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Caristi's Fixed Point Theorem and Ekeland's Variational Principle for Set Valued Mapping using the LZ-functions

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ABSTRACT

The aims of this paper is to give some new theorems in the field of fixed point theory. For that, we establish a generalized result of Caristi's fixed point theorem by introducing a new type of functions that will be called the LZ-functions. And since that theorem is equivalent to Ekeland's variational principle, we derive also an ε-variational-type principle, which generalizes the latter. As application, we study the existence of solution for a system of equilibrium problem.

General Terms

Fixed point

Keywords

Fixed point, Set valued map, LZ-function, Caristi, Ekeland

1. INTRODUCTION

Fixed point theory has several applications in many domains. It has applications in the study of market stability in economics. In dynamic systems it is used to deterministic timed systems on feedback semantics, and in the theory of integral and differential equations to demonstrate the existence and uniqueness of solutions [1, 2, 3, 4]
In 1922, the Polish mathematician Stefan Banach established a remarkable fixed point theorem, the famous contraction principle [6], which is one of the most important results of analysis. It is the most widely applied fixed point result in different areas of mathematics and applications. It requires the structure of a complete metric space with contractive condition on the map which is easy to test in many situations. It has been generalized in many different directions **[13, 14, 19, 21, 23]**. Moreover, the proof of the Banach contraction principle gives a sequence of approximate solutions and useful information as regards the rate of convergence toward the fixed point.
There are in the literature many different versions of the well-known theorems due to Banach [6] and Nadler [7] and Caristi **[17]**, concerning fixed points for single-valued and set-valued dynamic systems, respectively, in complete metric spaces [8, 9, 10, 11, 12]. For instance, the problem of existence and uniqueness of fixed points in partially ordered sets has been studied thoroughly because of its interesting nature. In this direction, a result was given by Turinici [5], where he extended the Banach contraction principle in partially ordered sets.
One of its most important extensions of Banach contraction principle is known as Caristi's fixed point theorem, since it is closely connected with the variational principle due to I. Ekeland **[31]**.
The proofs given to Caristi's result vary and use different techniques **[15, 16, 17, 18]**. In 1977, Siegel **[19]** found that Caristi's fixed point theorem based on the work of Brondsted **[21],** which implicitly use a partially order and ensure the existence of fixed point as a maximal element.
Brézis and Browder **[23]** proved a very general principle concerning order relations which include the Caristi's **[17]** theorem. In 2007, À. Szàz **[25]** generalizes the Brézis-Browder principle in abstract setting, and gives an abstract generalized of Caristi's and Ekeland's theorem as well.
On the other hand, many authors have extend Ekeland variational principle **[26]** in several directions, because of the important applications of this result in applied mathematics : control theory, convex analysis, etc**. [31, 27, 28, 29]**.
Following this direction of research, in this paper, we use the Szàz principle in a metric space to give a more generalized versions of Caristi's fixed point theorem. For that, we introduce a new class of functions called LZ-functions which generalizes the notion of dominated function in Caristi's theorem. Since Ekeland Variational Principle is equivalent to Caristi's Theorem, we derive an ε-variational principle which is also a generalization of Ekeland's variational principle. Furthermore, we give an application of our results for a system of equilibrium problem.
The rest of the manuscript is organized as follows: In Section II, some standard assumptions are introduced, with the main theoreti-

Fig. 1. Representation of the set valued map T of the example 1. The colored area represents the sets $T(x)$ over $[0, 1]$. The red line represents the first bisecting plane. The two circles are the fixed points of T over $[0, 1]$

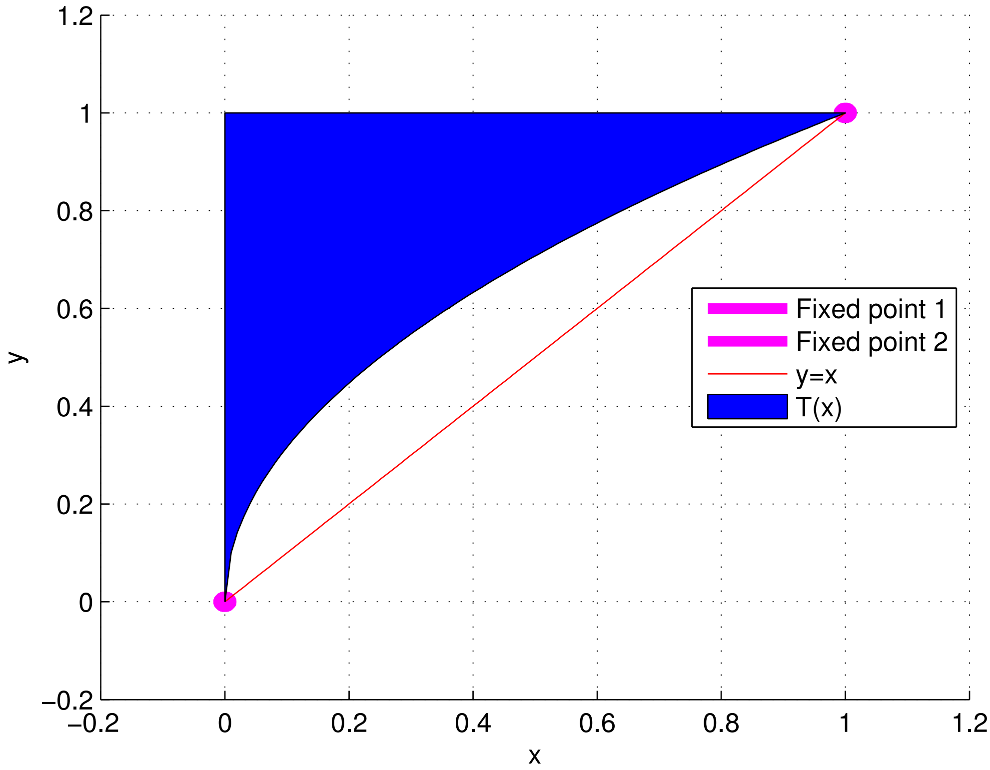
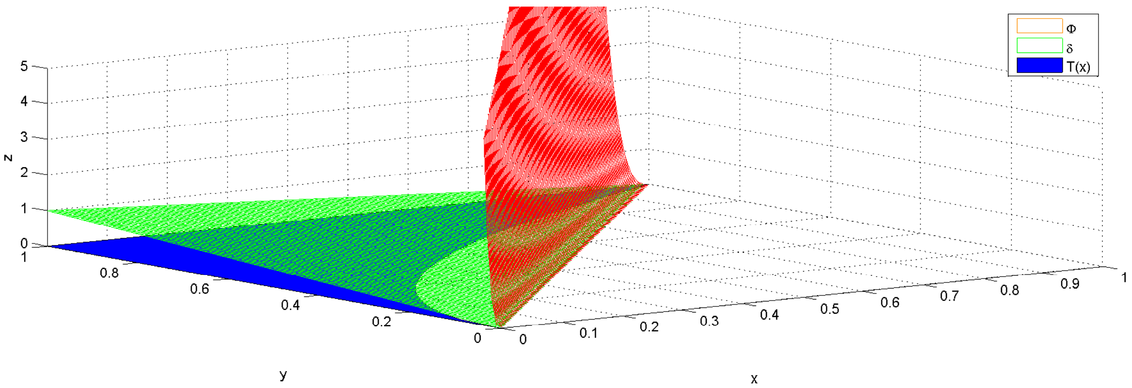


Fig. 2. Geometric verification of the condition (1) in the example 1. The blue domain of xy-plane is $T([0, 1])$. The lower green surface is $\delta(x, y)$ and the upper red one is $\Phi(x, y)$ for $(x, y) \in [0, 1]^2$ with $y \geq x$.



cal results. In Section III, an application to prove the the existence of the solution for a system of equilibrium problem.

2. MAIN RESULTS

In a nonempty set X, we define a reflexive and transitive relation $\preccurlyeq$ called a preorder and we said that $(X, \preccurlyeq)$ is preorder set. If in addition, $\preccurlyeq$ is antisymmetric, we will called it a partial order.

$x \in X$ is maximal element if $x \preccurlyeq y \Rightarrow x = y$ for all $y \in X$. We recall that

$$S^{+}(x) = \{z \in X\,;\, x \preccurlyeq z\}\,.$$

The following result will be useful in the sequel.

Fig. 3. Other view of $\delta(x, y)$ and $\Phi(x, y)$ of example 1, for $(x, y) \in [0, 1]^2$ with $y \geq x$.

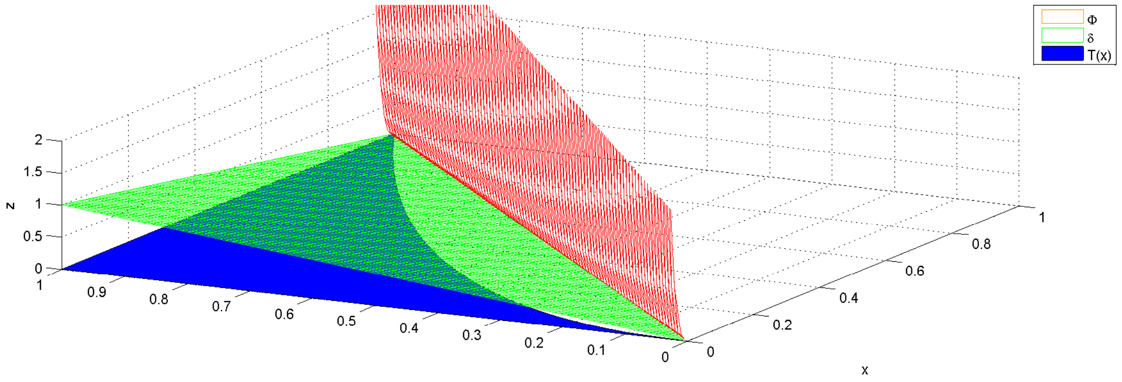
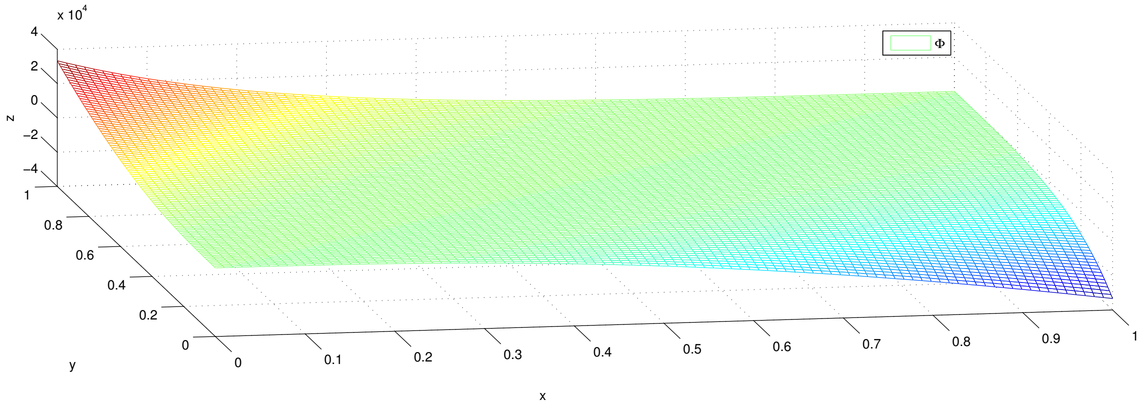


Fig. 4. Representation of the LZ-function $\Phi(x, y)$ of example 1, for $(x, y) \in [0, 1]^2$



Theorem 1. *Let $(X, \preccurlyeq)$ be a quasi-order set and let $\Phi : X \times X \longrightarrow]-\infty, \infty]$ be a function satisfying :*
(S1) $x \longmapsto \sup\limits_{y \in S^+(x)} \Phi(x, y)$ is decreasing;
(S2) $-\infty < \sup\limits_{y \in S^+(x)} \Phi(x, y)$ for all $x \in X$;
(S3) $\sup\limits_{y \in S^+(a)} \Phi(a, y) < \infty$ for some $a \in X$;
(S4) For every non-decreasing sequence $(x_n)_{n \in \mathbb{N}} \subset X$, with $x_0 = a$, there exists some $x \in X$ such that $x_n \preccurlyeq x$ for all $n \in \mathbb{N}$, and $\liminf\limits_{n \to \infty} \Phi(x_n, x_{n+1}) = 0$;
(S5) $0 < \Phi(x, y)$ for all $x, y \in X$ with $x \prec y$.
Then there exists a maximal element $\hat{x} \in X$.

Definition 2. A function $\Phi : X \times X \longrightarrow]-\infty, \infty]$ will be called a LZ-function if the following hold :
(C1) super-additivity : $\Phi(x, y) + \Phi(y, z) \leq \Phi(x, z)$ for each $x, y, z \in X$.
(C2) $y \longmapsto \Phi(x, y)$ is upper semi continuous for each $x \in X$.
(C3) there exists $x_0 \in X$ such that $\sup\limits_{y \in X} \Phi(x_0, y) < \infty$.
(C4) $x \longmapsto \Phi(x, y)$ is bounded bellow for each $y \in X$.

Lemma 3. Let (X, δ) be a metric space, $\Phi : X \times X \longrightarrow]-\infty, \infty]$ a LZ-function. A binary relation defined by

$$x \preceq y \Leftrightarrow x = y \text{ or } \delta(x, y) \leq \Phi(x, y)$$

is a partial order on X.

Theorem 4. *Let (X, δ) be a complete metric space, $T : X \longrightarrow 2^X$ a set valued map. If there exists a LZ-function $\Phi : X \times X \longrightarrow]-\infty, \infty]$ satisfying for each $x \in X$ there exists $y \in Tx$ such that*

$$\delta(x, y) \leq \Phi(x, y) \tag{1}$$

Then T has a fixed point in X.

Proof. Let $x_0 \in X$ be as in (C3) and by assumption, there exists $x_1 \in Tx_0$ such that

$$\delta(x_0, x_1) \leq \Phi(x_0, x_1)$$

We construct inductively a sequence $(x_n)_n$ satisfies for each $n \in \mathbb{N}$

$$\delta(x_n, x_{n+1}) \leq \Phi(x_n, x_{n+1}) \tag{2}$$

Note that if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for all $n \geq n_0$ then x_{n_0} is a fixed point of T. Assume that for all $n, m \in \mathbb{N}$ we get $x_n \neq x_m$ hence $(x_n)_n$ is an increasing sequence with respect to $\preceq$.
Let $n \in \mathbb{N}$, by inequality (2) we get

$$\sum_{k=0}^{n} \delta(x_k, x_{k+1}) \leq \sum_{k=0}^{n} \Phi(x_k, x_{k+1}) \leq \Phi(x_0, x_{n+1}) < \infty$$

then $\lim_{n\to\infty} \Phi(x_n, x_{n+1}) = 0$ and (x_n) is a Cauchy sequence in X so, converge to some $\overline{x} \in X$ and since $y \longmapsto \Phi(x,y)$ is upper semi continuous we get for each $n, m \in \mathbb{N}$,

$$\delta(x_n, x_m) \le \Phi(x_n, x_m)$$

so taking the limit with respect to m yields

$$\delta(x_n, \overline{x}) \le \Phi(x_n, \overline{x})$$

This proves that $x_n \preceq \overline{x}$ for all $n \in \mathbb{N}$.
Define a function $\gamma_\Phi : X \longrightarrow]-\infty, \infty]$ by $\gamma_\Phi(x) = \sup_{z\in X} \Phi(x,z)$ for each $x \in X$ and let $x \preceq y$ then we have $x = y$ or $\delta(x,y) \le \Phi(x,y)$:
if $x = y$ then $\gamma_\Phi(x) = \gamma_\Phi(y)$;
if $x \ne y$ we get $\Phi(x,y) > 0$ and by (C1) we get for all $z \in X$

$$\Phi(x,y) + \Phi(y,z) \le \Phi(x,z)$$

hence

$$\sup_{z\in X} \Phi(y,z) \le \sup_{z\in X} \Phi(x,z) \Leftrightarrow \gamma_\Phi(y) \le \gamma_\Phi(x)$$

that is γ_Φ is decreasing function.
By Theorem 1 $(X, \preceq)$ has a maximal element, say x^* . Since condition (1) implies there exists $y^* \in Tx^*$ such that $x^* \preceq y^*$ it must be the case that $x^* = y^*$.

□

Example 1.
In this example, we choose $X = [0,1]$ and T is a set valued map defined as follows

$$\begin{aligned} T : X &\to 2^X \\ x &\longmapsto [\sqrt{x}, 1] \end{aligned} \tag{3}$$

and Φ is a LZ-function $\Phi : X \times X \longrightarrow]-\infty, \infty]$ given by

$$\Phi(x,y) = 10^5 \ln^3\left(\frac{y+1}{x+1}\right) \tag{4}$$

with for all $x,\ y,\ z \in X$, we have

$$\begin{aligned} \Phi(x,y) + \Phi(y,z) &= 10^5 \ln^3\left(\frac{y+1}{x+1}\right) + 10^5 \ln^3\left(\frac{z+1}{y+1}\right) \\ &\le \left(10^5 \ln\left(\frac{y+1}{x+1}\right) + 10^5 \ln\left(\frac{z+1}{y+1}\right)\right)^3 \\ &\le \left(10^5 \ln\left(\frac{z+1}{x+1}\right)\right)^3 \\ &= \Phi(x,z) \end{aligned}$$

then the super additivity (C1) is well verified. It is clear from figures (2), (3) and 4 that Φ verify the conditions (C2), (C3) and (C4). Then Φ is well a LZ-function.
The metric δ is defined by

$$\delta(x,y) = |y - x| \tag{5}$$

Figure 1 shows the set valued map T represented by the colored domain of the plane, as the union of segments $[\sqrt{x}, 1]$ with $x \in [0,1]$.
Figure 2 shows the surfaces of Φ and δ over $[0,1]$, in order to verify the conditions of theorem 4, we choose to plot the LZ-function Φ and the metric δ for x and y in $[0,1]$ with $x \le y$, or more precisely, for $x \in [0,1]$ and $y \in T(x) = [\sqrt{x}, 1]$. The segments $T(x)$ are represented by the colored domain of the xy-plane (see Figures (2) and (3)). So it is clear that the condition (1) is well verified, that is

$$\delta(x,y) \le \Phi(x,y)$$

for all $x \in [0,1]$ and $y \in T(x) = [\sqrt{x}, 1]$. The conditions of theorem (4) are verified, then by this theorem, T has at least a fixed point in $X = [0,1]$. Figure 1 shows that the set valued map T has a two fixed points represented by the points of intersection with the red line $y = x$, that are $x = 0 \in [\sqrt{0}, 1] = [0,1]$ and $x = 1 \in [\sqrt{1}, 1] = \{1\}$.

Note that if $\Phi(x,y) = \varphi(x) - \varphi(y)$ we get the Caristi's theorem given by

Corollary 1. *Let (X, δ) be a complete metric space and $\varphi : X \longrightarrow [0, \infty)$ a lower semicontinuous function. If the mapping $T : X \longrightarrow X$ satisfies for each $x \in X$ the condition :*

$$\delta(x, Tx) \le \varphi(x) - \varphi(Tx)$$

then T has a fixed point in X.

On the other hand, and to give a generalized version of Ekeland's variational principle, we provide the following theorem

Theorem 5. *Let (X, δ) be a complete metric space and $\Phi : X \times X \longrightarrow]-\infty, \infty]$ a LZ-function. For each $\varepsilon > 0$ and $x_0 \in X$ such that $\Phi(x_0, x_0) \le \inf_{x\in X} \Phi(x, x_0) + \varepsilon$, then there exists $\overline{x} \in X$ such that*
(1) $\varepsilon\delta(x_0, \overline{x}) \le \Phi(x_0, \overline{x})$;
(2) $\varepsilon\delta(\overline{x}, y) > \Phi(\overline{x}, y)$ for each $y \in X \setminus \{\overline{x}\}$.

Proof. For each $x \in X$ we define a nonempty set $S(x)$ by

$$S(x) = \{y \in X;\ x = y \text{ or } \varepsilon\delta(x,y) \le \Phi(x,y)\}$$

and by continuity of $y \longmapsto \delta(x,y)$ and lower semi continuity of $y \longmapsto -\Phi(x,y)$ the set $S(x)$ is closed subset of X and then complete metric space.
Let $x_0 \in X$ as in (C3). For each $x \in S(x_0)$ set

$$H(x) = \{y \in X \setminus \{x\}\,;\, \varepsilon\delta(x,y) \le \Phi(x,y)\}$$

and define a set valued mapping

$$Tx = \begin{cases} \{x\} & \text{if, } H(x) = \emptyset \\ H(x) & \text{if, } H(x) \ne \emptyset \end{cases}$$

then T is a self set valued mapping from $S(x_0)$ to $2^{S(x_0)}$. Indeed, if $H(x) = \emptyset$ then $Tx \in 2^{S(x_0)}$ by definition and if $H(x) \ne \emptyset$ let $y \in H(x)$ then $y \ne x$ and $\delta(x,y) \le \Phi(x,y)$ which implies that $x \preceq y$ and since $x \in S(x_0)$ i.e. $x_0 \preceq x$ then $x_0 \preceq y$ which leads to

$$x_0 = y \text{ or } \varepsilon\delta(x_0, y) \le \Phi(x_0, y)$$

hence $y \in S(x_0)$.
Note that for each $x \in S(x_0)$ there exists $y \in Tx$ such that

$$\varepsilon\delta(x,y) \le \Phi(x,y)$$

by theorem 2, T has a fixed point $\overline{x} \in S(x_0)$, it follows that $H(\overline{x}) = \emptyset$. That is $\varepsilon\delta(\overline{x}, y) > \Phi(\overline{x}, y)$ for each $y \in X \setminus \{\overline{x}\}$ and since $\overline{x} \in S(x_0)$ we get $\varepsilon\delta(x_0, \overline{x}) \le \Phi(x_0, \overline{x})$. This complete the proof.

□

It is well known that theorem 1 is equivalent to Ekeland's variational principle, so, we get the following :

Corollary 2. Let (X,δ) be a complete metric space and $\varphi : X \longrightarrow]-\infty,\infty]$ a proper lower semi continuous function bounded below. For each $\varepsilon > 0$ and $x_0 \in X$ such that $\varphi(x_0) \leq \inf_{x\in X}\varphi(x)+\varepsilon$, then there exists $\overline{x} \in X$ such that
(1) $\varepsilon\delta(x_0,\overline{x}) \leq \varphi(x_0) - \varphi(\overline{x})$;
(2) $\varepsilon\delta(\overline{x},y) > \varphi(\overline{x}) - \varphi(y)$ for each $y \in X \setminus \{\overline{x}\}$.

3. APPLICATION

In this section, we propose to solve an equilibrium problem arising in variational inequality theory.
Let m be a positive integer and (X_i,δ_i) be a complete metric space. By a system of equilibrium problems we understand the problem of finding $\overline{x} = (\overline{x}_1,\overline{x}_2,\ldots,\overline{x}_m) \in A$ such that

$$(\mathcal{P}) : \quad f_i(y_i,\overline{x}) \leq 0 \qquad \forall i \in I,\ \forall y_i \in A_i,$$

where $f_i : A \times A_i \longrightarrow \mathbb{R}$, $A = \prod_{i=1}^{m} A_i$, with A_i some given sets in X_i. An element of the set $A^i = \prod_{j\neq i} A_j$ will be represented by x^i; therefore, $x \in A$ can be written as $x = (x_i,x^i) \in A_i \times A^i$. We denote by

$$\delta = \max_{i\in I}\delta_i$$

then $X = \prod_{i=1}^{m} X_i$
It is clear that there is no chance that the problem $(\mathcal{P})$ has a solution, so we will give a suitable set of conditions on the functions that do not involve convexity and lead to apply theorem 5.
The following result is an extension in complete metric space of Theorem 4.2. in **[30]**.

Proposition 6. Assume that for every $i \in I$, A_i is compact and $f_i : A_i \times A \longrightarrow \mathbb{R}$ is a function satisfying the assumptions :
(a) $y \longmapsto f_i(x_i,y)$ is bounded above and upper semi continuous, $\forall x_i \in A_i$.
(b) $f_i(x_i,x) = 0$, for every $x = (x_i,x^i) \in A$.
(c) $f_i(x_i,y) + f_i(y_i,z) \leq f_i(x_i,z)$, for every $x,y,z \in A$, where $y = (y^i,y_i)$.
Then, the set of solutions of $(\mathcal{P})$ is nonempty.

Proof. Let $i \in I$ be given. And put for each $x,y \in A$

$$\Phi(x,y) = -f_i(x_i,y)$$

by theorem 5, for each $n \in \mathbb{N}^*$ there is $x_n \in A$ such that

$$-f_i(y_i,x_n) \geq \frac{1}{n}\delta(y_i,x_n) \quad \forall y_i \in A_i.$$

Since A_i is compact for each $i \in I$, $A = \prod_{i=1}^{m} A_i$ is also a compact subset of X, then we can choose a sub-sequence $\left(x_{n_k}\right)_k$ of $(x_n)_n$ such that $x_{n_k} \longrightarrow \overline{x}$ as $n \longrightarrow \infty$. Then, by (a),

$$-f_i(y_i,\overline{x}) \geq \limsup_{k\to\infty}\left(-f_i\left(y_i,x_{n_k}\right) - \tfrac{1}{n_k}\|y_i,x_n\|\right) \quad \forall y_i \in A_i$$

then $\overline{x}$ is a solution of $(\mathcal{P})$. □

Next, we drop the assumption that A_i is compact and we assume that X_i is an Euclidean space and, for each $i \in I$, A_i is closed subset of X_i.

Let us consider the following coercivity condition $(\mathcal{C})$

$$(\mathcal{C}) : \begin{cases} \exists r > 0; \forall x \in A \text{ such that } \|x_i\|_i > r \text{ for some } i \in I, \\ \exists y_i \in A_i,\ \|y_i\|_i < \|x_i\|_i \quad \text{and } f_i(y_i,x) \geq 0. \end{cases}$$

Theorem 7. *Assume that for all $i \in I$, $f_i : A_i \times A \longrightarrow \mathbb{R}$ is a function satisfying:*
(a) $y \longmapsto f_i(x_i,y)$ is bounded above and upper semi continuous, $\forall x_i \in A_i$;
(b) $f_i(x_i,x) = 0$, for every $x = (x_i,x^i) \in A$;
(c) $f_i(x_i,y) + f_i(y_i,z) \leq f_i(x_i,z)$, for every $x,y,z \in A$, where $y = (y^i,y_i)$.
(d) $x_i \longmapsto f_i(x_i,y)$ bounded bellow.
If $(\mathcal{C})$ holds, then $(\mathcal{P})$ admits a solution.

Proof. For each $x \in A$ and every $i \in I$ consider the set

$$S_i(x) = \{y_i \in A_i;\ \|y_i\|_i \leq \|x_i\|_i \quad \text{and } f_i(y_i,x) \geq 0\}$$

Note that, by (c), for every x and $y = (y^i,y_i) \in A$, $y_i \in S_i(x)$ implies $S_i(y) \subseteq S_i(x)$. Indeed, we get for $z_i \in S_i(y) : f_i(z_i,y) \geq 0$ and

$$f_i(z_i,y) + f_i(y_i,x) \leq f_i(z_i,x) \Rightarrow 0 \leq f_i(z_i,x)$$

then $z_i \in S_i(x)$.
Let $B_i(r) = \{y_i \in A_i;\ \|y_i\| \leq r\}$ then it is a compact subset of A_i. By (a), $S_i(x)$ is bounded closed subset of A_i then, it is compact subset for all $x \in A$. Furthermore, by proposition 6, there exists an element $x_r \in \prod_{i\in I} B_i(r)$ (we may suppose that $B_i(r) \neq \emptyset$ for all $i \in I$) such that

$$f_i(y_i,x_r) \leq 0 \quad \forall y_i \in B_i(r),\ \forall i \in I. \tag{6}$$

Suppose that x_r is not a solution of $(\mathcal{P})$. In this case, there exists $j \in I$ and $z_j \in A_j$ with $f_i(z_i,x_r) > 0$. Let $z^j \in A^j$ be arbitrary and put $z = (z^j,z_j) \in A$. Define

$$a_j := \min_{y_j\in S_j(z)} \|y_j\|_j .$$

Case 1: $a_j \leq r$. Let $\overline{y}_j = \overline{y}_j(z) \in S_j(z)$ such that $\left\|\overline{y}_j\right\| = a_j \leq r$. Then we have

$$f_i\left(\overline{y}_j,z\right) \geq 0$$

and since $f_i(z_i,x_r) > 0$ we get by (c)

$$0 < f_j\left(\overline{y}_j,z\right) + f_j(z_j,x_r) \leq f_j\left(\overline{y}_j,x_r\right)$$

which contradict 6.
Case 2: $a_j > 0$. Let again $\overline{y}_j = \overline{y}_j(z) \in S_j(z)$ such that $\left\|\overline{y}_j\right\| = a_j > r$. Let$\overline{y}^j \in A^j$ be arbitrary and put $\overline{y} = \left(\overline{y}^j,\overline{y}_j\right) \in A$. Then, by $(\mathcal{C})$ we can choose an element $y_j \in A_j$ with $\|y_j\|_j < \left\|\overline{y}_j\right\|_j = a_j$ such that $f_j(y_j,\overline{y}) \geq 0$. Clearly, $y_j \in S_j(\overline{y}) \subseteq S_j(z)$, a contradiction since

$$\|y_j\|_j < \left\|\overline{y}_j\right\|_j = \min_{y_j\in S_j(z)} \|y_j\|_j$$

This completes the proof. □

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