

# Generalized (k, r) – Lucas Numbers

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## ABSTRACT

In this paper, we have defined new kinds of (k, r)-Lucas number. But the difference among these sequences comes to the forefront not only through the value of the natural number k but also through the value of new parameter which we find involved in the definition of this distance. Consequently we have various properties of these numbers for study.

## General Terms

Number theory, Fibonacci numbers, Lucas numbers, Pell numbers, Golden Ratio, Golden Sections, Applied Mathematics et. al..

## Keywords

k- Lucas Number, (k, r) Lucas Number, Binet's Formula, Generating engines.

## 1. INTRODUCTION

Classic Fibonacci numbers and classic Lucas numbers have charismatic properties and researcher have been shown their generalization and implemented in every field of science and arts. Classical Lucas numbers have been generalized in various way [6]. One of these generalizations that have been of a great interest lately among mathematical researchers is one that paves a way to the k- Lucas number.

### 1.1 Definition

For every natural numbers, the k- Lucas sequence  $L_k = \{L_{k,n}\}$  is defined by the recurrence relation[6]

$$L_{k,n+1} = kL_{k,n} + L_{k,n-1}, \text{ for } n \geq 1. \quad (1)$$

with initial conditions  $L_{k,0} = 2, L_{k,1} = k$

From above definition the general expression of first k- Lucas numbers are presented in Table 1:

**Table 1: k - Lucas numbers**

$L_{k,1} = k$
$L_{k,2} = k^2 + 2$
$L_{k,3} = k^3 + 3k$
$L_{k,4} = k^4 + 4k$
$L_{k,5} = k^5 + 5k^3 + 5k$
$L_{k,6} = k^6 + 6k^4 + 9k^2 + 2$
$L_{k,7} = k^7 + 7k^5 + 14k^3 + 7k$

If k = 1, the classical Lucas sequence is obtained {2, 1, 3, 4, 7, 11, 18,...}

If k = 2, the Pell – Lucas sequence is obtained {2, 2, 6, 14, 34, 82, 198,...}

## 2. GENERALIZED (k, r) LUCAS NUMBERS

This section witnesses the application of the definition of r – distance to the k- Lucas numbers in a such peculiar way that it generalizes the earlier findings [2, 4]. The intensely appealing formula are those which are used to calculate the general term of the sequences produced by the above definition. Moreover it supports in finding out the sum of the first n terms.

### 2.1 Definition

We define the generalized (k, r) Lucas number  $L_{k,n}(r)$  for the natural numbers  $k \geq 1, n \geq 0, r \geq 1$  by the recurrence relation

$$L_{k,n}(r) = kL_{k,n-r}(r) + L_{k,n-2}(r) \text{ for } n \geq r \quad (2)$$

with the initial conditions

$$L_{k,n}(r) = 2, n = 0, 1, 2, \dots, r - 1, \text{ except } L_{k,1}(1) = k.$$

So, if  $L_k(r) = \{L_{k,n}(r)/n \in \mathbb{N}\}$ , the expressions of the sequences obtained for  $r = 1, 2, \dots, 5$ .

$$L_k(1) = \{2, k, k^2 + 2, k^3 + 3k, k^4 + 4k^2 + 2, k^5 + 5k^3 + 5k, k^6 + 6k^4 + 9k^2 + 2, k^7 + 7k^5 + 14k^3 + 7k, \dots\}$$

$$L_k(2) = \{2, 2, 2k + 2, 2k + 2, 2k^2 + 4k + 2, 2k^2 + 4k + 2, 2k^3 + 6k^2 + 6k + 2, \dots\}$$

$$L_k(3) = \{2, 2, 2, 2k + 2, 2k + 2, 4k + 2, 2k^2 + 4k + 2, 2k^2 + 6k + 2, \dots\}$$

$$L_k(4) = \{2, 2, 2, 2, 2k + 2, 2k + 2, 4k + 2, 4k + 2, 2k^2 + 6k + 2, 2k^2 + 6k + 2, \dots\}$$

$$L_k(5) = \{2, 2, 2, 2, 2, 2k + 2, 2k + 2, 4k + 2, 4k + 2, 6k + 2, 2k^2 + 6k + 2, \dots\}$$

### 2.2 (k, r) – Lucas Numbers for k = 1, 2, 3

For k = 1, 2, 3, we obtain distinct integers sequences which is

For k = 1, the following sequences are obtained

**Table 2 (1, r) –Lucas numbers**

n	0	1	2	3	4	5	6	7...
$L_{1,n}(1)$	2	1	3	4	7	11	18	29 ...
$L_{1,n}(2)$	2	2	4	4	8	8	16	16 ...
$L_{1,n}(3)$	2	2	2	4	4	6	8	10 ...
$L_{1,n}(4)$	2	2	2	2	4	4	6	6 ...
$L_{1,n}(5)$	2	2	2	2	2	4	4	6 ...

$L_1(1)$  is the classical Lucas sequence  $L = \{ L_n \} = \{ 2, 1, 3, 4, 7, 11, 18, 29, \dots \}$

**Table 3 (2, r) Lucas Numbers**

n	0	1	2	3	4	5	6	7...
$L_{2,n}(1)$	2	2	6	14	34	82	198	478 ...
$L_{2,n}(2)$	2	2	6	6	18	18	54	54 ...
$L_{2,n}(3)$	2	2	2	6	6	10	20	22 ...
$L_{2,n}(4)$	2	2	2	2	6	6	10	10 ...
$L_{2,n}(5)$	2	2	2	2	2	6	6	10 ...

**Table 4 (3, r) Lucas Number**

n	0	1	2	3	4	5	6	7...
$L_{3,n}(1)$	2	3	11	36	119	393	1298	...
$L_{3,n}(2)$	2	2	8	8	32	32	128	128 ...
$L_{3,n}(3)$	2	2	2	8	8	14	32	38 ...
$L_{3,n}(4)$	2	2	2	2	8	8	14	14 ...
$L_{3,n}(5)$	2	2	2	2	2	8	8	14 ...

### 3. PROPERTIES OF (k, r) GENERALIZED LUCAS SEQUENCE

#### 3.1 First Explicit Formula for (k, r) – Generalized Lucas Numbers

The French mathematician Binet's dropped out two analytical formulas for the Fibonacci and Lucas numbers. In our case, Binet's formula favors us to express the generalized (k, r) Lucas numbers in function of the roots  $\alpha_1$  and  $\alpha_2$  of the following characteristic equation, associated to the recurrence relation

$$\alpha^2 - k\alpha - 1 = 0(3)$$

$$\text{where } \alpha_1 = \frac{k + \sqrt{k^2 + 4}}{2}, \alpha_2 = \frac{k - \sqrt{k^2 + 4}}{2},$$

$$\alpha_1 + \alpha_2 = k,$$

$$\alpha_1 - \alpha_2 = \sqrt{k^2 + 4}, \alpha_1 \alpha_2 = -1.$$

#### Proposition 1.(Binet's Formula)

The  $n^{\text{th}}$  Generalized (k, r) Lucas number is given by

$$L_{k,n}(r) = \alpha_1^n + \alpha_2^n. \quad (4)$$

#### Proof.

Since equation (3) has two distinct roots, then the sequence is

$$L_{k,n}(r) = C_1 \alpha_1^n + C_2 \alpha_2^n (5)$$

is the solution of the equation (3), Giving to n the values n = 0, 1 and r = 1, 2 respectively. Then we get values  $C_1 = C_2 = 1$

using in (5), we get

$$L_{k,n}(r) = \alpha_1^n + \alpha_2^n.$$

### 3.2 Catalan's Identity

In 1879 a Belgian Mathematician Eugene Charles Catalan inaugurated the Catalan's identity. Now we introduce Catalan's identity for (k, r) Lucas number.

#### Proposition 2. (Catalan's Identity)

$$L_{k,n-m}(r) \cdot L_{k,n+m}(r) - L_{k,n}^2(r) = (-1)^{n-m} L_{k,2m}(r) + 2(-1)^{n+1}. \quad (6)$$

#### Proof.

Using Binet's formula, we get

$$\begin{aligned} L_{k,n-m}(r) \cdot L_{k,n+m}(r) - L_{k,n}^2(r) &= [\alpha_1^{n-m} + \alpha_2^{n+m}] \cdot [\alpha_1^{n+m} + \alpha_2^{n+m}] - [\alpha_1^n + \alpha_2^n]^2 \\ &= (-1)^{n-m} [\alpha_1^{2m} + \alpha_2^{2m}] + 2(-1)^{n+1} \\ &= (-1)^{n-m} L_{k,2m}(r) + 2(-1)^{n+1}. \end{aligned}$$

### 3.3 Cassini's Identity

Jean-Dominique Cassini brought to light this identity in 1680.

#### Proposition 3. (Cassini's Identity)

$$L_{k,n-1}(r) \cdot L_{k,n+1}(r) - L_{k,n}^2(r) = (-1)^{n-1} L_{k,2}(r) + 2(-1)^{n+1} \quad (7)$$

#### Proof.

In (6), put m = 1, to get above result.

### 3.4 d'Ocagne's Identity

#### Proposition 4. (d'Ocagne's Identity)

If  $m > n$ , then

$$L_{k,m}(r) \cdot L_{k,n+1}(r) - L_{k,m+1}(r) \cdot L_{k,n}(r) = (-1)^n (\sqrt{k^2 + 4}) [L_{k,m}(r) - (k + \sqrt{k^2 + 4})^{m-n}]. \quad (8)$$

#### Proof.

$$\begin{aligned} L_{k,m}(r) \cdot L_{k,n+1}(r) - L_{k,m+1}(r) \cdot L_{k,n}(r) &= [\alpha_1^m + \alpha_2^m] \cdot [\alpha_1^{n+1} + \alpha_2^{n+1}] - [\alpha_1^{m+1} + \alpha_2^{m+1}] \cdot [\alpha_1^n + \alpha_2^n] \\ &= (\alpha_1 \alpha_2)^n \cdot (\alpha_1 - \alpha_2) (\alpha_2^{m-n} - \alpha_1^{m-n}) \\ &= (-1)^n (\sqrt{k^2 + 4}) [L_{k,m}(r) - 2(\alpha_1)^{m-n}] \\ &= (-1)^n (\sqrt{k^2 + 4}) [L_{k,m}(r) - (k + \sqrt{k^2 + 4})^{m-n}]. \end{aligned}$$

### 3.5 Limit of the two consecutive terms

In this section we break down that the limit of the quotient of two consecutive terms is equal to the positive root of the corresponding characteristic equation.

#### Proposition 5.

$$\lim_{n \rightarrow \infty} \frac{L_{k,n+1}(r)}{L_{k,n}(r)} = \alpha_1. \quad (9)$$

#### Proof. We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{L_{k,n+1}(r)}{L_{k,n}(r)} &= \lim_{n \rightarrow \infty} \left( \frac{\alpha_1^{n+1} + \alpha_2^{n+1}}{\alpha_1^n + \alpha_2^n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{\alpha_1^{n+1} \left( 1 + \left( \frac{\alpha_2}{\alpha_1} \right)^{n+1} \right)}{\alpha_1^n \left( 1 + \left( \frac{\alpha_2}{\alpha_1} \right)^n \right)} \end{aligned}$$

Since  $\alpha_2 < \alpha_1$ , then  $\lim_{n \rightarrow \infty} \left( \frac{\alpha_2}{\alpha_1} \right)^n = 0$ ,

We get

$$\lim_{n \rightarrow \infty} \frac{L_{k,n+1}(r)}{L_{k,n}(r)} = \alpha_1.$$

Also it can be shown easily

**Proposition 6.**

$$\lim_{n \rightarrow \infty} \frac{L_{k,n}(r)}{L_{k,n+1}(r)} = \frac{1}{\alpha_1} \quad (10)$$

**Proof .**

It can be also shown like Proposition 5.

### 3.6 Generating function for the (k, r)- Generalized Lucas Number

In below paragraph, the generating function for the (k, r)-  
Lucas sequences is given.

**Proposition 7.**

**Generating function of sequence**

**of  $L_k(r) = \{L_{k,n}(r)\}$  is given by**

$$G(L_{k,n}(r): x) = \frac{2+2x}{1-x^2-kx^r} \quad (11)$$

**Proof.**

$$\begin{aligned} G(L_{k,n}(r): x) &= \sum_{n=0}^{\infty} L_{k,n}(r) x^n \\ &= L_{k,0}(r)x^0 + L_{k,1}(r)x^1 + L_{k,2}(r)x^2 + \\ &\quad L_{k,3}(r)x^3 + \dots \\ &= 2 + 2x + \sum_{n=2}^{\infty} L_{k,n}(r)x^n \\ &= 2 + 2x + \sum_{n=2}^{\infty} [kL_{k,n-r}(r) + L_{k,n-2}(r)]x^n \\ &= 2 + 2x + kx^r \sum_{n=2}^{\infty} kL_{k,n-r}(r)x^{n-r} + \\ &\quad x^2 \sum_{n=2}^{\infty} L_{k,n-2}(r)x^{n-2} \\ &= 2 + 2x + kx^r \sum_{j=0}^{\infty} kL_{k,p}(r)x^p + x^2 \sum_{j=0}^{\infty} L_{k,j}(r)x^j \end{aligned}$$

where  $p = n - r$ ,  $n = 0, 1, 2, 3, \dots$  and  $r = 1, 2, 3, \dots$  and  $j = n - 2$

$$\text{so } (1 - kx^r - x^2) \sum_{n=0}^{\infty} L_{k,n}(r)x^n = 2 + 2x$$

$$\text{we have } L_{k,n}(r) = \frac{2+2x}{1-x^2-kx^r}.$$

## 4. ACKNOWLEDGEMENT

We would like to thank the anonymous referees for numerous helpful suggestions

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