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Some Characterizations on Soft Uni-groups and Normal Soft Uni-groups

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ABSTRACT

In this paper, we first give the definition of soft uni-product and characterize soft uni-groups as regards this definition and we prove a number of results and give some alternative formulations about soft uni-groups by using the the concepts of normal soft uni-subgroups, characteristic soft uni-groups, conjugate soft uni-groups, soft normalizer and commutator of a group, which are analogs of significant results in group theory.

General Terms

2010 AMS Classification: *03G25,20A15,03H05,03C55,03E75*

Keywords

Soft sets, soft uni-groups, soft uni-product, normal soft uni-subgroups, characteristic soft uni-groups, conjugate soft uni-groups, soft normalizer of a soft set.

1. INTRODUCTION

There have been a great amount of research and applications in the literature concerning some special tools like probability theory, fuzzy set theory [1, 2], rough set theory [3, 4], vague set theory [5], interval mathematics [6, 7] and intuitionistic fuzzy set theory [8, 9], which were established by researches for modeling uncertainties. But these theories has its advantages as well as limitations in dealing with uncertainties as mentioned by Molodtsov [10]. Soft set theory was introduced by Molodtsov [10] for modeling vagueness and uncertainty and it has received much attention since Maji et al. [11] and Ali et al. [12] introduced and studied several operations of soft sets. Soft set theory has many potential applications in [13, 14, 15, 16, 17].
Firstly, Rosenfeld [18] studied fuzzy sets in the structure of groups and obtained some analog results in group theory. Since then, many followers [19, 20, 21, 22, 23, 24] studied the fuzzy substructures of different algebraic structures.

Çağman et al. [25] introduced a new kind of soft group, called *soft int-group*, which is based on the inclusion relation and intersection of sets and studied its basic properties. Some more studies are available especially for normal soft int-groups [26]. Muştuoğlu et al. [27] defined one more new kind of soft group, called *soft uni-group*, which is in fact based on the inclusion relation and union of sets and more functional for obtaining results in the mean of soft group theory. In the study, they introduced the concepts soft uni-subgroups, soft normal uni-subgroups, e-left coset set of a soft set and anti image of a soft set and investigated these notions with respect to soft uni-groups and they obtained some relations and significant characterizations between soft int-groups and soft uni-groups.
In this paper, first we give the definition of soft uni-product for soft uni-groups and characterize soft uni-groups by using this new notions. Moreover, we extend the study of soft uni-groups by introducing the concepts of characteristic soft uni-groups, soft conjugate, conjugate soft uni-groups and soft normalizer. And we prove a number of results about soft uni-groups by using these concepts which correspond to significant results in group theory. We also give some alternative formulation for uni-soft group in terms of commutators.

2. PRELIMINARIES

In this section, we present the basic definition of soft set theory and operations defined on soft sets. Throughout this paper, U refers to an initial universe, E is a set of parameters, $P(U)$ is the power set of U and $A, B, C \subseteq E$.

DEFINITION 1. *([13]) A soft set f_A over U is a set defined by*

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Here f_A is also called approximate function. A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x, f_A(x)) : x \in E,\ f_A(x) \in P(U)\}.$$

DEFINITION 2. *([13]) Let f_A and f_B be soft sets over U. Then, f_A is a soft subset of f_B, denoted by $f_A \tilde{\subseteq} f_B$, if $f_A(x) \subseteq f_B(x)$ for all $x \in E$.*

DEFINITION 3. *([13]) Let f_A and f_B be soft sets over U. Then, f_A is a soft equal to f_B, denoted by $f_A = f_B$, if $f_A(x) = f_B(x)$ for all $x \in E$.*

DEFINITION 4. *([13]) Let f_A and f_B be soft sets over U. Then, union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$, is defined as $f_A \tilde{\cup} f_B = f_{A\tilde{\cup}B}$, where $f_{A\tilde{\cup}B}(x) = f_A(x) \cup f_B(x)$ for all $x \in E$.*
Intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$, is defined as $f_A \tilde{\cap} f_B = f_{A\tilde{\cap}B}$, where $f_{A\tilde{\cap}B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

From now on, G refers to a group structure.

DEFINITION 5. *([27]) Let G be a group and $f_G \in S(U)$. Then f_G is called an soft union-group over U if*

i) $f_G(xy) \subseteq f_G(x) \cup f_G(y)$ for all $x, y \in G$,
ii) $f_G(x^{-1}) = f_G(x)$ for all $x \in G$.

For the sake of ease in mathematical manipulation we denote a soft union-group by soft uni-group in what follows.

EXAMPLE 1. *([27]) Assume that $U = S_3$, symmetric group, is the universal set and $G = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$, dihedral group, is the subset of set of parameters. The group table of D_2 is known as*

$\cdot$	e	x	y	yx
e	e	x	y	yx
x	x	e	yx	y
y	y	yx	e	x
yx	yx	y	x	e

Now, we can construct a soft set f_G by

$$\begin{aligned} f_G(e) &= \{(13)\} \\ f_G(x) &= \{e, (12), (13)\} \\ f_G(y) &= \{e, (13), (23)\} \\ f_G(yx) &= \{e, (12), (13), (23)\}. \end{aligned}$$

Then, one can easily show that the soft set f_G is a soft uni-group over S_3.

PROPOSITION 1. *([27]) If f_G is a soft uni-group over U, then $f_G(e) \subseteq f_G(x)$ for all $x \in G$.*

THEOREM 6. *([27]) A soft set f_G over U is a soft uni-group over U if and only if $f_G(xy^{-1}) \subseteq f_G(x) \cup f_G(y)$ for all $x, y \in G$.*

DEFINITION 7. *([27]) Let G be a group, H be a subgroup of G and f_G be a soft uni-group over U. If f_H, the soft subset of f_G, itself is a soft uni-group over U, then f_H is said to be a soft uni-subgroup of f_G over U and denoted by $f_H \widetilde{\leq}_u f_G$.*

DEFINITION 8. *([27]) Let f_G be a soft uni-group over U. Then, f_G is called an abelian soft uni-group over U, if $f_G(xy) = f_G(yx)$ for all $x, y \in G$.*

DEFINITION 9. *([27]) Let f_G be a soft uni-group over U and $f_N \widetilde{\leq}_u f_G$. f_N is said to be a normal soft uni-subgroup of f_G over U if f_N is an abelian soft uni-group over U. This is denoted by $f_N \widetilde{\lhd}_u f_G$.*

3. SOFT UNION PRODUCT AND SOFT UNI-GROUPS

In this section, we define soft union product and study its properties as regards soft uni-groups.

DEFINITION 10. *Let $(G, \cdot)$ be a group and f_G and h_G be soft sets over the common universe U. Then, soft union product $f_G * h_G$ is defined by*

$$(f_G * h_G)(x) = \bigcap \{f_G(u) \cup h_G(v) \mid u \cdot v = x, \quad u, v \in G\},$$

and the inverse f^{-1} of f is defined by

$$f^{-1}(x) = f(x^{-1})$$

for all $x \in G$

Note that soft union product is abbreviated by soft uni-product in what follows. Soft uni-product was defined for semigroups in [28] and for rings in [29] before. However, since the algebraic structure of semigroup and ring differ from group, the definition and the properties of soft uni-product change, too. Especially, since every group has a unit element and every element in the group has its inverse, the definition of soft uni-product for soft groups has its characteristic properties.

EXAMPLE 2. *Consider the additive group $G = \mathbb{Z}_3$. Let $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ be the universal set. Let f_G and h_G be soft sets over U such that $f_G(0) = \{e, y, yx\}$, $f_G(1) = \{e, x\}$, $f_G(2) = \{y, yx\}$ and $h_G(0) = \{x, y\}$, $h_G(1) = \{e, yx\}$, $h_G(2) = \{yx\}$. Since $1 = 1 + 0 = 0 + 1 = 2 + 2$, then*

$$(f_G * h_G)(1) = \{f_G(1) \cup h_G(0)\} \cap \{f_G(0) \cup h_G(1)\} \cap \{f_G(2) \cup h_G(2)\} = \{y\}$$

Similarly, since $0 = 0 + 0 = 1 + 2 = 2 + 1$ and $2 = 1 + 1 = 0 + 2 = 2 + 0$, then $(f_G * h_G)(0) = \{e, yx\}$, $(f_G * h_G)(2) = \{yx\}$.

THEOREM 11. *Let f_G, g_G, h_G be soft sets over U. Then,*

i) $(f_G * g_G) * h_G = f_G * (g_G * h_G)$.
ii) $f_G * g_G \neq g_G * f_G$, generally. However, if G is abelian, then $f_G * g_G = g_G * f_G$.
iii) $f_G * (g_G \tilde{\cup} h_G) = (f_G * g_G) \tilde{\cup} (f_G * h_G)$ and $(f_G \tilde{\cup} g_G) * h_G = (f_G * h_G) \tilde{\cup} (g_G * h_G)$.
iv) $f_G * (g_G \tilde{\cap} h_G) = (f_G * g_G) \tilde{\cap} (f_G * h_G)$ and $(f_G \tilde{\cap} g_G) * h_G = (f_G * h_G) \tilde{\cap} (g_G * h_G)$.
v) If $f_G \tilde{\subseteq} g_G$, then $f_G * h_G \tilde{\subseteq} g_G * h_G$ and $h_G * f_G \tilde{\subseteq} h_G * g_G$.
vi) If $t_G, l_G \in S(U)$ such that $t_G \tilde{\subseteq} f_G$ and $l_G \tilde{\subseteq} g_G$, then $t_G * l_G \tilde{\subseteq} f_G * g_G$.

Proof: $i)$ follows from Definition 10 and Example 2. Let G be an abelian group and $x \in G$ such that $x = uv$. Then,

$$\begin{aligned} (f_G * g_G)(x) &= \bigcap (f_S(u) \cup g_S(v) \mid uv = x) \\ &= \bigcap (g_S(v) \cup f_S(u) \mid vu = x) \\ &= (g_G * f_G)(x) \end{aligned}$$

Hence $f_G * g_G = g_G * f_G$.
$iii)$ Let $a \in G$ such that $a = xy$. Then,

$$\begin{aligned} (f_G * (g_G \tilde{\cup} h_G))(a) &= \bigcap \{f_G(x) \cup (g_G \tilde{\cup} h_G)(y)\} \\ &= \bigcap \{f_G(x) \cup (g_G(y) \cup h_G(y)\} \\ &= \bigcap [(f_G(x) \cup g_G(y)) \cup (f_G(x) \cup h_G(y))] \\ &= [\bigcap (f_G(x) \cup g_G(y))] \cup [\bigcap (f_G(x) \cup h_G(y))] \\ &= (f_G * g_G)(a) \cup (f_G * h_G)(a) \\ &= [(f_G * g_G) \tilde{\cup} (f_G * h_G)](a) \end{aligned}$$

Thus, $(f_G \tilde{\cup} g_G) * h_G = (f_G * h_G) \tilde{\cup} (g_G * h_G)$ and (iv) can be proved similarly.

v) Let $x \in G$ such that $x = yz$, then

$$\begin{aligned}(f_G * h_G)(x) &= \bigcap (f_G(y) \cup h_G(z)) \\ &\subseteq \bigcap (g_S(y) \cup h_G(z)) \ (since\ f_G(y) \subseteq g_G(y)) \\ &= (g_G * h_G)(x)\end{aligned}$$

Similarly, one can show that $h_G * f_G \tilde{\subseteq} h_G * g_G$.

(vi) can be proved similar to (v).
In Definition 5, the soft uni-group is defined as regards the elements of G. Now, we give an equivalent definition for soft uni-groups in terms of soft uni-product.

THEOREM 12. *Let f_G be a soft set over U. Then, f_G is a soft uni-group over U if and only if $f_G * f_G \tilde{\supseteq} f_G$ and $f_G^{-1} = f_G$.*

Proof: Assume that f_G is a soft uni-group over U and $a \in G$ such that $a = xy$. Then,

$$\begin{aligned}(f_G * f_G)(a) &= \bigcap_{a=xy} (f_G(x) \cup f_G(y)) \\ &\supseteq \bigcap_{a=xy} f_G(xy) \\ &= \bigcap_{a=xy} f_G(a) \\ &= f_G(a)\end{aligned}$$

Thus, $f_G * f_G \tilde{\supseteq} f_G$. Moreover, by the definition of soft uni-group $f_G^{-1}(x) = f_G(x^{-1}) = f_G(x)$. Hence, $f_G^{-1} = f_G$.
Conversely, assume that $f_G * f_G \tilde{\supseteq} f_G$ and $f_G^{-1} = f_G$. Let $x, y \in G$ and $a = xy^{-1}$ (since G is a group, every element has an inverse). Then, we have:

$$\begin{aligned}f_G(xy^{-1}) &= f_G(a) \\ &\subseteq (f_G * f_G)(a) \\ &= \bigcap_{a=xy^{-1}} (f_G(x) \cup f_G(y^{-1})) \\ &\subseteq f_G(x) \cup f_G(y^{-1}) \\ &= f_G(x) \cup f_G(y)\end{aligned}$$

Hence, f_G is a soft uni-group over U. This completes the proof.

THEOREM 13. *Let $A, B \subset G$ and f_A, f_B be soft uni-groups. Then*

$$(f_A * f_B)^{-1} = f_B^{-1} * f_A^{-1}.$$

Proof: Let $x \in G$. Since G is a group,

$$\begin{aligned}(f_A * f_B)^{-1}(x) &= (f_A * f_B)(x^{-1}) \\ &= \bigcap\{(f_A(u) \cup f_B(v) \mid uv = x^{-1}, \ \ u, v \in G\} \\ &= \bigcap\{(f_B(v) \cup f_A(u) \mid uv = x^{-1}, \ \ u, v \in G\} \\ &= \bigcap\{f_B(v^{-1})^{-1} \cup f_A(u^{-1})^{-1} \mid v^{-1}u^{-1} = x\} \\ &= \bigcap\{f_B^{-1}(v^{-1}) \cup f_A^{-1}(u^{-1}) \mid v^{-1}u^{-1} = x\} \\ &= (f_B^{-1} * f_A^{-1})(x)\end{aligned}$$

In group theory, we know that if H and K are subgroubs of G, then HK is a subgroup of G if and only if $HK = KH$. Now , we have an analog theorem for soft uni-groups.

THEOREM 14. *Let $A, B \subset G$ and f_A, f_B be soft uni-groups. Then, $f_A * f_B$ is a soft uni-group if and only if $f_A * f_B = f_B * f_A$.*

Proof: Let f_A and f_B be soft uni-groups. First, assume that $f_A * f_B$ is a soft uni-group. Then;

$$f_B * f_A = f_B^{-1} * f_A^{-1} = (f_A * f_B)^{-1} = f_A * f_B.$$

Conversely, suppose that $f_A * f_B = f_B * f_A$. Then;

$$\begin{aligned}(f_A * f_B) * (f_A * f_B) &= f_A * (f_B * f_A) * f_B \\ &= f_A * (f_A * f_B) * f_B \\ &= (f_A * f_A) * (f_B * f_B) \\ &\supseteq f_A * f_B\end{aligned}$$

and $(f_A * f_B)^{-1} = (f_B * f_A)^{-1} = f_A^{-1} * f_B^{-1} = f_A * f_B$. Hence, $f_A * f_B$ is a soft uni-group.
In group theory, we know that if N is a normal subgroup of G and H is any subset of G, then $NH = HN$. Now , we have an analog theorem for normal soft uni-groups.

THEOREM 15. *Let $A, B \subset G$ and f_A be a normal soft uni-group and f_B be any soft set in G. Then, $f_A * f_B = f_B * f_A$.*

Proof: Let $x \in G$. Then,

$$(f_A * f_B)(x) = \bigcap (f_A(u) \cup f_B(v) \mid uv = x \ \ u, v \in G)$$

Since f_A is a normal soft uni-group (that is, abelian) and $uv = x$ implies $u = xv^{-1}$, then

$$\begin{aligned}(f_A * f_B)(x) &= \bigcap\{f_A(xv^{-1}) \cup f_B(v) \mid (xv^{-1})v = x \ \ v \in G\} \\ &= \bigcap\{f_B(v) \cup f_A(v^{-1}x) \mid v(v^{-1}x) = x, \ \ v \in G\} \\ &= (f_B * f_A)(x)\end{aligned}$$

In group theory, we know that if N is a normal subgroup of G and H is a subgroup of G, then NH is a subgroup of G. Now , we have an analog theorem for normal soft uni-groups.

THEOREM 16. *Let f_A be a normal soft uni-group and f_B be a soft uni-group. Then $f_A * f_B$ is a soft uni-group.*

Proof:

$$\begin{aligned}(f_A * f_B) * (f_A * f_B) &= f_A * (f_B * f_A) * f_B \\ &= f_A * (f_A * f_B) * f_B \\ &= (f_A * f_A) * (f_B * f_B) \\ &\supseteq f_A * f_B\end{aligned}$$

Now, let $x \in G$.Then,

$$\begin{aligned}(f_A * f_B)(x^{-1}) &= \bigcap\{f_A(u) \cup f_B(v) \mid uv = x^{-1}, \ \ u, v \in G\} \\ &= \bigcap\{f_A(u^{-1})^{-1} \cup f_B(v^{-1})^{-1} \mid v^{-1}u^{-1} = x\} \\ &= \bigcap\{f_B(v^{-1})^{-1} \cup f_A(u^{-1})^{-1} \mid v^{-1}u^{-1} = x\} \\ &= \bigcap\{f_B^{-1}(v^{-1}) \cup f_A^{-1}(u^{-1}) \mid v^{-1}u^{-1} = x\} \\ &= \bigcap\{f_B(v^{-1}) \cup f_A(u^{-1}) \mid v^{-1}u^{-1} = x\} \\ &= (f_B * f_A)(x) \\ &= (f_A * f_B)(x)\end{aligned} \quad (1)$$

Thus, $f_A * f_B$ is a soft uni-group.

4. SOFT CHARACTERISTIC FUNCTION AND SOFT UNI-GROUP

Soft characteristic function was defined for semigroups in [28] and for rings in [29] before. Now, we give the definition for group structure and we obtain the subgroups of o group by using this definition.

DEFINITION 17. *Let X be a subset of G. We denote by $\mathcal{S}_{X^c}$ the soft characteristic function of the complement X and define as*

$$\mathcal{S}_{X^c}(x) = \begin{cases} \emptyset, & \text{if } x \in X, \\ U, & \text{if } x \in G \setminus X \end{cases}$$

THEOREM 18. *Let X and Y be nonempty subsets of a group G. Then, the following properties hold:*

i) If $Y \subseteq X$, then $\mathcal{S}_{X^c} \widetilde{\subseteq} \mathcal{S}_{Y^c}$.
ii) $\mathcal{S}_{X^c} \widetilde{\cap} \mathcal{S}_{Y^c} = \mathcal{S}_{X^c \cap Y^c}$, $\mathcal{S}_{X^c} \widetilde{\cup} \mathcal{S}_{Y^c} = \mathcal{S}_{X^c \cup Y^c}$.

Proof: $i)$ is straightforward by Definition 17.

$ii)$ Let g be any element of G. Suppose $g \in X^c \cap Y^c$. Then, $g \in X^c$ and $g \in Y^c$. Thus, we have

$$(\mathcal{S}_{X^c} \widetilde{\cap} \mathcal{S}_{Y^c})(g) = \mathcal{S}_{X^c}(g) \cap \mathcal{S}_{Y^c}(g) = U \cap U = U = \mathcal{S}_{X^c \cap Y^c}(g)$$

Suppose $g \notin X^c \cap Y^c$. Then, $g \notin X^c$ or $g \notin Y^c$. Hence, we have

$$(\mathcal{S}_{X^c} \widetilde{\cap} \mathcal{S}_{Y^c})(g) = \mathcal{S}_{X^c}(g) \cap \mathcal{S}_{Y^c}(g) = \emptyset = \mathcal{S}_{X^c \cap Y^c}(s)$$

Let g be any element of G. Suppose $g \in X^c \cup Y^c$. Then, $g \in X^c$ or $g \in Y^c$. Thus, we have

$$(\mathcal{S}_{X^c} \widetilde{\cup} \mathcal{S}_{Y^c})(g) = \mathcal{S}_{X^c}(g) \cup \mathcal{S}_{Y^c}(g) = U = \mathcal{S}_{X^c \cup Y^c}(g)$$

Suppose $g \notin X^c \cup Y^c$. Then, $g \in X$ and $g \in Y$. Hence, we have

$$(\mathcal{S}_{X^c} \widetilde{\cup} \mathcal{S}_{Y^c})(g) = \mathcal{S}_{X^c}(g) \cup \mathcal{S}_{Y^c}(g) = \emptyset = \mathcal{S}_{X^c \cup Y^c}(g)$$

It is easy to see that if $f_G(x) = \emptyset$ for all $x \in G$, then f_G is a soft uni-group over U. We denote such a kind of soft uni-group by $\widetilde{\theta}$. It is obvious that $\widetilde{\theta} = \mathcal{S}_{G^c}$, i.e. $\widetilde{\theta}(x) = \emptyset$ for all $x \in G$.

LEMMA 19. *Let f_G be any soft uni-group over U. Then, we have the followings:*

i) $\widetilde{\theta} * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$.
ii) $f_G * \widetilde{\theta} \widetilde{\supseteq} \widetilde{\theta}$ and $\widetilde{\theta} * f_G \widetilde{\supseteq} \widetilde{\theta}$.
iii) $f_G \widetilde{\cap} \widetilde{\theta} = \widetilde{\theta}$ and $f_G \widetilde{\cup} \widetilde{\theta} = f_G$.

THEOREM 20. *A non-empty subset H of a group of G is a subgroup of G if and only if the soft subset f_G defined by*

$$f_G(x) = \begin{cases} \alpha, & \text{if } x \in G \setminus H, \\ \beta, & \text{if } x \in H \end{cases}$$

is a soft uni-group, where $\alpha, \beta \subseteq U$ such that $\alpha \supseteq \beta$.

Proof: Suppose H is a subgroup of G and $x, y \in G$. If $x, y \in H$, then $xy^{-1} \in H$. Hence, $f_G(xy^{-1}) = f_G(x) = f_G(y^{-1}) = f_G(y) = \beta$ and so, $f_G(xy^{-1}) \subseteq f_G(x) \cup f_G(y)$. If $x, y \notin H$, then $xy^{-1} \in H$ or $xy^{-1} \notin H$. In any case, $f_G(xy^{-1}) \subseteq f_G(x) \cup f_G(y) = \alpha$. Thus, f_G is a soft uni-group.
Conversely assume that f_G is a soft uni-group of G. Let $x, y \in H$. Then, $f_G(xy^{-1}) \subseteq f_G(x) \cup f_G(y) = \beta$. This implies that $f_G(xy^{-1}) = \beta$. Hence, $xy^{-1} \in H$ and so H is a subgroup of G.

THEOREM 21. *Let X be a nonempty subset of a group G. Then, X is a subgroup of G if and only if $\mathcal{S}_{X^c}$ is a soft uni-group of G.*

Proof: Since

$$\mathcal{S}_{X^c}(x) = \begin{cases} U, & \text{if } x \in G \setminus X, \\ \emptyset, & \text{if } x \in X \end{cases}$$

and $U \supseteq \emptyset$, the rest of the proof follows from Theorem 20.

5. CHARACTERISTIC SOFT UNI-GROUPS, CONJUGATE SOFT UNI-GROUPS AND SOFT NORMALIZER

In this section, we define characteristic soft uni-groups, conjugate soft uni-groups and soft normalizer and study their basic properties.

DEFINITION 22. *Let f_G be a soft uni-group over U and Θ be a map from G into itself. We define a map*

$$f_G^{\Theta} : G \to P(U)$$

by

$$f_G^{\Theta}(x) = f_G(\Theta(x)), \forall x \in G.$$

Then, f_G is called a characteristic soft uni-group over U if $f_G^{\Theta} = f_G$ for every automorphism Θ of G.

THEOREM 23. *Let f_G be a soft uni-group over U and Θ be a homomorphism of G. Then, f_G^{Θ} is a soft uni-group over U.*

Proof: We need to show that $f_G^{\Theta}(xy) \subseteq f_G^{\Theta}(x) \cup f_G^{\Theta}(y)$ for all $x, y \in G$. Let $x, y \in G$. Then,

$$\begin{aligned} f_G^{\Theta}(xy) &= f_G(\Theta(xy)) \\ &= f_G(\Theta(x)\Theta(y)), \ \textit{since } \Theta \textit{ is a homomorphism.} \\ &\subseteq f_G(\Theta(x)) \cup f_G(\Theta(y)) \\ &= f_G^{\Theta}(x) \cup f_G^{\Theta}(y) \end{aligned}$$

Again,

$$\begin{aligned} f_G^{\Theta}(x^{-1}) &= f_G(\Theta(x^{-1})) \\ &= f_G(\Theta(x))^{-1} \\ &= f_G(\Theta(x)) \\ &= f_G^{\Theta}(x) \end{aligned}$$

THEOREM 24. *Let f_G be a soft uni-group over U, where f_G is a bijective mapping and Θ be an epimorphism of G. Then, f_G^{Θ} is an abelian soft uni-group over U if and only if G is an abelian group.*

Proof: Let f_G be a soft uni-group over U, Θ be an epimorphism of G and G be an abelian group. In above theorem, we show that if f_G is a soft uni-group over U and Θ is a homomorphism of G, then f_G^{Θ} is a soft uni-group over U. Thus, we only show that f_G^{Θ} is an abelian soft uni-group over U. Let $x, y \in G$, then

$$\begin{aligned} f_G^{\Theta}(xy) &= f_G(\Theta(xy)) \\ &= f_G(\Theta(yx)) \\ &= f_G^{\Theta}(yx), \end{aligned}$$

which shows that f_G^{Θ} is an abelian soft uni-group over U.
Conversely, let f_G^{Θ} be an abelian soft uni-group over U where Θ is an epimorphism of G. Let $x, y \in G$, then

$$\begin{aligned} f_G^{\Theta}(xy) = f_G^{\Theta}(yx) &\Rightarrow f_G(\Theta(x)\Theta(y)) = f_G(\Theta(y)\Theta(x)) \\ &\Rightarrow \Theta(x)\Theta(y) = \Theta(y)\Theta(x). \end{aligned}$$

It follows that G is an abelian group, since Θ is an epimorphism of G.

THEOREM 25. *If f_G be a characteristic soft uni-group over U, then f_G is an abelian soft uni-group.*

Proof: Let $x, y \in G$. We need to show that $f_G(xy) = f_G(yx)$ for all $x, y \in G$. Let Θ be an automorphism of G defined by,

$$\Theta(x) = x^{-1}gx, \forall g \in G.$$

It is well-known that Θ is an automorphism of G, called *inner automorphism*. Since f_G is a characteristic soft uni-group over U, then $f_G^{\Theta} = f_G$. Thus, we have

$$\begin{aligned} f_G(xy) &= f_G^{\Theta}(xy) \\ &= f_G(\Theta(xy)) \\ &= f_G(x^{-1}(xy)x) \\ &= f_G((x^{-1}x)(yx)) \\ &= f_G(yx) \end{aligned}$$

Hence, f_G is an abelian soft uni-group.

DEFINITION 26. *Let f_G be a soft uni-group over U, f_H and f_K be soft uni-subgroups of f_G. We say that f_H is soft conjugate to f_K if for some $x \in G$, we have*

$$f_H(g) = f_K(x^{-1}gx), \forall g \in G.$$

THEOREM 27. *Conjugacy is an equivalence relation in the family of soft uni-subgroups of a soft uni-group.*

The family of soft uni-subgroups of a soft uni-group is a union of pairwise disjoint classes of soft uni-subgroups each consisting of soft uni-subgroups which are equivalent to one another. We shall now obtain an expression giving the number of distinct conjugates of a soft uni-group. First we give some preliminaries.
Let f_G be a soft uni-group over U and $g \in G$. We denote by f_G^g the map

$$f_G^g : G \to P(U)$$

by

$$f_G^g(x) = f_G(g^{-1}xg), \forall x \in G.$$

THEOREM 28. *Let f_G be a soft uni-group over U. Then, f_G^g is a soft uni-group over U for all $g \in G$.*

Proof: We need to show that $f_G^g(xy) \subseteq f_G^g(x) \cup f_G^g(y)$ for all $x, y \in G$. Let $x, y \in G$, then

$$\begin{aligned} f_G^g(xy) &= f_G(g^{-1}(xy)g) \\ &= f_G(g^{-1}x(gg^{-1})yg) \\ &= f_G((g^{-1}xg)(g^{-1}yg)) \\ &\subseteq f_G(g^{-1}xg) \cup f_G(g^{-1}yg)) \\ &= f_G^g(x) \cup f_G^g(y) \end{aligned}$$

Again,

$$\begin{aligned} f_G^g(x^{-1}) &= f_G(g^{-1}x^{-1}g) \\ &= f_G((g^{-1}xg)^{-1}) \\ &= f_G(g^{-1}xg), \\ &= f_G^g(x) \end{aligned}$$

DEFINITION 29. *The soft uni-group f_G^g in the above theorem is called conjugate soft uni-group over U determined by f_G and $x \in G$.*

DEFINITION 30. *Let f_G be a soft uni-group over U. Then,*

$$N(f_G) = \{g \in G : f_G^g = f_G\}$$

is called the soft normalizer of f_G.

THEOREM 31. *Let f_G be a soft uni-group over U. Then $N(f_G)$ is a subgroup of G.*

Proof: Let $a, b \in N(f_G)$. Then,

$$\begin{aligned} f_G^{ab}(x) &= f_G((ab)^{-1}xab) \\ &= f_G(b^{-1}a^{-1}xab) \\ &= f_G(b^{-1}(a^{-1}xa)b) \\ &= f_G^b(a^{-1}xa) \\ &= f_G(a^{-1}xa), \quad since \;\; b \in N(f_G) \\ &= f_G^a(x) \\ &= f_G(x), \quad since \;\; a \in N(f_G) \end{aligned}$$

Thus, $f_G^{ab} = f_G$, implying that $ab \in N(f_G)$.
Again, let $x \in N(f_G)$ and let $y = x^{-1}$. We show that $y \in N(f_G)$. For any $w \in G$, we have

$$\begin{aligned} f_G^y(w) &= f_G(y^{-1}wy) \\ &= f_G(xwx^{-1}) \\ &= f_G((x^{-1}w^{-1}x)^{-1}) \\ &= f_G(x^{-1}w^{-1}x) \\ &= f_G^x(w^{-1}) \\ &= f_G(w^{-1}), since \;\; f_G^x = f_G. \\ &= f_G(w) \end{aligned}$$

Thus, $f_G^y = f_G$, so $y \in N(f_G)$. It follows that $N(f_G)$ is a subgroup of G.

THEOREM 32. *Let f_G be a soft uni-group over U. Then, f_G is an abelian soft uni-group over U if and only is $N(f_G) = G$.*

Proof: Let f_G be an abelian soft uni-group over U, $g \in G$. Then for any $w \in G$, we have

$$\begin{aligned} f_G^g(w) &= f_G(g^{-1}wg) \\ &= f_G((g^{-1}g)w), \quad since \;\; f_G \;\; is \;\; abelian. \\ &= f_G(w) \end{aligned}$$

Thus, $f_G^g = f_G$ implying that $g \in N(f_G)$. It follows that $N(f_G) = G$.
Conversely, let $N(f_G) = G$ and $x, y \in G$. To prove that f_G is an abelian soft uni-group, we need to show that

$$f_G(xy) = f_G(yx)$$

for all $x, y \in G$. We have

$$\begin{aligned} f_G(xy) &= f_G(xy(xx^{-1})) \\ &= f_G(x(yx)x^{-1}) \\ &= f_G^{x^{-1}}(yx), \;\; by \;\; definition \;\; of \;\; f_G^{x^{-1}}. \\ &= f_G(yx) \end{aligned}$$

since $N(f_G) = G$ and so $x^{-1} \in N(f_G)$, implying that $f_G^{x^{-1}} = f_G$.
Thus, f_G is an abelian soft uni-group.
The above theorem illustrates the motivation behind the term normalizer, and also it shows the analogy with the fact that a subgroup H of a group G is normal in G if and only if the normalizer of H in G is equal to G itself. Again, the following theorem is an analog of a basic result in group theory.

THEOREM 33. *Let f_G be a soft uni-group over U. Then, the number of distinct conjugates of f_G is equal to $[G : N(f_G)]$, that is, the index of $N(f_G)$ in G, provided that G is a finite group.*

Proof: The proof of this result is based on the same technique used to prove the corresponding analogous result for groups. Consider the decomposition of G as a union of cosets of $N(f_G)$:

$$G = x_1N(f_G) \cup x_2N(f_G) \cup ... \cup x_kN(f_G),$$

where k is the number of distinct cosets, that is $[G : N(f_G)]$. Let $x \in N(f_G)$ and choose i such that $1 \leq i \leq k$. Then for $g \in G$,

$$\begin{aligned} f_G^{x_i x}(g) &= f_G((x_i x)^{-1} g(x_i x)) \\ &= f_G(x^{-1}(x_i^{-1} g x_i)x) \\ &= f_G^{x}(x_i^{-1} g x_i) \\ &= f_G(x_i^{-1} g x_i) \;\; since \;\; x \in N(f_G) \\ &= f_G^{x_i}(g) \end{aligned}$$

Thus, we have

$$f_G^{x_i x}(g) = f_G^{x_i}(g), \text{ for all } x \in N(f_G), 1 \leq i \leq k.$$

So any two elements of G which lie in the same coset $x_iN(f_G)$ give rise to the same conjugate of $f_G^{x_i}$ of f_G. Now we show that two distinct cosets give two distinct conjugates of f_G. For this, suppose that

$$f_G^{x_i} = f_G^{x_j},$$

where $i \neq j$ and $1 \leq i \leq k$, $1 \leq j \leq k$. Thus,

$$\begin{aligned} f_G^{x_i} = f_G^{x_j} &\Leftrightarrow f_G^{x_i}(g) = f_G^{x_j}(g), \;\; \forall g \in G. \\ &\Leftrightarrow f_G(x_i^{-1} g x_i) = f_G(x_j^{-1} g x_j), \;\; \forall g \in G. \end{aligned}$$

If we choose $g = x_j t x_j^{-1}$, it follows that,

$$\begin{aligned} f_G(x_i^{-1} x_j t x_j^{-1} x_i) &= f_G(x_j^{-1} x_j t x_j^{-1} x_j) \\ &\Rightarrow f_G((x_j^{-1}x_i)^{-1} t (x_j^{-1} x_i)) = f_G(t), \;\; \forall t \in G. \\ &\Rightarrow f_G^{x_j^{-1} x_i}(t) = f_G(t), \;\; \forall t \in G. \\ &\Rightarrow x_j^{-1} x_i \in N(f_G) \\ &\Rightarrow x_i N(f_G) = x_j N(f_G) \end{aligned}$$

However, if $i \neq j$, this is not possible when we consider the decomposition of G as a union of cosets of $N(f_G)$. Hence the number of distinct conjugates of f_G is equal $[G : N(f_G)]$.

THEOREM 34. *Let f_G be a soft uni-group over U and f_N be a soft uni-subgroup of f_G. Then, f_N is a normal soft uni-subgroup of f_G if and only if f_N is constant on each conjugate classes of N.*

Proof: Suppose that $f_N \widetilde{\lhd}_u f_G$. Then,

$$f_N(y^{-1}xy) = f_N(y^{-1}yx) = f_N(x)$$

for all $x, y \in N$. Conversely, suppose that f_N is constant on each conjugate classes of N. Since $f_N \widetilde{\leq}_u f_G$, then it is itself a soft uni-group, therefore it is enough to show that f_N is abelian. Let $x, y \in N$, then

$$\begin{aligned} f_N(xy) &= f_N(xy(xx^{-1})) \\ &= f_N(x(yx)x^{-1}) \\ &= f_N(yx), \end{aligned}$$

which shows that f_N is a normal soft uni-subgroup of f_G.

THEOREM 35. *Let f_G be a soft uni-group over U and f_N be a soft uni-subgroup of f_G. Then,*

$$f_N^{\subseteq\beta} = \{x \in N \mid f_N(x) \subseteq \beta\}$$

is a normal subgroup of N, where $\beta \subseteq U$ and $\beta \supseteq f_N(e)$.

Proof: Since $\beta \supseteq f_N(e)$, thus $e \in f_N^{\subseteq\beta}$ and $\emptyset \neq f_N^{\subseteq\beta} \subseteq N$. Now, let $x, y \in f_N^{\subseteq\beta}$. Then, $f_N(x) \subseteq \beta$ and $f_N(y) \subseteq \beta$. It follows that

$$\begin{aligned} f_N(xy^{-1}) &\subseteq f_N(x) \cup f_N(y) \\ &\subseteq \beta \cup \beta = \beta, \end{aligned}$$

implying that $xy^{-1} \in f_N^{\subseteq\beta}$. Now assume that $x \in f_N^{\subseteq\beta}$ and $n \in N$. Since $f_N \widetilde{\leq}_u f_G$, f_N is constant on each conjugate classes of N by the above theorem. It follows that

$$f_N(nxn^{-1}) = f_N(x) \subseteq \beta$$

for all $n \in N$ and $x \in f_N^{\subseteq\beta}$, which shows that $nxn^{-1} \in f_N^{\subseteq\beta}$. So, this completes the proof.
We now give an alternative formulation of soft uni-group in terms of "*commutators*" of a group. First we recall that if G is any group and $x, y \in G$, then the element $x^{-1}y^{-1}xy$ is usually denoted by $[x, y]$ and called the *commutator* of x and y. If G is abelian, then $[x, y] = e$ for all $x, y \in G$. This motivates the following proposition:

PROPOSITION 2. *Let f_G be a soft uni-group over U. Then we have the following:*

i) If f_G is an abelian soft uni-group over U, then $f_G[x, y] = f_G(e)$ for all $x, y \in G$.
ii) If $f_G[x, y] = f_G(e)$ for all $x, y \in G$, where f_G is a bijective function, then f_G is an abelian soft uni-group over U.

Proof: *i)* Let f_G be an abelian soft uni-group over U. Then,

$$\begin{aligned} f_G[x, y] &= f_G(x^{-1}y^{-1}xy) \\ &= f_G(x^{-1}(y^{-1}y)x) \\ &= f_G(e) \end{aligned}$$

ii) Let $x, y \in G$. Then,

$$\begin{aligned} f_G[x, y] = f_G(x^{-1}y^{-1}xy) = f_G(e) &\Leftrightarrow x^{-1}y^{-1}xy = e \\ &\Leftrightarrow yx = xy, \\ &\Leftrightarrow f_G(yx) = f_G(xy) \end{aligned}$$

implying that f_G is an abelian soft uni-group over U.

THEOREM 36. *Let f_G be a soft uni-group over U. Then, for all $x, y \in N$, f_N is a normal soft uni-normal subgroup of f_G if and only is $f_N[x, y] \subseteq f_N(x)$.*

Proof: Suppose that f_N is a normal soft uni-subgroup of f_G and $x, y \in N$. Then,

$$\begin{aligned} f_N[x, y] = f_N(x^{-1}y^{-1}xy) &\subseteq f_N(x^{-1}) \cup f_N(y^{-1}xy) \\ &= f_N(x) \cup f_N((y^{-1}y)x) \\ &= f_N(x) \cup f_N(x) \\ &= f_N(x) \end{aligned}$$

Conversely, suppose that $f_N[x, y] \subseteq f_N(x)$ for all $x, y \in N$. In order to show that $f_N \widetilde{\lhd}_u f_G$, it is enough to show f_N is constant on each conjugate classes of N. Let $x, z \in N$, then we have

$$\begin{aligned} f_N(x^{-1}zx) &= f_N((zz^{-1})x^{-1}zx) \\ &\subseteq f_N(z(z^{-1}x^{-1}zx)) \\ &\subseteq f_N(z) \cup f_N[z,x] \\ &= f_N(z) \end{aligned}$$

Thus,

$$f_N(x^{-1}zx) \subseteq f_N(z)$$

Again, we get

$$\begin{aligned} f_N(z) &= f_N((xx^{-1})z(xx^{-1})) \\ &= f_N(x(x^{-1}zx)x^{-1}) \\ &\subseteq f_N(x) \cup f_N(x^{-1}zx) \cup f_N(x^{-1}) \\ &= f_N(x) \cup f_N(x^{-1}zx) \cup f_N(x) \\ &= f_N(x) \cup f_N(x^{-1}zx) \end{aligned}$$

If $f_N(x) \cup f_N(x^{-1}zx) = f_N(x)$, then we obtain that $f_N(z) \subseteq f_N(x)$ for all $x, z \in N$, implying that f_N is a constant function and in this case, obviously $f_N(x^{-1}zx) \subseteq f_N(z)$ is satisfied as well, thus the result holds immediately. So we consider the case when $f_N(x) \cup f_N(x^{-1}zx) = f_N(x^{-1}zx)$. Then,

$$f_N(z) \subseteq f_N(x^{-1}zx), \text{ and so } f_N(x^{-1}zx) = f_N(z).$$

Hence, f_N is a normal soft uni-subgroup of f_G.
Let $f_N \widetilde{\leq}_u f_G$ and f_N be an abelian soft uni-group over U. Then it is obvious that $f_N \widetilde{\lhd_u} f_G$. In fact, since f_N is an abelian soft uni-group, then by Proposition 2 (i),

$$f_N[x,y] = f_N(e) \subseteq f_N(x) \text{ for all } x, y \in N.$$

Since

$f_N[x,y] \subseteq f_N(x)$, it follows that $f_N \widetilde{\lhd_u} f_G$ by the above theorem.

$$\begin{aligned} f_G(e) &= \{(13)\} \\ f_G(x) &= \{e, (12), (13)\} \\ f_G(y) &= \{e, (13), (23)\} \\ f_G(yx) &= \{e, (12), (13), (23)\}. \end{aligned}$$

6. CONCLUSION

Soft int-groups and its related properties was first introduced and studied in [25] and soft uni-groups and its related properties in [27]. In this paper, first by defining soft uni-product, we have investigated this concept as regards sof uni-groups and we have obtained some significant relations between subgroups of a group and and soft uni-groups. Moreover, we have extended the study of soft uni-groups. We introduced the concepts of characteristic soft uni-group, soft normalizer, soft conjugate and conjugate soft uni-groups. We have proved several results and made some characterizations about soft uni-groups by using these concepts which correspond to significant results in group theory. We also gave some formulations for uni-soft groups in terms of commutators. To extend this study, one can further study the soft cosets and normal soft uni-subgroups in the mean of quotient groups.

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