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Independent Resolving Number of Fibonacci Cubes and Extended Fibonacci Cubes

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ABSTRACT
A subset S of vertices in a graph G is said to be an independent set of G if each edge in the graph has at most one endpoint in S and a set $W \subseteq V$ is said to be a resolving set of G, if the vertices in G have distinct representations with respect to W. A resolving set W is said to be an independent resolving set, or an *ir*-set, if it is both resolving and independent. The minimum cardinality of W is called the independent resolving number and is denoted by $ir(G)$. In this paper, we determine the independent resolving number of Fibonacci Cubes and Extended Fibonacci cubes.

General Terms
Graph theory, Interconnection network, Binary representation

Keywords
Resolving set, Independent resolving number, Fibonacci Cubes, Extended Fibonacci Cubes, Hamming distance.

1. INTRODUCTION
For an ordered set $W = \{w_1, w_2, ..., w_k\}$ subset of V and a vertex. $v \in V(G)$, the representation of v with respect to W is defined as the k-vector $r(v \mid W) = (d(v, w_1), d(v, w_2), ..., d(v, w_k))$. The set W is a *resolving set* of G if $r(x \mid W) \neq r(y \mid W)$ for any two distinct vertices $x, y \in V$. A *minimum resolving set* or a *basis* of G is a resolving set of G with minimum number of vertices. The metric (dimension) $dim(G)$ is the number of vertices in a basis for G. A resolving set W of G is connected if the sub graph $< W >$ induced by W is a connected sub graph of G and the minimum cardinality of a connected resolving set W in a graph G is called the *connected resolving number* which is denoted by $cr(G)$ [1]. A set $W \subseteq V$ is said to be an *independent set* of G if there is no edge connecting every two vertices in W. A resolving set W is said to be an *independent resolving set*, or an *ir-set*, if it is both resolving and independent. The cardinality of a minimum independent resolving set in a graph G is known as the *independent resolving number* $ir(G)$. [1]

For every connected graph G of order n, every independent resolving set is a resolving set. Saenpholphat et al. [1] have proved that $1 \leq dim(G) \leq ir(G) \leq \beta(G) \leq n-1$ where $dim(G)$ is the number of vertices in a basis for G and $\beta(G)$ is the number of vertices in a maximum independent set in a graph G.

One of the basic problems in chemistry is to provide distinct mathematical representations for a set of chemical compounds. The structure of a chemical compound is labeled graphically where the vertex and edge labels specify the atom and bond types, respectively. Under graph theoretical concept, this problem is to find a resolving set of the graph. Further this concept has wide applications in problems of network discovery and verification [2], pattern recognition and image processing, coin weighing problems, strategies for master mind game, geometrical routing protocols, sonar and loran stations [3], pharmaceutical chemistry, Combinatorial Search and Optimization, robot navigation, etc.

In this paper, we determine the independent resolving number of Fibonacci cube, Extended Fibonacci cubes $EFC_1(n)$ and $EFC_2(n)$.

2. FIBONACCI CUBES AND EXTENDED FIBONACCI CUBES
One of the most popular and efficient topological structure of interconnection network is hypercube [4]. This led to the introduction of a special sub cube of hypercube, called Fibonacci cube proposed by Hsu [5]. Fibonacci cubes are the sub graphs of hypercube induced by the vertices that no two consecutive 1's are there in the binary representation. When comparing the hypercube of dimension n with an n-dimensional Fibonacci cubes, it is found that there are 1/5 fewer edges [6] and does not increase rapidly in size as the dimension increases. Without affecting the properties of Fibonacci cubes a new cube, known as extended Fibonacci cube, [7], has been introduced. Extended Fibonacci cubes are used to construct parallel machines with arbitrary size since it eliminates the restriction on the number of vertices.

A Fibonacci sequence is defined as $f_0 = 0$. $f_1 = 1$, $f_i = f_{i-1} + f_{i-2}$ for $i \geq 2$. The symbol $\|$ denotes concatenation operation for, $01 \| \{0, 1\} = \{010, 011\}$ and $01 \| \{\} = 01$. The Fibonacci cube $\Gamma_n = (V_n, E_n)$ of order n, $n > 1$, is defined recursively as $V_n = 0 \| V_{n-1} \cup 10 \| V_{n-2}$, where V_{n-1} and V_{n-2} are the set of vertices of the order $n-1$ and $n-2$ respectively and there is an edge between two vertices if their binary representations differ exactly in one position. The initial conditions are $\Gamma_2 = (\{\lambda\}, \emptyset)$ and $\Gamma_2 = (\{0, 1\}, \{(0, 1)\})$. A Fibonacci cube of order n has f_n vertices where f_n denotes the n^{th} Fibonacci number [5]. The Fibonacci cube Γ_n contains two disjoint subgraphs that are isomorphic to Γ_{n-1} and Γ_{n-2}, and there are exactly f_{n-2} edges connecting the two subgraphs. The degree of a vertex in the Fibonacci cube Γ_n, lies between $\left\lfloor \frac{n-2}{3} \right\rfloor$ and $n-2$, for $n \geq 3$ [5]. Fibonacci cubes of orders 2, 3, 4 are depicted in Figure 1.

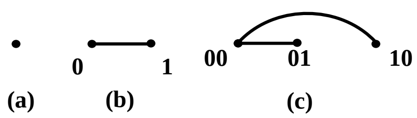

Figure 1: (a) Γ_2 (b) Γ_3 (c) Γ_4

Based on the Fibonacci sequence the extended Fibonacci cubes are defined by changing the initial conditions.

A series of Extended Fibonacci Cubes, denoted by EFC_k, $k \geq 1$ is defined as $EFC_k(n) = \{V_k(n), E_k(n)\}$, $EFC_k(n-1) = (V_k(n-1), E_k(n-1))$, and $EFC_k(n-2) = (V_k(n-2), E_k(n-2))$. Then $V_k(n) = 0 \parallel V_k(n-1) \cup 10 \parallel V_k(n-2)$. Two vertices in $EFC_k(n)$ are connected by an edge in $E_k(n)$ if and only if their labels differ in exactly one bit. As initial conditions for recursion, $V_k(k+2) = \{0,1\}^k$, $V_k(k+3) = \{0,1\}^{k+1}$ where $\{0,1\}^k$ denotes the set of binary strings of length k [7].

A Fibonacci cube Γ_n is a proper subgraph of $EFC_1(n)$, for $n \geq 4$. For any $n \geq i + 3$ and $n \geq j + 3$, $EFC_i(n)$ is a proper subgraph of $EFC_j(n)$ if $i < j$. The degree of vertices in $EFC_k(n)$ is between $\left\lceil \frac{n-(k-1)}{3} \right\rceil + (k-1)$ and $n-2$.

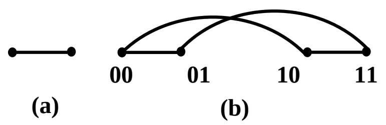

Figure 2: (a) $EFC_1(3)$ (b) $EFC_1(4)$

The vertices of an $EFC_k(n)$ are labelled with binary strings of length $n-2$,where the first $n-k-2$ bits represent a Fibonacci number and the last k represents a binary number. The number of vertices in $EFC_k(n)$ is $2^k f_{n-k}$, where f_{n-k} is the $(n-k)^{th}$ Fibonacci number, $n > k+1$. The diameter of $EFC_k(n)$ is $n-2$ for all $k \geq 0$ [8]. An extended Fibonacci cube $EFC_0(n)$ is a Fibonacci cube Γ_n. $EFC_1(n)$ and $EFC_2(n)$ are shown in Figures 2 and 3.

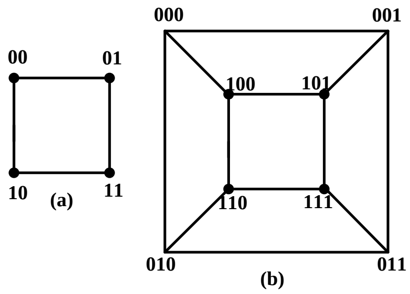

Figure 3: (a) $EFC_2(3)$ (b) $EFC_2(4)$

Theorem 2.1: Let $G = EFC_2(n)$. Then $ir(G) = 3$ for $n \geq 5$.

Proof: Vertices of G are denoted by $a_1, a_2, a_3, \ldots, a_m$ where m is the number of vertices in G and any two vertices in G are connected if and only if their Hamming distance is one. Consider $W = \{a_1\}$. Then there exist vertices a_i, $2 \leq i \leq m$ and $i \neq m-1$ with identical representation with respect to W. Thus W does not resolve G as an ir-set. Therefore, $ir(G) > 1$.

Include a_3 into W, that is, $W = \{a_1, a_3\}$. The vertices a_{2i}, $1 \leq i \leq \frac{m}{2}$ are at equidistant from both a_1 and a_3. Hence $ir(G) > 2$.

Now include a_m in W. It follows that $d(a_{m-4}, a_m) = 1$.
Then for $1 \leq i \leq \frac{m}{4}$,

$$d(a_{4i-2}, a_m) = (m/4) - (i-1) + d(a_{m-4}, a_m)$$

Similarly for $1 \leq i \leq \left(\frac{m}{4}\right) - 2$,

$$d(a_{4i}, a_m) = (m/4) - 2 - (i-1) + d(a_{m-4}, a_m)$$

$$d(a_{m-2}, a_m) = 3 + d(a_{m-4}, a_m)$$

Thus any two vertices have distinct W-coordinates and W resolves G.
Further the vertices a_1, a_3 and a_m are non-adjacent vertices in G. Therefore W is a minimum independent set. Hence W resolves G as an independent resolving set and $ir(G) = 3$.

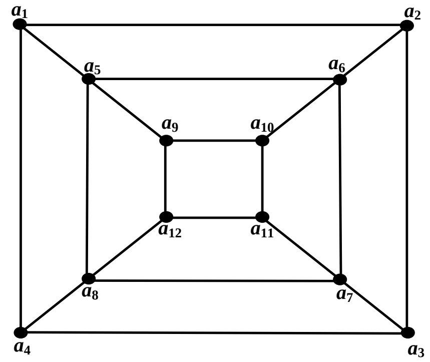

Figure 4: $EFC_2(6)$ with its resolving set $\{a_1, a_3, a_{12}\}$

Lemma 2.2: Let $G = EFC_1(n)$. Then $ir(G) \geq n-3$ for $n \geq 5$.

Proof: Let $W = \{a_k\}$ where $k = \begin{cases} 2i+2, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 4i+4, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \end{cases}$ be an independent set in G such that $|W| \leq n-4$. Then there exist vertices $\{a_{2n-9}, a_m\}$ which have identical representation with respect to W. Thus W does not resolve G and $ir(G) \geq n-3$.

Theorem 2.2: For $G = EFC_1(n)$, $ir(G) = n-3$ for $n \geq 5$.

Proof: By Lemma 2.2, $ir(G) \geq n-3$.

Let us assume that $ir(G) = n-3$. We will prove this by induction on the number of vertices n.

Let $G = EFC_1(5)$ and $W = \{a_4, a_6\}$ be an independent resolving set in G. Then it can be easily verified that every vertex in G is resolved by W. Thus W is an *ir-set* and $ir(G) = 2$. Figure 5 represents the distinct representation in W with respect to G.
Now let us assume that the result is true for $EFC_1(n)$.

$W = \{a_k, a_j\}$ where $k = \begin{cases} 2i+2, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 4i+4, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \end{cases}$

and $j = m(EFC_1(n-1)) + 2$ is an independent resolving set of G where $m(EFC_1(n))$ denotes the number of vertices in $EFC_1(n)$.

By the structure of extended Fibonacci cubes, $EFC_1(n+1)$, contains a copy of $EFC_1(n)$ and $EFC_1(n-1)$. Now we have to show that W is an independent resolving set of $EFC_1(n+1)$. That is, we have to prove that any two vertices, say, x and y, do not have same distance with respect to W.

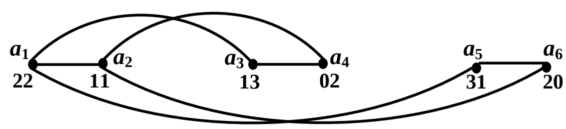

Figure 5: $EFC_1(5)$ and its distinct representations with respect to W

Case i: If $x, y \in EFC_1(n)$ (or) $x, y \in EFC_1(n-1)$, by induction hypothesis W resolves x and y.

Case ii: If $x \in EFC_1(n)$ and $y \in EFC_1(n-1)$.

In this case,

$$d(x, a_j) = d(x, a_{2p+j-1}) + d(a_{2p+j-1}, a_{j-1}) + d(a_{j-1}, a_j)$$

$$= d(x, a_{2p+j-1}) + d(a_{2p+j-1}, y) + d(y, a_j)$$

$$\neq d(y, a_j), where\ p = \left\lfloor \frac{x}{2} \right\rfloor$$

Thus representations of x and y with respect to W are distinct. Thus W is an *ir-set* and $ir(G) = n - 3$.

Theorem 2.3: The independent resolving number of Γ_n is given by

$$ir(\Gamma_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor - 1, if\ n\ is\ even \\ \left\lfloor \frac{n}{2} \right\rfloor, \quad if\ n\ is\ odd \end{cases}$$

The proof of this theorem is obtained by using the same strategy as in Theorem 2.2.

3. CONCLUSION

The exact values of independent resolving number for Fibonacci and extended Fibonacci cubes are determined. The problem is open for other networks like hypercube, hexagonal, etc.

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