantic spline of continuity class C^2 by solving the above mentioned interpolation scheme.

Furthermore, it is also proved that this spline function converges to the given function with the desired order of accuracy.

Keywords
Lacunary interpolation, splines.

1. INTRODUCTION
In this paper we have discussed (0,2;0,3) lacunary interpolation problem where the function value is prescribed at every node of the partitionof the unit interval whereas third and second derivatives are prescribed alternately at even and odd nodes.

For more related workone is referred to [1] – [6].Let us denotes by $S^3_{n,5}$, the class of quintic splines s(x) on the unit interval [0, 1] such that

(i) $s(x) \in C3\ [0, 1]$

(ii) $s(x) \in \pi 5$ on each [v/n, (v +1)/n] , $0 \le v \le n - 1$.

We shall prove the following theorems.

Theorem 1

Foreveryodd integer n and for every given set of 2n + 4 real numbersf_0 , $f_1 , \dots , f_n$; $f_0''' , f_2''' , \dots , f_{n-1}'''$; $f_1'' , f_3'' , \dots , f_n''$; f_0' ; f_n'there exists a unique spline $s(x) \in$ Sn,5(3) such that

(1.1) $s(v/n) = f_v$; v = 0,1,…,n,

(1.2) $s'''(2v/n) = f_{2v}'''$; v = 0,1,…,(n – 1)/2,

(1.3) $s''((2v+1)/n) = f_{2v+1}''$; v = 0,1,…,(n–1)/2.

(1.4) $s'(0) = f_0$, $s'(1) = f_n'$

2. THEOREM2
Let $f \in C4\ [0,1]$ and n be an odd integer. Then for the unique quintic spline Sn(x) satisfying conditions of Theorem1with

$f_v = f(v/n)$, $v = 0,1,\dots,n$,
$f_{2v}''' = f'''(2v/n)$, $v = 0,1,\dots,(n-1)/2$,

$f_{2v+1}'' = f''(2v+1/n)$, $v = 0,1,\dots,(n-1)/2$,

$f_0' = f'(0)$ and $f_n' = f'(1)$;

We have

(2.1)
$$\| S_n^{(r)}(x) - f^{(r)}(x) \|_\infty \le K_v n^{r-3} \omega_4(1/n) + 2n^{r-4} \| f^{(4)} \|_\infty .$$
Here Kv are different constants depending on v and ω4(.)denotes the modulus of continuity of f(4).

3. PRELIMINARIES
It can be verified that if P(x) is a quantic on [0,1] then

(3.1) P(x) = P (0) A0(x) +. P(1) A1(x) + P'' (0) A2 (x) + P'' (1) A3 (x) + P(4) (0) A4(x) + P(4)(1)A5(x).

Where

$A_0(x) = x$,

$A_1(x) = 1/6\ (x^3 - 3x^2 + 2x)$,

$A_2(x) = 1/2\ (x^2 - x)$,

$A_3(x) = -1/120\ (x^5 - 5x^4 + 20x^2 - 16x)$,

$A_4(x) = 1/120\ (x^5 - 10x^2 + 9x)$,

$A_5(x) = 1/120\ (2x^5 + x^2 + 2x)$.

A quintic Q(x) on [1,2] can be expressed as

(3.2) Q(x) = Q (2) A0(2 – x) + Q (1) A1 (2 – x) + + Q' (2) A2(2 – x) + Q'' (1) A3(2 – x) + + Q''' (2) A4 (2 – x) + Q'' (1) A5(2– x).

For later reference we note that

(3.3)

$$
\begin{cases}
A_0'(0)=1, & A_0''(0)=0, & A_0'''(0)=0, & A_0^{(4)}(0)=0, & A_0^{(5)}(0)=0,\\
A_0'(1)=1, & A_0''(1)=0, & A_0'''(1)=0, & A_0^{(4)}(1)=0, & A_0^{(5)}(1)=0,\\
A_1'(0)=1/3, & A_1''(0)=-1, & A_1'''(0)=1, & A_1^{(4)}(0)=0, & A_1^{(5)}(0)=0,\\
A_1'(1)=-1/6, & A_1''(1)=0, & A_1'''(1)=1, & A_1^{(4)}(1)=0, & A_1^{(5)}(1)=0,\\
A_2'(0)=1/2, & A_2''(0)=-1, & A_2'''(0)=0, & A_2^{(4)}(0)=0, & A_2^{(5)}(0)=0,\\
A_2'(1)=-1/2, & A_2''(0)=1, & A_2'''(1)=0, & A_2^{(4)}(1)=0, & A_2^{(5)}(1)=0,\\
A_3'(0)=2/15, & A_3''(0)=-1/3, & A_3'''(0)=0, & A_3^{(4)}(0)=1/5, & A_3^{(5)}(0)=-1,\\
A_3'(1)=-3/40, & A_3''(0)=0, & A_3'''(1)=1/2, & A_3^{(4)}(1)=-4/5, & A_3^{(5)}(1)=-1,\\
A_4'(0)=3/40, & A_4''(0)=-1/6, & A_4'''(0)=0, & A_4^{(4)}(0)=0, & A_4^{(5)}(0)=1,\\
A_4'(1)=-1/20, & A_4''(1)=0, & A_4'''(1)=-1/2, & A_4^{(4)}(1)=0, & A_4^{(5)}(1)=1.
\end{cases}
$$

A quintic P(x) in [0,1] can be expressed in the following form.

(3.4) $P(x) = P(0)\,B0(x) + P(1)\,B1(x) + P'(0)\,B2(x) + P'(1)\,B3(x) + P'''(0)\,B4(x) + P''(1)\,B5(x)$,

Where
$B_0(x) = 1/3(-8x^5 + 15x^4 - 10x^2 + 3)$,
$B_1(x) = 1/3(8x^5 - 15x^4 + 10x^2)$,
$B_2(x) = -x^5 + 2x^4 - 2x^2 + x$,
$B_3(x) = 1/3(-5x^5 + 9x^4 - 4x^2)$,
$B_4(x) = 1/18(x^5 - 3x^4 + 3x^3 - x^2)$,
$B_5(x) = 1/6(2x^5 - 3x^4 + x^2)$,

Also a quintic Q (x) in [1,2] can be written as

(3.5) $Q(x) = Q(2)\,B_0(2-x) + Q(1)\,B_1(2-x) - Q'(2)\,B_2(2-x) - Q'(1)\,B_3(2-x) - Q'''(2)\,B_4(2-x) + Q''(1)\,B_5(2-x)$.

For later reference we have

$$
(3.6)\begin{cases}
B_0'(0)=0, & B_0''(0)=-20/3, & B_0'''(0)=120, & B_0^{(4)}(0)=120,\\
B_0'(1)=0, & B_0''(1)=0, & B_0'''(1)=-40, & B_0^{(4)}(1)=-200,\\
B_1'(0)=0, & B_1''(0)=20/3, & B_1'''(0)=-120, & B_1^{(4)}(0)=-120,\\
B_1'(1)=0, & B_1''(1)=0, & B_1'''(1)=40, & B_1^{(4)}(1)=200,\\
B_2'(0)=1, & B_2''(0)=0, & B_2'''(0)=0, & B_2^{(4)}(0)=48,\\
B_2'(1)=0, & B_2''(1)=0 & B_2'''(1)=-12, & B_2^{(4)}(1)=-72,\\
B_3'(0)=0, & B_3''(0)=-8/3 & B_3'''(0)=0, & B_3^{(4)}(0)=72,\\
B_3'(1)=1, & B_3''(1)=0, & B_3'''(1)=-28, & B_3^{(4)}(1)=-128,\\
B_4'(0)=0, & B_4''(0)=-1/9, & B_4'''(0)=1, & B_4^{(4)}(0)=-4,\\
B_4'(1)=0, & B_4''(1)=0, & B_4'''(1)=1/3, & B_4^{(4)}(1)=8/3,\\
B_5'(0)=0 & B_5''(0)=1/3 & B_5'''(0)=0 & B_5^{(4)}(1)=-12\\
B_5'(1)=0 & B_5''(1)=1 & B_5'''(1)=8 & B_5^{(4)}(1)=28
\end{cases}
$$

Using equation (3.4) and (3.6) we have

(3.7) $P''(0) = -20/3\,P(0) + 20/3\,P(1) - 4P'(0) - 8/3P'(1) - -1/9\,P'''(0) + 1/3\,P''(1)$,

(3.8) $P'''(1) = -40\,P(0) + 40\,P(1) - 12P'(0) - 28P'(1) + +1/3\,P'''(0) + 8P''(1)$,

(3.9) $P^{(4)}(0) = 120\,P(0) - 120P(1) + 48P'(0) + 72\,P'(1) - -4P'''(0) - 12P''(1)$,

(3.10) $P^{(4)}(1) = -200\,P(0) + 200\,P(1) - 72P(0) - 128P'(1) + +8/3P'''(0) + 28P''(1)$

Similarly from equation (3.5) and (3.6) we get

(3.11) $Q''(2) = -20/3\,Q(2) + 20/3\,Q(1) + 4Q'(2) + 8/3\,Q'(1) + +1/9\,Q'''(2) + 1/3\,Q''(1)$, (3.12) $Q'''(1) = 40\,Q(2) - 40\,Q(1) - 12\,Q'(2) - 28Q'(1) + 1/3\,Q'''(2) - 8\,Q''(1)$,

(3.13) $Q^{(4)}(2) = 120\,Q(2) - 120\,Q(1) - 48\,Q'(2) - 72Q'(1) + 4\,Q'''(2) - 12\,Q''(1)$,

(3.14) $Q^{(4)}(1) = -200\,Q(2) + 200\,Q(1) + 72\,Q'(2) + 128\,Q'(1) - 8/3\,Q'''(2) + 28\,Q''(1)$,

Proof of Theorem1:

For a given $s(x) \in S_{n,5}^{(3)}$ set $h = 1/n$ and

$M_v = s^{(4)}(vh+)$, $v = 0, 1, \ldots, n-1$,

$N_v = s^{(4)}(vh-)$, $v = 0, 1, \ldots, n$.

Since $S^{(4)}(x)$ is linear in each interval $[vh, (v+1)h]$, it is completely determined by the 2n constants $\{M_v\}^{n-1}_{0}$ and $\{N_v\}_1^n$. Also if s(x) satisfies the requirements of Theorem 1, it follows from equation (1.1) – (1.3) and (3.1) – (3.2) that for

$2vh \le x \le (2v+1)h$, $v = 0, 1, \ldots, (n-1)/2$, it must have the form

(1.5) $$s(x) = f_{2v}\, A_0\left(\frac{(2v+1)h - x}{h}\right) + f_{2v+1}\, A_0\left(\frac{x - 2vh}{h}\right) + h^3 f''_{2v}\, A_1\frac{(x-2vh)}{h} + h^2 f''_{2v+1}\, A_2\frac{(x-2vh)}{h} + h^4 M_{2v}\, A_3\frac{(x-2vh)}{h} + h^4 N_{2v+1}\, A_4\frac{(x-2vh)}{h}$$

For $(2v+1)h \le x \le (2v+2)h$, $v = 0,1,\ldots,(n-3)/2$

(1.6) $$s(x) = f_{2v+1}\, A_0\left(\frac{(2v+2)h - x}{h}\right) + f_{2v+2}\, A_0\left(\frac{x-(2v+1)h}{h}\right) - h^3 f''_{2v+2}\, A_1\left(\frac{(2v+2)h - x}{h}\right) + h^2 f''_{2v+1}\, A_2\left(\frac{(2v+2)h - x}{h}\right) + h^4 N_{2v+2}\, A_3\left(\frac{(2v-2)h - x}{h}\right) + h^4 M_{2v+1} A_4\left(\frac{(2v+2)h - x}{h}\right)$$

Since we have $s'(0) = f'_0$; therefore equation (1.1) implies

(1.7) $16M_0 + 9N_1 = 120\,h^{-4}\,[f_0 - f_1 + h f'_0 + h^2/2\, f''_1 - h^3 f''_0]$.

Similarly using condition $s'(1) = f'_0$ we have from equation (1.6)

(1.8) $3M_{n-1} + 2N_n = 40\,h^{-4}\,[-f_{n-1}' + f''_n - hf_n' + h^2/2\, f_n'' - h^3/3\, f_{n-1}''']$.

Also using $s'((2v+1)/h-) = s'((2v+1)/h+)$

and $s'''((2v+1)/h-) = s'''((2v+1)/h+)$, we get

(1.9) $3(M_{2v} + N_{2v+2}) + 2(M_{2v+1} + N_{2v+1}) = -40^{-4}\,[f_{2v} - 2f_{2v+1} + f_{2v+2} + h^2 f''_{2v+1} - h^3/6\,(f'''_{2v} - f'''_{2v+2})]$

and

(1.10) $(M_{2v} + N_{2v+2}) + (M_{2v+1} + N_{2v+1}) = -2h^{-1}\,(f'''_{2v} - f'''_{2v+2})$

Similarly from $s'((2v+2)/h-) = s'((2v+2)/h+)$

and $s''((2v+2)/h-) = s''((2v+2)/h+)$, we get

(1.11) $16\,(M_{2v+1} + N_{2v+2}) + 9(M_{2v+1} + N_{2v+3}) = -120\,h^{-4}\,[f_{2v+2} - 2f_{2v+2} + f_{2v+3} - h^2/2\,(f''_{2v+1} + f''_{2v+3})]$

and

(1.12) $2(M_{2v+2} - N_{2v+2}) - (M_{2v+1} + N_{2v+3}) = 6\,h^{-4}\,[h^2(f''_{2v+3} - f''_{2v+1}) - 2h^3 f'''_{2v+2}]$

The constants $\{M_v\}_0^{n-1}$ and $\{N_v\}_1^n$ will be determined uniquely from equations (1.7) to (1.12) if the corresponding homogeneous system of equations given by

(1.13) $16\,M_0 + 9N_1 = 0$,

(1.14) $3\,M_{n-1} + 2N_n = 0$,

(1.15) $3(M_{2v} + N_{2v+2}) + 2(M_{2v+1} + N_{2v+1}) = 0$,

(1.16) $(M_{2v} + N_{2v+2}) + 2(M_{2v+1} + N_{2v+1}) = 0$,

(1.17) $16(M_{2v+2} + N_{2v+2}) + 9(M_{2v+1} + N_{2v+3}) = 0$,

(1.18) $2(M_{2v+2} - N_{2v+2}) - (M_{2v+1} - N_{2v+3}) = 0$,

has only the zero solution.Now from equation (1.15) and (1.16) we have

(1.19) $M_{2v} + N_{2v+2} = 0$

(1.20) $M_{2v+1} + N_{2v+1} = 0$.

Putting $M_{2v+1} = -N_{2v+1}$ and $N_{2v+2} = -M_{2v}$ from equation (1.19) in equation (1.17) we have

(1.21) $16(M_{2v+2} - N_{2v}) + 9(N_{2v+3} - N_{2v+1}) = 0$

Taking sum for $v = 0,1,\ldots,(n-3)/2$, we get

$$16(M_{n-1} - M_0) + 9(N_n - N_1) = 0,$$

or

$$16(M_0 - M_{n-1}) + 9(N_1 - N_n) = 0,$$

Using equation (1.13) we obtain

(1.22) $16\,M_{n-1} + 9N_n = 0$,

This gives, on using equation (1.13) $M_{n-1} = 0$, $N_n = 0$, Then putting $v = (n-3)/2$ and $M_{n-1} = 0$ $N_n = 0$ in equations (1.17) and (1.18) we have

(1.23) $16 N_{n-1} + 9M_{n-2} = 0$

and

(1.24) $2N_{n-1} + M_{n-2} = 0$

From these two equations we get

$$M_{n-2} = 0,\ N_{n-1} = 0$$

Following similar arguments for different values of v, we have

$$M_{n-3} = 0, \quad N_{n-2} = 0;$$
$$M_{n-4} = 0, \quad N_{n-3} = 0;$$
$$\dots$$
$$\dots$$
$$M_2 = 0, \quad N_3 = 0;$$
$$M_1 = 0, \quad N_2 = 0;$$
$$M_0 = 0, \quad N_1 = 0;$$

Therefore

$M_0 = M_1 = \dots = M_{n-1} = 0$

and

$N_1 = N_2 = \dots = N_n = 0$.

This completes the proof of Theorem1.

For the proof of Theorem 2 we need to prove the following two lemmas.

Lemma 1

Let $f \in C^4 [0,1]$, n any odd integer and $h = 1/n$. Then for the unique spline $s_n(x)$ Theorem (1),

we have

(L.1) $|A_{2v}| = | s_n'(2vh) - f'(2vh)|$

$\le 41 h^2/18\ \omega_4(h)$, $v = 0,1,\dots,(n-1)/2$

and

(L.2) $|A_{2v+1}| = | s_n'((2v+1)h) - f'(2v+1)h)|$

$\le C_v h^2\ \omega_4(h)$, $v = 0,1,\dots,(n-1)/2$

whereC_v are the different constants depending on *v*.

Proof

From equation (P.8) we have for $2vh \le x \le (2v+1)h$

(L.3) $h^3 s'''((2v+1)h) = -40 f_{2v} + 40 f_{2v+1} - 12 h s'(2vh) -$

$-28 h s'((2v+1)h) + h^3/3 f_{2v}''' + 8h^2 f_{2v+1}'''$

Similarly for $(2v+1)h \le x \le (2v+2)h$, we have

(L.4) $h^3 s'''((2v+1)h) = 40 f_{2v+2} + 40 f_{2v+1} - 12 h s'(2v+2)h)$

$-28 h s'((2v+1)h) + h^3/3 f_{2v+2}''' - 8h^2 f_{2v+1}'''$,

Subtracting (1.8) from (L.3) we get

$12 (A_{2v} - A_{2v+2}) = -40 (f_{2v} + f_{2v+2}) + 80 f_{2v+1} + 16 h^2 f_{2v+1}'''$

$+ h^3/3 (f_{2v}''' - f_{2v+2}''') + 12h (f_{2v+1}' - f_{2v}')$.

Using Taylor expansion for the function f and its derivatives on R.H. S., we get

$12(A_{2v} - A_{2v+2}) = h^3 [-80/3 f^{(4)}(\gamma) + 10/3 f^{(4)}(\alpha) + 24/3 f^{(4)}(\beta)$

$- 2/3 f^{(4)}(\delta) + 45/3 f^{(4)}(\eta)]$

$2vh \le \alpha, \beta \le (2v+1)h$; $(2v+1)h \le \gamma, \delta, \eta \le (2v+2)h$.

Fixing $k, 0 \le k \le (n-2)/2$, and summing both sides of the above equation for

$v = k, k+1, \dots\dots, (n-2)/2$, we have

$12A_{2k} = h^3/3 \sum_{r=k}^{(n-2)/2} [-80 f^{(4)}(\gamma)$ $+10 f^{(4)}(\alpha) + 24 f^{(4)}(\beta) - 2 f^{(4)}(\delta) + 48 f^{(4)}(\eta)]$,

Here we use the fact that $A_n = 0$

Therefore
$|A_{2k}| \le 41/18\ \theta_0 h^2 \omega_4(h)$, $|\theta_0| \le 1$, $k = 0,1\dots,(n-1)/2$

This proves equation (L.1).

Again from equation (P.7) for $2vh \le x \le (2v+1)h$ we have

(L.5) $h^3 S''(2vh) = -20/3 f_{2v} + 20/3 f_{2v+1} - 4 h s'(2vh) -$

$-8h/3 s'((2v+1)h) - h^3/9 f_{2v}''' + h^2/3 f_{2v+1}''$

Similarly for $(2v+1)h \le x \le (2v+2)h$, we obtain

(L.6) $h^2 S''((2v+2)h) = -20/3 f_{2v+2} + 20/3 f_{2v+1} + 4h s'(2v+2)h) +$

$+ 8h/3 s'((2v+1)h) + h^3/9 f_{2v+2}''' + h^2/9 f_{2v+1}''$.

Writing equation (L.5) for $v + 1$ and subtracting from equation (L.6) .We have

$8h A_{2v+2} + 8h/3 A_{2v+1} + 8h/3 A_{2v+3} = -20/3 f_{2v+1} + 20/3 f_{2v+3} - 2h^2/9 f_{2v+2}''' - h^2/3 f_{2v+1}'' +$

$+ h^2/3 f_{2v+3}'' - 8h f_{2v+2}' - 8h/3 f_{2v+1}' - 8h/3 f_{2v+3}' - 8h f_{2v+2}' -$

$- 8h/3 f_{2v+1}' - 8h/3 f_{2v+3}'$,

or

(L.7) $8/3 (A_{2v+1} + A_{2v+3}) = -8A_{2v+2} - 20/3 f_{2v+1} + 20/3 f_{2v+3} - 2h^2/9 s' f_{2v+2}''' -$

$- h^2/3 f_{2v+1}'' + h^2/3 f_{2v+3}'' - 8h f_{2v+2}' - 8h/3 f_{2v+1}' - 8h/3 f_{2v+3}$

For $v = (n-3)/2$, $A_n = 0$. This gives

$8/3 A_{n-2} = -8A_{n-1} - 20/3h [f_{n-2} - f_n] - 2h^2/9 f_{n-1}''' - h/3 [f_{n-2}'' - f_n''] - 8/3[3f_{n-1}' + f_{n-2}' + f_n']$, Using Taylor expansion for the function f and its derivatives on R.H.S. of this equation

We have

$8/3\ A_{n-2} = -8A_{n-1} + h^3\ [-40/9\ f^{(4)}(\alpha_1) + 2/9\ f^{(4)}(\alpha_2) - 2/3\ f^{(4)}(\alpha_3) + 4/3\ f^{(4)}(\alpha_4) + 32/9\ f^{(4)}(\alpha_5)]$,

Therefore

$$8/3\ |A_{n-2}| \le 8\ |A_{n-1}| + 46/9\ \theta_1 h^2 \omega_4(h), \quad |\theta_1| \le 1.$$

Using $|A_{n-1}| \le 41/18\ \theta_0 h^2 \omega_4(h)$,

we obtain

$$|A_{n-2}| \le 35/4\ \theta h^2 \omega_4(h), \quad |\theta| \le 1.$$

Following the similar arguments for other values of *v*, we have equation(L.2) of Lemma1.

Lemma 2

Let $f \in C^4\ [0,1]$, n any odd integer and h = 1/n. Then for $s_n(x) = S_n(f,x)$ ofTheorem1, we have

(L.8) $|s'''((2v+1)h) - f_{2v+1}'''| \le K_{1,v}\omega_4(h)$,

(L.9) $h|M_{2v} - N_{2v+1}| \le K_{2,v}\omega_4(h)$,

and

(L.10) $h|M_{2v+1} - N_{2v+2}| \le K_{3,v}\omega_4(h)$;

where $K_{1,v}$, $K_{2,v}$ and $K_{3,v}$ are different constants depending on *v*.

Proof

From equation (P.8) we have

$h^3 S'''((2v+1)h) = -40\ f_{2v} + 40\ f_{2v+1} - 12h\ s'(2vh) -$

$- 28h\ s'((2v+1)h) + h^3/3\ f_{2v}''' + 8h^2\ f_{2v+1}''$.

This gives on using Taylor's expansion

$h^3 |S'''((2v+1)h) - f_{2v+1}'''| \le 12h\ |A_{2v}| + 28h\ |A_{2v+1}| + 17h^4/3\ \omega_4(h)$.

Using Lemma1 we have

$|s'''((2v+1)h) - f_{2v+1}'''| \le K_{1,v}\omega_4(h)$,

This proves equation (L.8).

From equation (P.9) and (P.10) we have,

$h^4(M_{2v} - N_{2v+1}) = 320\ f_{2v} - 320\ f_{2v+1} + 120h\ A_{2v} + 200h\ A_{2v+1} - 20h^3/3\ f_{2v}''' -$

$- 40h^2\ f_{2v+1}'' + 120h\ s'(2vh) + 200h\ s'((2v+1)h)$.

This can be rewritten as

$h^4(M_{2v} - N_{2v+1}) = 120h\ A_{2v} + 200h\ A_{2v+1} + 640(f_{2v} - f_{2v+1}) + 240h\ f_{2v}' +$

$+ 400h\ f_{2v+1}' - 40h^3/3\ f_{2v}''' - 80h^2\ f_{2v+1}''' - h^4(M_{2v} - N_{2v+1})$,

or

$h^4(M_{2v} - N_{2v+1}) = 60h\ A_{2v} + 100h\ A_{2v+1} + 320(f_{2v} - f_{2v+1}) + 120h\ f_{2v}' +$

$+ 200h\ f_{2v+1}' - 20h^3/3\ f_{2v}''' - 40h^2\ f_{2v+1}'''$.

Therefore using Taylor expansion we have

$|h^4(M_{2v} - N_{2v+1})| \le 60h\ |A_{2v}| + 100h\ |A_{2v+1}| + 100h^4/3\ \omega_4(h)$.

plugging in the values of A_{2v} and A_{2v+1} from Lemma 1, we have

$|h^4(M_{2v} - N_{2v+1})| \le K_{2,v}\omega_4(h)$.

proof for equation (L.10) can be carried out on similar lines as for (L.8) and (L.9) so we omit the details.

4. PROOF OF THEOREM 2

For $2vh \le x \le (2v+1)h$, $v = 0, 1, \ldots, (n-1)/2$, we have from equation (1.5)

$$s'''(x) = s'''(2vh)\ A_0\left(\frac{(2v+1)h - x}{h}\right) + s'''((2v+1)h)\ A_0\left(\frac{x - 2vh}{h}\right) + h^2 s^{(5)}(2vh)\ A_1\left(\frac{x - 2vh}{h}\right).$$

Now from equation (1.5) and (P.3) we have

$$s^{(5)}(2vh+) = -h^{-1}(M_{2v} - N_{2v+1}).$$

Since

$$A_0\left(\frac{(2v+1)h - x}{h}\right) + A_0\left(\frac{x - 2vh}{h}\right) = 1,$$

We have

(2.2) $s'''(x) - f'''(x) =$

$$\left(s'''(2vh+) - f'''(x)\right) A_0\left(\frac{(2v+1)h - x}{h}\right) + \left(s'''((2v+1)h-) - f'''(x)\right) A_0\left(\frac{x - 2vh}{h}\right) - h(M_{2v} - N_{2v+1})A_1\left(\frac{x - 2vh}{h}\right).$$

$$= I_1 + I_2 + I_3, \quad \text{say.}$$

Here $|A_0| \le 1$, $|A_1| \le 1$.

$$|I_1| = |(s'''(2vh+) - f'''(x)|$$

$$= |(s'''(2vh+) - f'''(2vh) + (x - 2vh)f^{(4)}(\alpha)|, \quad 2vh \le \alpha \le x$$

$$= |(s'''(2vh+) - f'''(2vh)| + h\Omega, \quad \text{where } \Omega = \|f^{(4)}\|_\infty$$

or

(2.3) $|I_1| \le h\Omega$.

$$|I_2| = |(s'''((2vh+1)h-) - f'''(x)|$$

$= | s^{'''} (2vh + 1) – f^{'''} (2vh + 1) + (x–(2v+1)h)f^{(4)}(\beta) |$,

Using Lemma 2, we have

(2.4) $| I_2 | \le K_{1,v}\omega_4 (h) + h\Omega$

and

$| I_3 | = – h (M_{2v} – N_{2v+1})$

(2.5) $| I_3 | \le K_{2,v}\omega_4 (h)$.

Thus from equations (2.2) – (2.5) we have the theorem for $2vh \le x \le (2v+1)h$ and $r = 3$.

Further let $(2v+1)h \le x \le (2v+2)h$, $v = 0, 1,…,(n –3)/2$.

Form equation (1.6) we have

$$s'''(x) = s'''((2v+1)h) A_0 \frac{((2v+2)h-x)}{h} + s'''((2v+2)h) A_0 \frac{(x-(2v+1)h)}{h} +$$

$$+ h^2 s^{(5)}((2v+2)h) A_1 \frac{((2v+2)h-x)}{h} .$$

Using equation (1.5) and (1.3) we get

$$s'''(x) = s'''((2v+1)h) A_0 \frac{((2v+2)h-x}{h} + s'''((2v+2)h) A_0 \frac{(x-(2v+1)h)}{h} +$$

$$+ h (M_{2v+1} – N_{2v+2}) A_1 \frac{((2v+2)h-x)}{h} .$$

Following similar arguments we can prove the result for $(2v+1)h \le x \le (2vh+2)h$ and $r = 3$.

Next for $r = 0, 1, 2$, usinginterpolatory condition we can write

$$|s''(x) – f''(x)| = |\int_x^{(2v+1)h} (s'''(t) – f'''(t))\, dt|$$

$$\le \int_x^{(2v+1)h} |s'''(t) – f'''(t)| dt$$

$\le K_{1,v}\, h\, \omega_4 (h)$.

Also we can write

$$|s'(x) – f'(x)| = |\int_\lambda^x (s''(t) – f''(t))\, dt |, \quad vh\le\lambda\le (v+1)h,$$

Therefore,

$|s'(x) – f'(x)| \le h | s''(t) – f''(t)$

$\le K_{1,v}\, h^2\omega_4 (h)$.

Similarly,

$$|s(x) – f(x)| = |\int_{2vh}^x (s'(t) – f'(t))\, dt|$$

$\le K_{1,v}\, h^3\omega_4 (h)$.

This proves Theorem 2 completely.

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