Truncated Compound Normal with Gamma Mixture Model for Mixture Density Estimation

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ABSTRACT

In this paper, the truncated compound normal with gamma distribution model is formally presented and its density function has been derived for defining a mixture model(TCNGM) based on this as an extension work to the proposed compound normal with gamma mixture(CNGM) model introduced in our earlier work for image segmentation. Update equations for this model have been derived in the context of maximum likelihood estimation(MLE) procedure under Expectation Maximization(EM) framework.

General Terms

Probabilistic models, Compound Distributions, Mixture Distributions, Truncated Distributions, Maximum Likelihood Estimation, EM Framework.

Keywords

TCNGM, CNGM, NM, GM, MLE, EM

1. INTRODUCTION

In this paper, a formal treatment of the truncated compound normal with gamma distribution model, its density function, and a mixture model based on this(TCNGM) is presented as an extension work to the proposed compound normal with gamma mixture(CNGM) model introduced in **[1],[2]** for image segmentation. A truncated distribution has been introduced in **[2]** citing the reasons for truncation and the problems that are sometimes solved using such a distribution model.

1.1 Compound Normal With Gamma Distribution

As given in **[3]** by Normal L. Johnson et al, a compound normal with gamma distribution or $Normal(\mu,\sigma^2) \underset{\sigma^{-2}}{\Lambda} Gamma(c\chi_v^2)$ is formed by ascribing a distribution to σ^2 i.e., variance by considering it as a random variable and fitting a new distribution. The corresponding distribution is defined to have a density function given as

$$f(x) = (2c)^{\frac{-v}{2}}\left[\Gamma\left(\frac{v}{2}\right)\right]^{-1}\int_0^\infty\left[\sqrt{2\pi}\sigma\right]^{-1}(\sigma^{-2})^{\left(\frac{v}{2}-1\right)} .exp[-(2c\sigma^2)^{-1}(2\sigma^2)^{-1}(x-\mu)^2]d\sigma^{-2} \quad (1)$$

After some mathematical transformations and further treatment, Equation (1) reduces to

$$f(x) = \frac{1}{c^{1/2}B(1/2, v/2)}\left[1+\frac{(x-\mu)^2}{c}\right]^{-(v+1)/2} \quad (2)$$

The compound normal with gamma distribution model that has been introduced has formed the basis for our work**[1],[2]** and a mixture model for this is used to solve the image segmentation problem.

1.2 Truncated Distributions

As stated in **[4]**, truncated distributions are formed by restricting the domain of some other probability distribution. Truncated distributions are useful to solve problems where the values lie above or below a given threshold or within a specified range.

In general, if *X* is a random variable with density $f_x(.)$ and cumulative distribution $F_x(.)$, then the density of *X* truncated on the left at *a* and on the right at *b* is given by**[5]**

$$\frac{f_x(x)}{F_x(b)-F_x(a)} \quad (3)$$

For example, image segmentation problem may be viewed as mixture density estimation problem and since gray level images are spatially represented using an eight bit intensity or pixel value, the pixels only take values ranging between 0 and 255, each representing a particular gray value ranging between black and white. This strongly suggests to define a truncated mixture model, with $0 \le x \le 255$ in place of the more general case of $-\infty < x < +\infty$ for the random variable *x* that represents intensity value, for image segmentation because truncated distributions model finite range data well in comparison to the more general model.

1.3 Mixture Distribution

A brief introduction as given by Mood et al in **[5]** to the concept of contagious distribution or a mixture is given here. If $f_0(.), f_1(.), \ldots, f_n(.), \ldots$ is a sequence of density functions which are either all discrete density functions or all probability density functions which may or may not depend on parameters, and $p_0, p_1, \ldots, p_n, \ldots$ is a sequence of parameters satisfying $p_i \ge 0$ and $\sum_{i=0}^{\infty} p_i = 1$, then $\sum_{i=0}^{\infty} p_i f_i(x)$ is a density function, which is sometimes called contagious distribution or a mixture.

Physical considerations of the random experiment at hand can sometimes persuade one to consider modeling the experiment with a mixture. The experimenter may know that the phenomena that he is observing are a mixture; for example, the radioactive particle emissions under observation might be a mixture of the emissions of two, or several, different types of radioactive materials **[5]**.

For example, the current literature on statistical image segmentation techniques mostly assumes the data describing the image as a mixture of component distributions, as shown in Fig. 1 **[6]**,**[7]**,**[8]**,**[9]**.

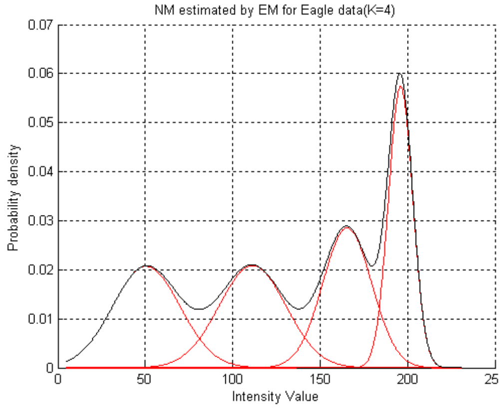

Figure 1 An Example Mixture Distribution

1.4 Clustering As Mixture Density Estimation Problem

Several researchers have viewed clustering as mixture density estimation problem in the framework of probabilistic modeling for cluster analysis. For example, image segmentation may be thought of as a clustering problem. The current literature on statistical image segmentation techniques mostly assumes the image as of containing a mixture of components each of which following normal distribution(Normal or Gaussian Mixture) i.e., $N(\mu,\sigma^2)$ with some weight **[6]**,**[7]**,**[8]**,**[9]**. In our previous work**[1]**, we assumed the image as of containing a mixture of components each of which following compound normal with gamma distribution(CNGM) i.e., $Normal(\mu,\sigma^2)\underset{\sigma^{-2}}{\Lambda}Gamma(c\chi_v^2)$ with some weight. And the whole image is thought of as following the weighted distribution where weighted distribution implies weighted average of the constituent components. In that work, we have studied the feasibility of CNGM vis-a-vis normal mixture(NM) model in consideration to variations of normal distribution.

In this paper, we present the truncated version of compound normal with gamma distribution as a viable model for solving any problem that comes under the scope of cluster analysis. In particular, the scope of this paper is to describe the process of deriving the analytical expressions for model parameters in the context of maximum likelihood estimation which involved considerable mathematical rigor. The use of this model for solving image segmentation or other similar problems will be considered separately.

2. TRUNCATED COMPOUND NORMAL WITH GAMMA MIXTURE MODEL

We know that, for compound normal with gamma distribution, the equality that is given below holds.

$f(x) =$

$$\frac{1}{c^{1/2}B(v/2,1/2)}\int_{-\infty}^{+\infty}\left[1+\frac{(x-\mu)^2}{c}\right]^{-\frac{(v+1)}{2}}dx = 1 \qquad (4)$$

After transformations **(See Appendix)**, the above equation may be re written as

$$\frac{1}{B(1/2,v/2)}\int_0^1 t^{\left(\frac{v}{2}-1\right)}(1-t)^{\left(\frac{1}{2}-1\right)}dt = 1$$
(5)

where $t = \left[1+\frac{z^2}{c}\right]^{-1} = \frac{c}{c+z^2}$ and $z = (x-\mu)$

For the above, the cumulative distribution may be obtained **[3]**, given a value for $z \ge 0$ or $(x-\mu)\ge 0$ (by choosing a value for x) as

$\Pr[z_v \le z] = \Pr[z_v \le 0] + \Pr[0 \le z_v \le z]$

$$= \Pr[z_v \le 0] + \frac{1}{c^{1/2}B(1/2,v/2)}\int_0^z\left[1+\frac{y^2}{c}\right]^{-\frac{(v+1)}{2}}dy$$

$$\Rightarrow \frac{1}{2}+\frac{1}{2B(1/2,v/2)}\int_t^1 w^{\left(\frac{v}{2}-1\right)}(1-w)^{\left(\frac{1}{2}-1\right)}dw$$

$$\Rightarrow \frac{1}{2}+\frac{1}{2}-\frac{1}{2B(1/2,v/2)}\int_0^t w^{\left(\frac{v}{2}-1\right)}(1-w)^{\left(\frac{1}{2}-1\right)}dw$$

$$\Rightarrow 1-\frac{1}{2}I_t\left(\frac{v}{2},\frac{1}{2}\right)$$
(6)

where $I_t\left(\frac{v}{2},\frac{1}{2}\right)$ is incomplete beta function ratio defined as

$$I_t\left(\frac{v}{2},\frac{1}{2}\right) = \frac{1}{B(v/2,1/2)}\int_0^t w^{\left(\frac{v}{2}-1\right)}(1-w)^{\left(\frac{1}{2}-1\right)}dw$$
(7)

The cumulative distribution for $z \le 0$ or $(x-\mu)\le 0$ is

$$1-\left(1-\frac{1}{2}I_t\left(\frac{v}{2},\frac{1}{2}\right)\right) = \frac{1}{2}I_t\left(\frac{v}{2},\frac{1}{2}\right)$$
(8)

The probability density function of Truncated Compound Normal with Gamma Mixture(TCNGM) distribution after choosing left and right truncating points as a and b is defined as in Equation(3) where

$f(x) = \frac{1}{c^{1/2}B(v/2,1/2)}\left[1+\frac{(x-\mu)^2}{c}\right]^{-\frac{(v+1)}{2}}$ is the density function defined for the compound normal with gamma distribution,

$$F(b) = 1-\frac{1}{2}I_{b_1}\left(\frac{v}{2},\frac{1}{2}\right) \qquad (9)$$

is the cumulative distribution function for some x taking value b such that $x \ge \mu$, and

$$F(a) = \frac{1}{2}I_{a_1}\left(\frac{v}{2},\frac{1}{2}\right)$$
(10)

is the cumulative distribution function for some x taking value a such that $x \le \mu$.

In the Equations (9) and (10), $b_1 = \frac{c}{c+(b-\mu)^2}$ and $a_1 = \frac{c}{c+(a-\mu)^2}$

since $t = \frac{c}{c+(x-\mu)^2}$

Therefore, Equation (3) may be written as

$$f(x) = \frac{2}{c^{1/2}B(v/2,1/2)\left[2-\left[I_{a_1}\left(\frac{v}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x-\mu)^2}{c}\right]^{-\frac{(v+1)}{2}}$$

(11)

or

$$f(x) = \frac{2}{c^{1/2}\left[2B(v/2\,,1/2) - \left[B_{a_1}\left(\frac{v}{2},\frac{1}{2}\right) + B_{b_1}\left(\frac{v}{2},\frac{1}{2}\right)\right]\right]}\left[1 + \frac{(x-\mu)^2}{c}\right]^{-\frac{(v+1)}{2}} \quad (12)$$

where $B_{a_1}\left(\frac{v}{2},\frac{1}{2}\right)$ and $B_{b_1}\left(\frac{v}{2},\frac{1}{2}\right)$ are incomplete beta functions. The $f(x)$ in Equation (12) is the new density function for the truncated compound normal with gamma distribution with *a* and *b* as left and right truncation points.

3. ANALYTICAL EXPRESSIONS FOR MODEL PARAMETERS, $\theta_l(\mu_l, c_l, v_l)$, FOR TCNGM

In **[1],[2]**, the steps involved in the maximum likelihood estimation **[10]** of the model parameters under Expectation Maximization framework **[11]** for a mixture density problem have been formally treated in the context of compound normal with gamma mixture model. In this section, the analytical expressions for the maximum likelihood estimates for model parameters, $\theta_l(\mu_l, c_l, v_l)$ which describe partly the parameter set Θ are derived in the context of the use of TCNGM under EM framework. EM algorithm optimizes the expected value of the complete data likelihood using expectation and maximization steps iteratively until convergence is reached. This optimization function is formally defined as

$$Q(\Theta, \Theta^g) = \sum_{l=1}^{M}\sum_{i=1}^{N} \log\left(\alpha_l p_l(x_i|\theta_l)\right) p(l|x_i, \Theta^g)$$

$$= \sum_{l=1}^{M}\sum_{i=1}^{N} \log(\alpha_l)\, p(l|x_i,\Theta^g) + \sum_{l=1}^{M}\sum_{i=1}^{N} \log\left(p_l(x_i|\theta_l)\right) p(l|x_i,\Theta^g) \quad (13)$$

where α_l is the prior probability of *l*th component of the mixture, $p_l(x_i|\theta_l)$ is the conditional probability of x_i belonging to *l* and is defined for our model as in Equation (12) , and $p(l|x_i,\Theta^g)$ is the posterior probability of component *l* given x_i and current estimates of parameters Θ^g and is defined as

$$p(l|x_i,\Theta^g) = \frac{\alpha_l p_l(x_i|\theta_l)}{\sum_{l=1}^{M}\alpha_l p_l(x_i|\theta_l)} \quad (14)$$

A similar treatment is also required for the truncated version except that the modified version of the likelihood function that uses the density function defined as in Equation (12) in the previous section is used. This density function for *l*th component is nothing but $p_l(x_i|\theta_l)$ that appears in the second term in Equation(13). Hence, in this section, the steps involved for deriving analytical expressions for $\theta_l(\mu_l, c_l, v_l)$ for the truncated version effected by the new density function are only shown.

The partial derivatives with respect to the model parameters $\mu_l, c_l, and\ v_l$, after equating them to zero**[1],[2]**, are given as:

$$\frac{\partial}{\partial\mu_l}\left[\sum_{l=1}^{M}\sum_{i=1}^{N}\left[log\frac{2}{c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)\right]=0 \quad (15)$$

$$\frac{\partial}{\partial c_l}\left[\sum_{l=1}^{M}\sum_{i=1}^{N}\left[log\frac{2}{c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)\right]=0 \quad (16)$$

$$\frac{\partial}{\partial v_l}\left[\sum_{l=1}^{M}\sum_{i=1}^{N}\left[log\frac{2}{c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)\right]=0 \quad (17)$$

Or, for component *l*, the above equations take the form as:

$$\frac{\partial}{\partial\mu_l}\left[\sum_{i=1}^{N}\left[log\frac{2}{c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)\right]=0 \quad (18)$$

$$\frac{\partial}{\partial c_l}\left[\sum_{i=1}^{N}\left[log\frac{2}{c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)\right]=0 \quad (19)$$

$$\frac{\partial}{\partial v_l}\left[\sum_{i=1}^{N}\left[log\frac{2}{c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)\right]=0 \quad (20)$$

3.1 Derivation Of Expression For μ_l

Equation (18) can be rewritten as

$$\sum_{i=1}^{N}\left[-\frac{\partial}{\partial\mu_l}\log\left[c_l{}^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]\right]+\frac{\partial}{\partial\mu_l}\,log\,\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^g)=0$$

$$\Rightarrow \sum_{i=1}^{N}\left[-\frac{\frac{\partial}{\partial\mu_l}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+(v_l+1)\left(\frac{x_i-\mu_l}{c_l}\right)\right]p(l|x_i,\Theta^g)=0$$

In **Appendix**, we present the details for the second 'log' term approximation and derivation of, for example, $\frac{\partial}{\partial\mu_l}B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)$.

$$\Rightarrow \sum_{i=1}^{N}\left[\frac{\frac{2}{c_l{}^{1/2}}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+(v_l+1)\left(\frac{x_i-\mu_l}{c_l}\right)\right]p(l|x_i,\Theta^g)=0$$

$$\Rightarrow \sum_{i=1}^{N}\left[\frac{2\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{c_l^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+(v_l+1)\left(\frac{x_i-\mu_l}{c_l}\right)\right]p(l|x_i,\Theta^{g})=0$$

$$\Rightarrow \sum_{i=1}^{N}\left[\frac{2c_l^{1/2}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+(v_l+1)(x_i-\mu_l)\right]p(l|x_i,\Theta^{g})=0$$

$$\Rightarrow (v_l+1)\mu_l\sum_{i=1}^{N}p(l|x_i,\Theta^{g})=\sum_{i=1}^{N}\left((v_l+1)x_i+\frac{2c_l^{1/2}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\right)p(l|x_i,\Theta^{g})$$

$$\Rightarrow \mu_l\sum_{i=1}^{N}p(l|x_i,\Theta^{g})=\sum_{i=1}^{N}x_i\;p(l|x_i,\Theta^{g})+\sum_{i=1}^{N}\frac{2c_l^{1/2}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{(v_l+1)\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\;p(l|x_i,\Theta^{g})$$

$$\therefore\ \mu_l=\frac{\sum_{i=1}^{N}x_i\,p(l|x_i,\Theta^{g})}{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}+\frac{2c_l^{1/2}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{(v_l+1)\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\frac{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}\quad\text{or}$$

$$\mu_l=\frac{\sum_{i=1}^{N}x_i\,p(l|x_i,\Theta^{g})}{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}+\frac{2c_l^{1/2}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{(v_l+1)B\left(\frac{v_l}{2},\frac{1}{2}\right)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\qquad(21)$$

3.2 Derivation Of Expression For c_l

Equation (19) can be rewritten as

$$\sum_{i=1}^{N}\left[-\frac{\partial}{\partial c_l}\frac{\log c_l}{2}-\frac{\partial}{\partial c_l}log\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]+\frac{\partial}{\partial c_l}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^{g})=0$$

$$\Rightarrow \sum_{i=1}^{N}\left[-\frac{1}{2c_l}-\frac{\frac{\partial}{\partial c_l}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\frac{\partial}{\partial c_l}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]^{-\frac{(v_l+1)}{2}}\right]p(l|x_i,\Theta^{g})=0$$

The details of the second 'log' term approximation and derivation of, for example, $\frac{\partial}{\partial c_l}B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)$ are presented in **Appendix**. Therefore, the above equation changes to

$$\sum_{i=1}^{N}\left[-\frac{1}{2c_l}+\frac{\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{c_l c_l^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\frac{(v_l+1)(x_i-\mu_l)^2}{2c_l^2}\right]p(l|x_i,\Theta^{g})=0$$

$$\Rightarrow \sum_{i=1}^{N}\left[1-\frac{2\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{c_l^{1/2}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}-\frac{(v_l+1)(x_i-\mu_l)^2}{c_l}\right]p(l|x_i,\Theta^{g})=0\text{ , or}$$

$$\sum_{i=1}^{N}\left[c_l-\frac{2c_l^{1/2}\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}-(v_l+1)(x_i-\mu_l)^2\right]p(l|x_i,\Theta^{g})=0$$

$$\Rightarrow \left[c_l-\frac{2c_l^{1/2}\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\right]\sum_{i=1}^{N}p(l|x_i,\Theta^{g})=(v_l+1)\sum_{i=1}^{N}(x_i-\mu_l)^2\,p(l|x_i,\Theta^{g})$$

$$\Rightarrow c_l-\frac{2c_l^{1/2}\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}=\frac{(v_l+1)\sum_{i=1}^{N}(x_i-\mu_l)^2p(l|x_i,\Theta^{g})}{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}$$

$$\therefore c_l=\frac{(v_l+1)\sum_{i=1}^{N}(x_i-\mu_l)^2p(l|x_i,\Theta^{g})}{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}+\frac{2c_l^{1/2}\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\quad\text{or}$$

$$c_l=\frac{(v_l+1)\sum_{i=1}^{N}(x_i-\mu_l)^2p(l|x_i,\Theta^{g})}{\sum_{i=1}^{N}p(l|x_i,\Theta^{g})}+\frac{2c_l^{1/2}\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{B\left(\frac{v_l}{2},\frac{1}{2}\right)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\qquad(22)$$

3.3 Derivation Of Expression For v_l

Equation (20) can be rewritten as

$$\sum_{i=1}^{N}\left[-\frac{\partial}{\partial v_l} log\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]-\frac{\partial}{\partial v_l}\left(\frac{v_l}{2}\right)\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]\right]p(l|x_i,\Theta^g)=0$$

$$\Rightarrow \sum_{i=1}^{N}\left[\frac{\frac{\partial}{\partial v_l}\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\frac{1}{2}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]\right]p(l|x_i,\Theta^g)=0$$

Please see **Appendix** for the proof for $\frac{\partial}{\partial v_l}B\left(\frac{v_l}{2},\frac{1}{2}\right)$. Therefore, the above equation changes to

$$\sum_{i=1}^{N}\left[\frac{\left[2\left(-\frac{1}{2}B\left(\frac{v_l}{2},\frac{3}{2}\right)\right)-\left[-\frac{1}{2}B_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+\left(-\frac{1}{2}B_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right)\right]\right]}{\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-\left[B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+B_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\frac{1}{2}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]\right]p(l|x_i,\Theta^g)=0$$

$\Rightarrow$

$$\sum_{i=1}^{N}\left[\frac{-\left[2B\left(\frac{v_l}{2},\frac{3}{2}\right)-B\left(\frac{v_l}{2},\frac{3}{2}\right)\left[I_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right]\right]}{2\left[2B\left(\frac{v_l}{2},\frac{1}{2}\right)-B\left(\frac{v_l}{2},\frac{1}{2}\right)\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\frac{1}{2}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]\right]p(l|x_i,\Theta^g)=0$$

$\because \frac{\partial}{\partial v_l}B_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)=-\frac{1}{2}B_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)$

for which, the proof is given in **Appendix**

$$\Rightarrow \sum_{i=1}^{N}\left[\frac{-\mathrm{B}\left(\frac{v_l}{2},\frac{3}{2}\right)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right]\right]}{\mathrm{B}\left(\frac{v_l}{2},\frac{1}{2}\right)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]\right]p(l|x_i,\Theta^g)=0$$

The above equation further leads to

$$\sum_{i=1}^{N}\left[\frac{-\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right]\right]}{(v_l+1)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}+\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]\right]p(l|x_i,\Theta^g)=0$$

$\because \frac{\mathrm{B}\left(\frac{v_l}{2},\frac{3}{2}\right)}{\mathrm{B}\left(\frac{v_l}{2},\frac{1}{2}\right)}=\frac{1}{(v_l+1)}$ Please refer to **Appendix** for the proof.

The above equation may be written as

$$(v_l+1)\sum_{i=1}^{N}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]p(l|x_i,\Theta^g)=\frac{\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right]\right]}{\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\sum_{i=1}^{N}p(l|x_i,\Theta^g)$$

$$\therefore\ v_l=\frac{\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right]\right]}{\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\frac{\sum_{i=1}^{N}p(l|x_i,\Theta^g)}{\sum_{i=1}^{N}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]p(l|x_i,\Theta^g)}-1 \quad (23)$$

Therefore, the update equations for μ_l, c_l, and v_l after solving Equations (18), (19), and (20) are

$$\mu_l=\frac{\sum_{i=1}^{N}x_i\,p(l|x_i,\Theta^g)}{\sum_{i=1}^{N}p(l|x_i,\Theta^g)}+\frac{2c_l^{1/2}\left[\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{(v_l+1)B\left(\frac{v_l}{2},\frac{1}{2}\right)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]} \quad (24)$$

$$c_l=\frac{(v_l+1)\sum_{i=1}^{N}(x_i-\mu_l)^2p(l|x_i,\Theta^g)}{\sum_{i=1}^{N}p(l|x_i,\Theta^g)}+\frac{2c_l^{1/2}\left[(a-\mu_l)\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}+(b-\mu_l)\left[\frac{c_l}{c_l+(b-\mu_l)^2}\right]^{\frac{v_l+1}{2}}\right]}{B\left(\frac{v_l}{2},\frac{1}{2}\right)\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]} \quad (25)$$

$$v_l=\frac{\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{3}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{3}{2}\right)\right]\right]}{\left[2-\left[I_{a_1}\left(\frac{v_l}{2},\frac{1}{2}\right)+I_{b_1}\left(\frac{v_l}{2},\frac{1}{2}\right)\right]\right]}\frac{\sum_{i=1}^{N}p(l|x_i,\Theta^g)}{\sum_{i=1}^{N}\log\left[1+\frac{(x_i-\mu_l)^2}{c_l}\right]p(l|x_i,\Theta^g)}-1 \quad (26)$$

4. CONCLUSIONS

In this paper, a formal treatment of mixture density estimation procedure for the truncated compound normal with gamma mixture model is presented. The analytical expressions for the maximum likelihood estimates for model parameters, $\theta_l(\mu_l, c_l, v_l)$ which describe partly the parameter set Θ, have been derived since the derivation for these parameters involved some added complexity than that for the un truncated one. The derived expressions are similar in form to compound normal with gamma mixture model except that these include some additional terms due to the truncation done with respect to the left and right truncation defined by 'a' and 'b' respectively. These expressions can be embedded into the Expectation Maximization framework for solving mixture density estimation problem. The EM framework for this truncated mixture model that can be used to solve mixture density estimation problems like image segmentation and other clustering problems is considered as an extension to the work presented in this paper .

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6. APPENDIX

I. To Take Linear Term As Approximation For $\ln z$ (Sections 3.1 and 3.2 of the paper have reference to this)

Since for any real number z that satisfies $0<z<2$, the following formula holds:

$$\ln z=(z-1)-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-\frac{(z-1)^4}{4}+\cdots$$

Taking linear term as approximation,

$$\ln z=(z-1) \quad (A1)$$

II. To Prove That $\frac{\partial \log B(1/2, v_l/2)}{\partial v_l}=\frac{-1}{2(v_l+1)}$

(Section 3.3 of the paper has reference to this)

We know that $\frac{\partial \log B(1/2, v_l/2)}{\partial v_l}=\frac{\frac{\partial}{\partial v_l}B(1/2, v_l/2)}{B(1/2, v_l/2)}$

Since we know that beta function $B(a,b)$ is defined as

$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$ for $a > 0, b > 0$ (A2)

$\frac{\partial}{\partial v_l}B(1/2\ ,v_l/2) = \int_0^1 \frac{\partial}{\partial v_l}(1-x)^{-1/2}x^{(v_l-2)/2}dx$ (A3)

To solve the above equation, we have to first solve $\frac{\partial}{\partial v_l}x^{(v_l-2)/2}$

Let $y = x^{(v_l-2)/2} \Rightarrow \log y = \log x^{(v_l-2)/2} = \frac{(v_l-2)}{2}\log x$

$\Rightarrow \frac{1}{y}dy = \frac{\log x}{2}dv_l \Rightarrow \frac{dy}{dv_l} = y\frac{\log x}{2}$

$$\therefore \frac{\partial}{\partial v_l}x^{(v_l-2)/2} = x^{(v_l-2)/2}\left(\frac{\log x}{2}\right)$$

Hence Equation (A3) can be written as

$$\frac{\partial}{\partial v_l}B(1/2\ ,v_l/2) = \int_0^1 x^{(v_l-2)/2}\left(\frac{\log x}{2}\right)(1-x)^{-1/2}dx$$

Taking linear term as approximation of $\log x$ as $(x-1)$, given by Equation (A1), the above equation becomes

$$\frac{\partial}{\partial v_l}B(1/2\ ,v_l/2) = \int_0^1 x^{(v_l-2)/2}\left(\frac{x-1}{2}\right)(1-x)^{-1/2}dx$$

$$\Rightarrow -\frac{1}{2}\int_0^1 x^{(v_l-2)/2}(1-x)^{-1/2}dx$$

$$\Rightarrow -\frac{1}{2}\int_0^1 x^{\frac{v_l}{2}-1}(1-x)^{\frac{3}{2}-1}dx$$

$\Rightarrow -\frac{1}{2}B(v_l/2\ ,3/2)$ (Section 3.3 of the paper has reference to this)

$\therefore \frac{\partial}{\partial v_l}B(1/2\ ,v_l/2) = -\frac{1}{2}B(v_l/2\ ,3/2)$ (A4)

Hence $\frac{\partial \log B(1/2\ ,v_l/2)}{\partial v_l} = \frac{\frac{\partial}{\partial v_l}B(1/2\ ,v_l/2)}{B(1/2\ ,v_l/2)} = \frac{-\frac{1}{2}B(v_l/2\ ,3/2)}{B(1/2\ ,v_l/2)}$

$= -\frac{1}{2}\frac{\Gamma\left(\frac{v_l}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{v_l}{2}+\frac{3}{2}\right)}\frac{\Gamma\left(\frac{v_l}{2}+\frac{3}{2}\right)}{\Gamma\left(\frac{v_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} = -\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma\left(\frac{v_l}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{v_l}{2}+\frac{1}{2}+1\right)} = -\frac{1}{2}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{\Gamma\left(\frac{v_l}{2}+\frac{1}{2}\right)}{\frac{(v_l+1)}{2}\Gamma\left(\frac{v_l}{2}+\frac{1}{2}\right)} = \frac{-1}{2(v_l+1)}$

Since $B(a,b) = B(b,a) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (A5)

and the *gamma function,* denoted by $\Gamma(.)$, is defined by

$\Gamma(t) = \int_0^\infty x^{(t-1)}e^{-x}dx$ for $t > 0$ (A6)

The following definitions about gamma function are taken from **[5]** as given by Mood et al.

$\Gamma(t)$ is nothing more than a notation for the definite integral that appears on the right hand side of Equation (6). Integration by parts yields

$\Gamma(t+1) = t\Gamma(t)$ (A7)

and hence, if $t = n$(an integer), $\Gamma(n+1) = n!$.

If n is an integer,

$\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5\ldots(2n-1)}{2^n}\sqrt{\pi}$, and in particular $\Gamma\left(\frac{1}{2}\right) = 2\Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}$.

III. To Derive Partial Derivatives Of Incomplete Beta Functions With Respect To μ_l , c_l , and v_l .

Here, the partial derivatives of incomplete beta functions with respect to μ_l, c_l, and v_l for solving Equations (18), (19), (20) related to the truncated normal with gamma mixture distribution have been derived for the purpose of their use in sections 3.1, 3.2, and 3.3 of the paper.

$b_1 = \frac{c}{c+(b-\mu)^2}$ and $a_1 = \frac{c}{c+(a-\mu)^2}$ since $t = \frac{c}{c+(x-\mu)^2}$, details of which can be seen sections 5.2 and 5.3 in the said chapter.

$\frac{\partial}{\partial \mu_l}B_{a_1}\left(\frac{v_l}{2}\ ,\frac{1}{2}\right) = \frac{\partial}{\partial \mu_l}\int_0^{a_1} y^{\frac{v_l}{2}-1}(1-y)^{\frac{1}{2}-1}dy$

$= \frac{\partial}{\partial \mu_l}\int_0^{\frac{c_l}{c_l+(a-\mu_l)^2}} y^{\frac{v_l}{2}-1}(1-y)^{\frac{1}{2}-1}dy$ (A8)

The above equation is in the form of $\frac{\partial}{\partial x}\int_0^{\psi(x)} f(x)\,dx$, which is equal to $f(\psi(x))\psi'(x)$

Therefore, the solution for Equation (A8) is

$$\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l}{2}-1}\left[1-\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{-\frac{1}{2}}\frac{d}{d\mu_l}\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]$$

$$\Rightarrow \left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l}{2}-1}\left[1-\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{-\frac{1}{2}}\frac{2c_l(a-\mu_l)}{[c_l+(a-\mu_l)^2]^2}$$

since $\frac{d}{d\mu_l}\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right] = \frac{c_l'[c_l+(a-\mu_l)^2]-c_l[c_l+(a-\mu_l)^2]'}{[c_l+(a-\mu_l)^2]^2} = \frac{2c_l(a-\mu_l)}{[c_l+(a-\mu_l)^2]^2}$

Upon simplification, the above equation may be written as

$\frac{\partial}{\partial \mu_l}B_{a_1}\left(\frac{v_l}{2}\ ,\frac{1}{2}\right) = \frac{2}{c_l^{1/2}}\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}$ (A9)

Similarly
$\frac{\partial}{\partial c_l}B_{a_1}\left(\frac{v_l}{2}\ ,\frac{1}{2}\right) =$

$$\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l}{2}-1}\left[1-\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{-\frac{1}{2}}\frac{d}{dc_l}\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]$$

$$\Rightarrow \left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l}{2}-1}\left[1-\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{-\frac{1}{2}}\frac{(a-\mu_l)^2}{[c_l+(a-\mu_l)^2]^2}$$

since $\frac{d}{dc_l}\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right] = \frac{c_l'[c_l+(a-\mu_l)^2]-c_l[c_l+(a-\mu_l)^2]'}{[c_l+(a-\mu_l)^2]^2} = \frac{(a-\mu_l)^2}{[c_l+(a-\mu_l)^2]^2}$

Upon simplification, the above equation may be written as

$\frac{\partial}{\partial c_l}B_{a_1}\left(\frac{v_l}{2}\ ,\frac{1}{2}\right) = \frac{(a-\mu_l)}{c_l c_l^{1/2}}\left[\frac{c_l}{c_l+(a-\mu_l)^2}\right]^{\frac{v_l+1}{2}}$ (A10)

In respect of $\frac{\partial}{\partial v_l}B_{a_1}\left(\frac{v_l}{2}\ ,\frac{1}{2}\right)$, Equation (A4) holds good.

$\therefore \frac{\partial}{\partial v_l}B_{a_1}\left(\frac{v_l}{2}\ ,\frac{1}{2}\right) = -\frac{1}{2}B_{a_1}(v_l/2\ ,3/2)$ (A11)

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