Certain Combinatorial Identities of Twin Pairs Related to Tchebychev Polynomials

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ABSTRACT

In the present paper, a twin pair (x_n, y_n) and (X_n, Y_n) of numbers related to one and two variable Tchebychev polynomials of first and second kinds are proposed. Certain Combinatorial Identities of the twin pairs are stated and proved.

General Terms

AMS Clasiification: $05A19$, $30B70$ and $33C70$.

Keywords

Combinatorial Identities, Continued fractions and Functions of hypergeometric type in one and severable variables

1. INTRODUCTION

Tchebychev polynomials belong to hypergeometric family of functions [3, 6, 7, 8, 9, 11, 12, 13]. They play a very important role in both pure and applied Mathematics. For instance, they have many applications in Number Theory in the context of continued fractions and Combinatorial Identities [1, 2, 4, 5, 7, 10, 14, 15]. Also they are extensively used in Best Approximation Theory of Numerical Analysis [3, 6, 8, 11, 12, 13].

In the present paper, a pair $(T_n(x, y), U_n(x, y))$ of two variable homogenous polynomials of degree n are proposed. When y =1, the polynomials will reduce to $(T_n(x), U_n(x))$, Tchebychev polynomials of first and second kinds in single variable. The Recurrence Relations, Hypergeometric series represention, Rodrigue formula, Generating Functions and Determinant formulas are worked out.

In sections 3 and 4,certain Combinatorial Identities of twin pairs (x_n, y_n) and (X_n, Y_n) related to Tchebychev Polynomials are stated and proved respectively.

2. A GENERALIZATION OF TCHEBYCHEV POLYNOMIALS AND EXTENDED RESULTS

Definition: Generalized Tchebychev Polynomial of first kind in two variables x and y of degree n, denoted by $T_n(x, y)$ is

$$T_n(x, y) \;=\; \frac{1}{2}\left[[x + \sqrt{x^2 - y^2}]^n + [x - \sqrt{x^2 - y^2}]^n\right].$$

It is a homogeneous polynomial of degree n and hence

$$T_n(x, y) = y^n T_n\left(\frac{x}{y}\right).$$

Generalized Tchebychev Polynomial of Second kind in two variables x and y of degree n, denoted by $U_n(x, y)$ is

$$U_n(x, y) \;=\; \frac{1}{2\sqrt{x^2 - y^2}}\left[[x + \sqrt{x^2 - y^2}]^{n+1} - [x - \sqrt{x^2 - y^2}]^{n+1}\right].$$

It is also a homogeneous polynomial of degree n and hence

$$U_n(x, y) = y^n U_n\left(\frac{x}{y}\right).$$

When $y = 1$, $T_n(x, y)$ and $U_n(x, y)$ are nothing but $T_n(x)$ and $U_n(x)$ respectively.
Initial Polynomials: The initial polynomials of generalized Tchebychev polynomials of first and second kind in two variables are

$$\begin{aligned} T_n(x, y) &: 1,\ x,\ 2x^2 - y^2,\ 4x^3 - 3xy^2, \ldots, \\ U_n(x, y) &: 1,\ 2x,\ 4x^2 - y^2,\ 8x^3 - 4xy^2, \ldots \ . \end{aligned}$$

Three Term Recurrence Relations: By direct verification using the definition, one can show that the following Recurrence Relation is satisfied by the generalized Tchebychev Polynomials of first kind in two variables:

$$\begin{aligned} T_{n+1}(x, y) &= 2xT_n(x, y) - y^2 T_{n-1}(x, y), \\ T_0(x, y) &= 1,\ T_1(x, y) = x,\ n = 1, 2, 3, \ldots \ . \end{aligned}$$

Similarly, one can show that the following Recurrence relation is satisfied by the generalized Tchebychev Polynomials of second kind in two variables:

$$\begin{aligned} U_{n+1}(x, y) &= 2xU_n(x, y) - y^2 U_{n-1}(x, y), \\ U_0(x, y) &= 1,\ U_1(x, y) = 2x,\ n = 1, 2, 3, \ldots \ . \end{aligned}$$

$T_n(x,y)$ and $U_n(x,y)$ are also connected by the following three term recurence relation:

$$\begin{aligned} T_{n+1}(x,y) &= xU_n(x,y) - y^2U_{n-1}(x,y), \\ U_0(x,y) &= 1,\ U_1(x,y) = 2x,\ n = 1,2,3,\dots\ . \end{aligned}$$

Hypergeometric Series Representation: Tchebychev Polynomials of first and second kind in one variable can be represented in the form of hypergeometric series as follows [9].

$$\begin{aligned} T_n(x) &= {}_2F_1\left(-n,n;\frac{1}{2};\frac{1-x}{2}\right), \\ U_n(x) &= (n+1)\,{}_2F_1\left(-n,n+2;\frac{3}{2};\frac{1-x}{2}\right). \end{aligned}$$

The generalized Tchebychev polynomials of first and second kind in two variables also have the following extended result and the proof will be similar to one variable case.

THEOREM 1. *The hypergeometric representation for generalized Tchebychev polynomials are*

$$\begin{aligned} &T_n(x,y) = y^n\,{}_2F_1\left(-n,n;\frac{1}{2};\frac{y-x}{2y}\right), \\ \textit{and}\quad &U_n(x,y) = y^n(n+1)\,{}_2F_1\left(-n,n+2;\frac{3}{2};\frac{y-x}{2y}\right). \end{aligned}$$

Rodrigue Formula: The Rodrigue formula for Tchebychev Polynomials of first kind in one variable is

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}},$$

$$\begin{aligned} T_n(x,y) &= y^n T_n\left(\frac{x}{y}\right) \\ &= y^n\frac{(-1)^n 2^n n!}{(2n)!}\frac{(y^2-x^2)^{\frac{1}{2}}}{(y^2)^{\frac{1}{2}}}\frac{\partial^n}{\partial x^n}\left(\frac{(y^2-x^2)^{n-\frac{1}{2}}}{(y^2)^{n-\frac{1}{2}}}\right) \\ &= \frac{1}{y^n}\frac{(-1)^n 2^n n!}{(2n)!}(y^2-x^2)^{\frac{1}{2}}\frac{\partial^n}{\partial x^n}(y^2-x^2)^{n-\frac{1}{2}}. \end{aligned}$$

Similarly the Rodrigue formula for genarlised Tchebychev polynomials of second kind in two variables can be derived. The extended result is stated in the following theorem.

THEOREM 2. *The Rodrigue formula for generalized Tchebychev polynomials are*

$$T_n(x,y) = \frac{1}{y^n}\frac{(-1)^n 2^n n!}{(2n)!}(y^2-x^2)^{\frac{1}{2}}\frac{\partial^n}{\partial x^n}(y^2-x^2)^{n-\frac{1}{2}},$$

and

$$U_n(x,y) = \frac{1}{y^n}\frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(y^2-x^2)^{-\frac{1}{2}}\frac{\partial^n}{\partial x^n}(y^2-x^2)^{n+\frac{1}{2}}.$$

Generating Functions:

Put $\quad f(x,y,t) = \sum_{n=0}^{\infty} T_n(x,y)t^n.$

$$\begin{aligned} f(x,y,t) &= T_0(x,y) + T_1(x,y)t + \cdots \\ &+T_{n+1}(x,y)t^{n+1} + \cdots \\ -2xtf(x,y,t) &= -2xT_0(x,y)t - 2xT_1(x,y)t^2 - \cdots \\ &-2xT_n(x,y)t^{n+1} - \cdots \\ y^2t^2f(x,y,t) &= T_0(x,y)y^2t^2 + T_1(x,y)y^2t^3 + \cdots \\ &+T_{n-1}(x,y)y^2t^{n+1} + \cdots\ . \end{aligned}$$

Summing all the three expressions on both sides, we get

$$\begin{aligned} (1-2xt+t^2y^2)f(x,y,t) &= 1 + xt - 2xt \\ f(x,y,t) &= \frac{1-xt}{1-2xt+t^2y^2}. \end{aligned}$$

Similarly the generalized Tchebychev polynomials of second kind in two variables can be shown to have a genarating function stated in the following theorem combined with that of $T_n(x,y)$.

THEOREM 3. *The genarating function for generalized Tchebychev polynomials are*

$$\sum_{n=0}^{\infty} T_n(x,y)t^n = \frac{1-xt}{1-2xt+t^2y^2}$$

and

$$\sum_{n=0}^{\infty} U_n(x,y)t^n = \frac{1}{1-2xt+t^2y^2}.$$

Determinants Formulas: We state the following theorem for generalized Tchebychev polynomials of first and second kinds without proof because they follow directly from their three term recurrence relations.

THEOREM 4. *The determinants formulas for generalized Tchebychev polynomials are*

$$T_n(x,y) = \begin{vmatrix} x & -y & 0 & \cdots & 0 & 0 \\ -y & 2x & -y & 0 & \cdots & 0 \\ 0 & -y & \ddots & -y & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & 2x \end{vmatrix}$$

and

$$U_n(x,y) = \begin{vmatrix} 2x & -y & 0 & \cdots & 0 & 0 \\ -y & 2x & -y & 0 & \cdots & 0 \\ 0 & -y & \ddots & -y & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & -y & 2x \end{vmatrix}.$$

3. CERTAIN COMBINATORIAL IDENTITIES OF THE FIRST PAIR

Let $x_n := T_n(N)$ and $y_n := U_{n-1}(N)$, where N =2,3, $\dots$. Then they satsify the following three term recurrence relations:

$$x_{n+1} = 2Nx_n - x_{n-1}, \ x_0 = 1, \ x_1 = N \quad (1)$$
$$y_{n+1} = 2Ny_n - y_{n-1}, \ y_0 = 1, \ y_1 = 2N \quad (2)$$
$$x_{n+1} = Ny_{n+1} - y_n, \ y_1 = 1, \ y_2 = 2N \quad (3)$$

As a result, they have the following Binet forms:

$$x_n = \frac{1}{2}\left[[N+\sqrt{N^2-1}]^n + [N-\sqrt{N^2-1}]^n\right] \quad (4)$$
$$y_n = \frac{1}{2\sqrt{N^2-1}}\left[[N+\sqrt{N^2-1}]^n - [N-\sqrt{N^2-1}]^n\right] (5)$$

Using $(1) - (5)$, the following combinatorial identities are derived.

THEOREM 5. *The pair (x_n, y_n) satisfies the following identities:*

$$5(i) \quad \begin{bmatrix} x_n & y_n \\ x_{n+1} & y_{n+1} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} N & 1 \\ (N^2-1) & N \end{bmatrix}^{n-1}.$$
$$5(ii) \quad x_n y_{n+1} - y_n x_{n+1} = 1.$$
$$5(iii) \quad \frac{x_{n+1}}{y_{n+1}} = N_- \frac{1}{2N}{}_- \frac{1}{2N} \cdots\cdots \frac{1}{2N}.$$

PROOF.

5(i): The result is proved by using Mathematical Induction on n.
For $n = 1$ the result is obvious.
Suppose for $n = k$, the result is true.

$$\begin{bmatrix} x_k & y_k \\ x_{k+1} & y_{k+1} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} N & 1 \\ (N^2-1) & N \end{bmatrix}^{k-1}.$$

The result directly follows for $n = k + 1$, if the following identities are true:

$$x_{k+1} = Nx_k + (N^2-1)y_k \quad (6)$$
$$y_{k+1} = 1.x_k + Ny_k \quad (7)$$

Using (2) follwed by (3), one obtains (7)

$$\begin{aligned} y_{k+1} &= 2Ny_k - y_{k-1} \\ &= Ny_k + [Ny_k - y_{k-1}] \\ &= Ny_k + x_k. \end{aligned}$$

Using (7) and (3)

$$\begin{aligned} Ny_{k+1} &= Nx_k + N^2 y_k \ \ and \\ x_{k+1} - Ny_{k+1} &= -y_k. \end{aligned}$$

One obtains (6), by direct addition on both sides.
5(ii): The result directly follows by taking determinant on both sides of $5(ii)$ because

$$det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} = N(2N) - (2N^2-1)\cdot 1 = 1$$

5(iii): The three term recurrence relations (3) and (2)can be rewritten as follows.

$$\begin{aligned} \frac{x_{n+1}}{y_{n+1}} &= N_- \frac{1}{\frac{y_{n+1}}{y_n}} \\ \frac{y_{n+1}}{y_n} &= 2N_- \frac{1}{\frac{y_n}{y_{n-1}}}. \end{aligned}$$

As a result

$$\frac{x_{n+1}}{y_{n+1}} = N - \frac{1}{2N}{}_- \frac{1}{2N}{}_- \frac{1}{2N} \cdots\cdots \frac{1}{2N}, \ \text{because} \ \frac{y_1}{y_0} = 2N.$$

□

THEOREM 6. *The sequence $\{x_n\}$ satisfies the following identities*

$$6(i) \quad \begin{bmatrix} x_{n-1} & x_n \\ x_n & x_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & N \\ N & 2N^2-1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2N \end{bmatrix}^{n-1}.$$
$$6(ii) \quad x_{n-1}x_{n+1} - x_n^2 = N^2 - 1.$$
$$6(iii) \quad \frac{x_{n+1}}{x_n} = 2N_- \frac{1}{2N}{}_- \frac{1}{2N} \cdots\cdots \frac{1}{2N}{}_- \frac{1}{N}.$$

PROOF.

6(i): Again the result is proved by using Mathematical Induction. Put

$$A_k = \begin{bmatrix} x_{k-1} & x_k \\ x_k & x_{k+1} \end{bmatrix}; B = \begin{bmatrix} 0 & -1 \\ 1 & 2N \end{bmatrix},$$

then

$P(1) \ : \ A_1 = A_1 B^0$
$P(k) \ : \ A_k = A_1 B^{k-1}$

$P(k+1)$: Using the three term recurrence relation for x_{n+1} in (1), we get $A_{k+1} = A_k B$. Further using $P(k)$, one arrive at $A_{k+1} = A_1 B^k$.

6(ii): The identity can be directly deduced by applying determinant on both sides of $6(i)$.

6(iii): The three term recurrence relations (1) yields

$$\frac{x_{n+1}}{x_n} = 2N_- \frac{1}{2N}{}_- \frac{1}{2N} \cdots\cdots \frac{1}{2N}{}_- \frac{1}{N}, \ \text{because} \ \frac{x_1}{x_0} = N.$$

□

THEOREM 7. *The sequence $\{y_n\}$ satisfies the following identities*

$$7(i) \quad \begin{bmatrix} y_{n-1} & y_n \\ y_n & y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & N \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2N \end{bmatrix}^{n-1}.$$
$$7(ii) \quad y_{n-1}y_{n+1} - y_n^2 = -1.$$
$$7(iii) \quad \frac{y_{n+1}}{y_n} = 2N - \frac{1}{2N}{}_- \frac{1}{2N}{}_- \frac{1}{2N} \cdots\cdots \frac{1}{2N}.$$

PROOF.

7(i): Again the result is proved by using Mathematical Induction on n. Put

$$A_k = \begin{bmatrix} y_{k-1} & y_k \\ y_k & y_{k+1} \end{bmatrix}; B = \begin{bmatrix} 0 & -1 \\ 1 & 2N \end{bmatrix}.$$

Then $P(1) \ : \ A_1 = A_1 B^0$
$P(k) \ : \ A_k = A_1 B^{k-1}$

$P(k+1)$: Using the three term recurrence relation for y_{n+1} in (1), we get $A_{k+1} = A_k B$. Further using $P(k)$, one arrive at $A_{k+1} = A_1 B^k$.

7(ii): The identity can be directly deduced by applying determinant on both sides of $7(i)$.

7(iii): The three term recurrence relations (2) yields

$$\frac{y_{n+1}}{y_n} = 2N - \frac{1}{2N\,-}\,\frac{1}{2N\,-}\,\frac{1}{2N\,-\cdots-}\,\frac{1}{2N}, \text{ because } \frac{y_1}{y_0} = 2N.$$

□

4. CERTAIN COMBINATORIAL IDENTITIES OF THE SECOND PAIR

Let $X_n := T_n(N,K)$ and $Y_n := U_{n-1}(N,K)$, where $N = 2,3,\ldots$, $K = 1,2,\ldots,N-1$ and $N^2 - K^2$ is not a square number. When $K = 1, X_n = x_n$ and $Y_n = y_n$. Hence we can expect similar identities for X_n and Y_n and their proofs will also be quite similar to those of one variable case except for small adoptation to incopartion K^2. In the present section, we describe the similar identities for X_n and Y_n with out giving proofs.
The three term recurrence relations are as follows.

$$X_{n+1} = 2NX_n - K^2 X_{n-1}, X_0 = 1,\ X_1 = N. \tag{8}$$
$$Y_{n+1} = 2NY_n - K^2 Y_{n-1}, Y_0 = 0,\ Y_1 = 1. \tag{9}$$
$$X_{n+1} = NY_{n+1} - K^2 Y_n, Y_0 = 0,\ Y_1 = 1. \tag{10}$$

A pair of use full identities are

$$X_{n+1} = NX_n + (N^2 - K^2)Y_n. \tag{11}$$
$$Y_{n+1} = X_n + NY_n. \tag{12}$$

The Binet forms are as follows

$$X_n = \frac{1}{2}\left[[N + \sqrt{N^2 - K^2}]^n + [N - \sqrt{N^2 - K^2}]^n\right]. \tag{13}$$

$$Y_n = \frac{1}{2\sqrt{N^2 - K^2}}\left[[N + \sqrt{N^2 - K^2}]^n - [N - \sqrt{N^2 - K^2}]^n\right]. \tag{14}$$

An interesting special case: When $N = 3$ and $K = 2$,

$$X_1 = 3, X_2 = 14, X_3 = 72, \ldots, X_n = 2^{n-1} L_{2n}.$$
$$Y_1 = 1, Y_2 = 5, Y_3 = 32, \ldots, Y_n = 2^{n-1} F_{2n}.$$

Where L_n and F_n are well known Lucas and Fibonacci numbers [2, 5, 15].

THEOREM 8. *The pair (X_n, Y_n) satisfies the following identities*

$$8(i) \quad \begin{bmatrix} X_n & Y_n \\ X_{n+1} & Y_{n+1} \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix} \begin{bmatrix} N & 1 \\ N^2 - K^2 & N \end{bmatrix}^{n-1}.$$

$$8(ii) \quad X_n Y_{n+1} - Y_n X_{n+1} = K^2 + (K^2 - 1)\phi_{n-1}(k^2),$$

$$where\ \phi_{n-1}(k^2) = k^2 \frac{k^{2n-2} - 1}{k^2 - 1}.$$

$$8(iii) \quad \frac{X_{n+1}}{Y_{n+1}} = N\,-\,\frac{K^2}{2N\,-}\,\frac{K^2}{2N\,-\cdots-}\,\frac{K^2}{2N}.$$

THEOREM 9. *The sequence $\{X_n\}$ satisfies the following identities*

$$9(i) \quad \begin{bmatrix} X_{n-1} & X_n \\ X_n & X_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & N \\ N & 2N^2 - K^2 \end{bmatrix} \begin{bmatrix} 0 & -K^2 \\ 1 & 2N \end{bmatrix}^{n-1}.$$

$$9(ii) \quad X_{n-1}X_{n+1} - X_n^2 = (N^2 - K^2) + (K^2 - 1)\psi_{n-1}(k^2),$$

$$where\ \psi_{n-1}(k^2) = (N^2 - K^2)\frac{k^{2n-2} - 1}{k^2 - 1}.$$

$$9(iii) \quad \frac{X_{n+1}}{Y_n} = 2N\,-\,\frac{K^2}{2N\,-}\,\frac{K^2}{2N\,-\cdots-}\,\frac{K^2}{N}.$$

THEOREM 10. *The sequence $\{Y_n\}$ satisfies the following identities*

$$10(i) \quad \begin{bmatrix} Y_{n-1} & Y_n \\ Y_n & Y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ N & 2N \end{bmatrix} \begin{bmatrix} 0 & -K^2 \\ 1 & 2N \end{bmatrix}^{n-1}.$$

$$10(ii) \quad Y_{n-1}Y_{n+1} - Y_n^2 = (-1) + (K^2 - 1) + \chi_{n-2}(k^2),$$

$$where\ \chi_{n-2}(k^2) = -\frac{k^{2n-2} - 1}{k^2 - 1}.$$

$$10(iii) \quad \frac{Y_{n+1}}{Y_n} = 2N\,-\,\frac{K^2}{2N\,-}\,\frac{K^2}{2N\,-\cdots-}\,\frac{K^2}{2N}.$$

The results of this section are indeed non trivial generalizations of those results proved in the previous section.

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