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Legendre Wavelet Expansion of a Function $f(x, y)$ and its Approximation

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ABSTRACT

In this paper, two new Legendre Wavelet estimators of a function f of two variable x and y by $(2^{k-1}, M; 2^{k'-1}, M')^{th}$ partials sums of their Legendre wavelet series are obtained. These estimators are sharper and better in Wavelet Analysis.

General Terms

Continuous function, basic wavelet, partial derivatives of a function of two variable, Legendre polynomials, partial sums of Legendre expansion.

Keywords

Legendre wavelets, wavelet expansion, wavelet approximation, mother wavelet

1. INTRODUCTION

In wavelet analysis, orthogonal functions and wavelets are used to approximate certain functions. The orthogonal functions are generally classified into the following families:

(1) Sets of piecewise constant basis functions including Walsh function, block pulse function etc.
(2) Sets of orthogonal polynomials such as Legendre polynomials and Chebyshev polynomials etc.
(3) Sets of sine-cosine functions in Fouries series.

It is remarkable to note that the approximation of a continuous function by the help of a piecewise constant basis functions may not be continuous. Contrast to this, if continuous basis function are used to approximate a discontinuous function then the resulting approximant is continuous and it gives no proper informations for discontinuties. There are several functional equations whose solutions vary continuously in some regions and discontinuous in others. Continuous basis functions or piecewise constant basis functions taken alone can not accurately or efficiently model these dimensionally varying properties. Thus, it is necessary to use approximating basis functions that take in account the continuous and discontinuous phenomena. Wavelet functions are most efficient for these situations. Legendre wavelets possess the orthogonality property. Some results concerning to Legendre wavelet and Haar wavelet have been discussed by researchers Islam[3], Lal and Kumar[5], Lal and Kumar[6], Nanshan[4] and Razzaghi[2] etc. The Legendre wavelet approximation of functions of two variables have not been discussed so far. In an attempt to make an advance study in this direction, in this paper, wavelet approximations of function f of two variables x and y by Legendre wavelet method have been established.

2. DEFINITIONS

2.1 One dimensional Legendre Wavelet

Wavelets constitute a family of functions constructed from dilation and translation of a single function called mother wavelet. If we restrict the values of dilation and translation parameter to $a = a_0^{-k}$, $b = nb_0{a_0}^{-k}, a_0 > 1, b_0 > 0$, respectively, the following family of discrete wavelets are constructed.

$$\psi_{k,n} = |a_0|^{\frac{-k}{2}} \psi(a_0^{-k}x - nb_0)$$

The one dimensional Legendre wavelet over the interval [0,1] is defined as $\psi_{n,m}(x) = \begin{cases} \sqrt{m+\frac{1}{2}}\ 2^{\frac{k}{2}} L_m(2^k x - \hat{n}), & \hat{n} - 12^k \leq x < \hat{n} + 12^k \\ 0 & , otherwise. \end{cases}$

Where $n = 1, 2, ..., 2^{k-1}$ and $m = 0, 1, 2, ...M - 1$. In above definition, the polynomials L_m are Legendre polynomials of degree m over the interval [-1,1] which can be defined as follows,

$$L_0(x) = 1$$

$$L_1(x) = x$$

$$(m+1)L_{m+1}(x) = (2m+1)xL_m(x) - mL_{m-1}(x) \quad m = 1, 2, 3, ...$$

The set of $\{L_m(x) : m = 1, 2, 3, ...\}$ in the Hilbert space $L^2[-1, 1]$ is a complete orthogonal set. Orthogonality of Legendre polynomial on the interval [-1,1] implies that $\langle L_m(x), L_n(x)\rangle = \int_{-1}^{1} L_m(x)L_n(x)dx = \begin{cases} 22m+1 & , m = n \\ 0 & , otherwise. \end{cases}$ Furthermore, the set of wavelets $\psi_{n,m}(x)$ makes an orthonormal basis in $L^2[0, 1]$, $i.e$

$$\int_0^1 \psi_{n,m}\psi_{n',m'}dx = \delta_{n,n'}\delta_{m,m'}$$

in which δ denotes Kronecker delta function. The function $f(x) \in L^2[0,1]$ can be approximated by Legendre wavelet as

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\psi(x)$$

in which C and $\psi(x)$ are $2^{k-1}M$ vectors of the form

$$C^T = [c_{1,0}, c_{1,1}, ...c_{1,M-1}...c_{2,0}, c_{2,1}...c_{2^{k-1},0}, ...c_{2^{k-1},M-1}]$$

and

$$\psi(x) = [\psi_{1,0}, \psi_{1,1}, ...\psi_{1,M-1}...\psi_{2,0}, \psi_{2,1}...\psi_{2^{k-1},0}, ...\psi_{2^{k-1},M-1}]^T.$$

2.2 Two dimensional Legendre wavelet

One dimensional Legendre wavelets discussed in previous section is now generalized into two dimensional Legendre wavelets as following:
Two dimensional Legendre wavelets over the region $[0,1) \times [0,1)$ can be defined as follows

$$\begin{aligned}\psi_{n,m;n',m'}(x,y) &= \psi_{n,m}(x)\psi_{n',m'}(y)\\ &= \begin{cases} \sqrt{m+\frac{1}{2}}\sqrt{m'+\frac{1}{2}}\,2^{\frac{k+k'}{2}}L_m(2^kx-\hat{n}) \\ \times L_{m'}(2^{k'}y-\hat{n'}), \ \frac{\hat{n}-1}{2^k} \le x < \frac{\hat{n}+1}{2^k}, \\ \qquad\qquad \frac{\hat{n'}-1}{2^{k'}} \le y < \frac{\hat{n'}+1}{2^{k'}}; \\ 0, \qquad\qquad \textit{otherwise}, \end{cases}\end{aligned}$$

where $m = 0,1,2...,M-1, m' = 0,1,2...,M'-1, n = 1,2 ...,2^{k-1}, n' = 1,2...,2^{k'-1}$. By above definition, the region $[0,1)\times[0,1)$ is divided to $2^{k-1}\times 2^{k'-1}$ subregions. The paremeter M and M' denote the number of Legendre polynomials for variables x and y respectively. So, $M \times M'$ wavelets are constructed on each subregions. By considering $\psi_{n,m}(x)$ and $\psi_{n',m'}(y)$ as two sets of one dimensional Legendre Wavelets over variables x and y, respectively, the two dimensional Legendre wavelets over the region $[0,1)\times[0,1)$ may be written as

$$\psi_{n,m,n',m'}(x,y) = \psi_{n,m}(x)\psi_{n',m'}(y).$$

If

$$\begin{aligned}\Psi_{n,n'}(x,y) &= [\psi_{n,0;n',0}, ..., \psi_{n,0;n',m'-1}, \psi_{n,1;n',0}, ..., \psi_{n,1;n',m'-1}\\ &\quad \psi_{n,m-1;n',0}, ..., \psi_{n,m-1;n',m'-1}]^T\end{aligned}$$

be an MM' vector of $2D$ - LWs defined on $(nn')^{th}$ subregion, then

$$\Psi(x,y) = [\Psi_{1,1}^T, .., \Psi_{1,n'}^T, \Psi_{2,1}^T, .., \Psi_{2,n'}^T, .., \Psi_{n,1}^T, .., \Psi_{n,n'}^T]$$

is an $2^{k-1}2^{k'-1}MM'$ vector concluding $2D$ LWs. The set of $2D$ LWs is an orthogonal set over the region $[0,1)\times[0,1)]$ that is

$$\begin{aligned}\int_0^1\int_0^1 \psi_{n,m;n',m'}(x,y)\psi_{n_1,m_1;n'_1,m'_1}(x,y)dxdy &= \delta_{n,n_1}\delta_{m,m_1}\\ &\times\ \delta_{n',n'_1}\delta_{m,m'_1}.\end{aligned}$$

3. FUNCTION EXPANSION

Any function in $L^2([0,1)\times[0,1)$ can be expanded as,

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} c_{n,m;n',m'}\psi_{n,m;n',m'}(x,y) \quad (1)$$

Let $S_{2^{k-1},M;2^{k'-1},M'}(x,y)$ denotes the $(2^{k-1},M;2^{k'-1},M')^{th}$ partial sums of the series given by the equation (1), then

$$S_{2^{k-1},M,2^{k'-1},M}(x,y) = \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=0}^{M-1}\sum_{m'=0}^{M'-1} c_{n,m;n',m'}\psi_{n,m;n',m'}(x,y).$$

4. LEGENDRE WAVELET EXPANSION

Any function f in $L^2([0,1)\times[0,1)$ can be expanded in double Legendre wavelet series as

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} c_{n,m;n',m'}\psi_{n,m;n',m'}(x,y) \quad (2)$$

Let $S_{2^{k-1},M;2^{k'-1},M'}(x,y)$ denote the $(2^{k-1},M;2^{k'-1},M')^{th}$ partial sums of the series (2),then

$$S_{2^{k-1},M;2^{k'-1},M'}(x,y) = \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=0}^{M-1}\sum_{m'=0}^{M'-1} c_{n,m;n',m'}\psi_{n,m;n',m'}(x,y).$$

5. LEGENDRE WAVELET APPROXIMATION

The Legendre Wavelet Approximation $E_{2^{k-1},M;2^{k'-1},M'}(f)$ of a function $f \in L^2[0,1)\times[0,1)$ is given by

$$E_{2^{k-1},M;2^{k'-1},M'}(f) = inf f - S_{2^{k-1},M;2^{k'-1},M'}2.$$

$$Where, \quad f_2 = \left(\int_0^1\int_0^1 |f(x,y)|^2dxdy\right)^{\frac{1}{2}}.$$

If $E_{2^{k-1},M;2^{k'-1},M'}(f) \to 0$ as $k,k';M,M' \to \infty$ then $E_{2^{k-1},M;2^{k'-1},M'}(f)$ is called the best wavelet approximation of f of order $(2^{k-1},M;2^{k'-1},M')$.(Zygmund) [1])

6. THEOREMS

In this paper we prove the following theorems.

6.1 Theorem

If $f(x,y)$ is continuous function on $[0,1)\times[0,1)$ into R such that $0 < \frac{\partial^4 f}{\partial x^2\partial y^2} = N < \infty\ \forall(x,y)\in[0,1)\times[0,1)$. Then Legendre Wavelet approximation $E^{(1)}_{2^{k-1},M;2^{k'-1},M'}(f)$ of $f(x,y) = \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} c_{n,m;n',m'}\psi_{n,m;n',m'}(x,y)$ by $S_{2^{k-1},M;2^{k'-1},M} = \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=0}^{M-1}\sum_{m'=0}^{M'-1} c_{n,m;n',m'}\psi_{n,m;n',m'}$ is given by

$$\begin{aligned}E^{(1)}_{2^{k-1},M;2^{k'-1},M'}(f) &= inf f - S_{2^{k-1},M;2^{k'-1},M'}2\\ &= O\left[\left(\frac{1}{2^{2k+2k'+1}}\right)\right.\\ &\times\ \left.\left(\left(\frac{1}{2M-3}\right)^{\frac{3}{2}} + \left(\frac{1}{2M'-3}\right)^{\frac{3}{2}}\right)\right]\end{aligned}$$

, for $M \ge 2$ and $M' \ge 2$.

6.2 Theorem

If $f(x,y)$ is continuous function on $[0,1)\times[0,1)$ into R such that $0<\frac{\partial^4 f}{\partial x\partial y^3}=N_1<\infty\ \forall(x,y)\in[0,1)\times[0,1)$.Then Legendre Wavelet approximation $E^2_{2^{k-1},M;2^{k'-1},M'}(f)$ of $f(x,y)$ is given by

$$\begin{aligned} E^{(2)}_{2^{k-1},M;2^{k'-1},M'}(f) &= inf f - S_{2^{k-1},M;2^{k'-1},M'2} \\ &= O\left[\left(\frac{1}{2^{k+3k'+1}}\right) \right.\\ &\times \left.\left(\left(\frac{1}{2M-1}\right)^{\frac{1}{2}}+\left(\frac{1}{2M'-5}\right)^{\frac{5}{2}}\right)\right] \end{aligned}$$

for $M\geq 1$ and $M'\geq 3$.

7. PROOFS

7.1 Proof of Theorem 6.1

The Legendre Wavelet Series of $f(x,y)\in L^2[0,1)\times[0,1)$ is

$$f(x,y)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}c_{n,m;n',m'}\psi_{n,m;n',m'}(x,y)$$

.

Then,

$$\begin{aligned} \langle f,\psi_{p,q}\rangle &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}c_{n,m;n',m'}\langle\psi_{n,m;n',m'},\psi_{p,q}\rangle \\ &= \sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}c_{p,q;n',m'}\langle\psi_{p,q;n',m'},\psi_{p,q}\rangle \end{aligned}$$

$$\begin{aligned} \langle\langle f,\psi_{p,q}\rangle,\psi_{p',q'}\rangle &= \sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}c_{p,q;n',m'}\langle\langle\psi_{p,q;n',m'},\psi_{p,q}\rangle,\psi_{p',q'}\rangle \\ &= c_{p,q;p',q'}\langle\langle\psi_{p,q;p',q'},\psi_{p,q}\rangle,\psi_{p',q'}\rangle \end{aligned}$$

,by orthogonality of $\psi_{n,m;,n',m'}$

Thus,

$$\begin{aligned} c_{p,q;p',q'} &= \frac{\langle\langle f,\psi_{p,q}\rangle,\psi_{p',q'}\rangle}{\langle\langle\psi_{p,q;p',q'},\psi_{p,q}\rangle,\psi_{p',q'}\rangle} \\ &= \frac{\langle\langle f,\psi_{p,q}\rangle,\psi_{p',q'}\rangle}{\langle\psi_{p,q},\psi_{p,q}\rangle\langle\psi_{p',q'},\psi_{p',q'}\rangle} \end{aligned}$$

$$c_{p,q;p',q'}=\langle\langle f,\psi_{p,q}\rangle,\psi_{p',q'}\rangle \qquad (\psi_{p,q2}=\psi_{p',q'2}=1)$$

$$\begin{aligned} c_{n,m;n',m'} &= \langle\langle f,\psi_{n,m}\rangle,\psi_{n',m'}\rangle \\ &= \langle\langle f,\psi_{n',m'}\rangle,\psi_{n,m}\rangle \\ &= \langle\psi_{n,m},\langle f,\psi_{n',m'}\rangle\rangle \qquad (\langle x,y\rangle=\overline{\langle y,x\rangle}) \end{aligned}$$

Now,

$$\begin{aligned} c_{n,m;n',m'} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\psi_{n,m}(x)\psi_{n',m'}(y)dxdy \\ &= \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}\psi_{n,m}(x)\left(\int_{\frac{\hat{n}'-1}{2^k}}^{\frac{\hat{n}'+1}{2^k}}f(x,y)\psi_{n',m'}(y)dy\right)dx \\ &= \sqrt{m+\frac{1}{2}}\sqrt{m'+\frac{1}{2}}\ 2^{\frac{k+k'}{2}}\int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}L_m(2^kx-\hat{n}) \\ &\times \left(\int_{\frac{\hat{n}'-1}{2^{k'}}}^{\frac{\hat{n}'+1}{2^{k'}}}f(x,y)L_{m'}(2^{k'}y-\hat{n}')dy\right)dx \qquad (3) \end{aligned}$$

Let

$$I_1=\int_{\frac{\hat{n}'-1}{2^{k'}}}^{\frac{\hat{n}'+1}{2^{k'}}}f(x,y)L_{m'}(2^{k'}y-\hat{n}')dy \qquad (4)$$

Then

$$\begin{aligned} I_1 &= \int_{-1}^{1}f\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)L_{m'}(v)\frac{dv}{2^{k'}} \quad ,taking\ \ 2^{k'}y-\hat{n}'=v \\ &= \frac{1}{2^{k'}}\int_{-1}^{1}f\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L'_{m'+1}-L'_{m'-1}}{2^{k'}}\right)dv \\ &= \frac{-1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)(L_{m'+1}-L_{m'-1})dv \qquad (5) \\ &= \frac{-1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)L_{m'+1}dv \\ &+ \frac{1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)L_{m'-1}dv \\ &= \frac{-1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L'_{m'+2}-L'_{m'})}{2m'+3}dv \\ &+ \frac{1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L'_{m'}-L'_{m'-2})}{2m'-1}dv \\ &= \frac{-1}{2^{2k'}(2m'+1)}\left[0-\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L_{m'+2}-L_{m'})}{2^{k'}(2m'+3)}dv\right] \\ &+ \frac{1}{2^{2k'}(2m'+1)}\left[0-\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L_{m'}-L_{m'-2})}{2^{k'}(2m'-1)}dv\right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2^{3k'}}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L_{m'+2}-L_{m'})}{(2m'+3)(2m'+1)}dv \\ &+ \frac{1}{2^{3k'}}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L_{m'-2}-L_{m'})}{(2m'-1)(2m'+1)}dv \\ &= \frac{1}{2^{3k'}}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)E_{m'}(v)dv. \qquad (6) \end{aligned}$$

Where

$$E_{m'}(v)=\frac{(L_{m'+2}-L_{m'})}{(2m'+3)(2m'+1)}+\frac{(L_{m'-2}-L_{m'})}{(2m'-1)(2m'+1)}$$

By equation (3) and (6), We get

$$\begin{aligned} c_{n,m;n',m'} &= \sqrt{m+\frac{1}{2}}\sqrt{m'+\frac{1}{2}}\ 2^{\frac{k+k'}{2}}\ \frac{1}{2^{3k'}} \\ &\times \int_{\frac{\hat{n}-1}{2^k}}^{\frac{\hat{n}+1}{2^k}}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)L_m(2^kx-\hat{n})E_{m'}(v)dxdv \end{aligned}$$

Now putting $2^k x - \hat{n} = u$ and integrating as above, we get

$$\begin{aligned} c_{n,m;n',m'} &= \sqrt{m+\frac{1}{2}}\sqrt{m'+\frac{1}{2}}\; 2^{\frac{k+k'}{2}} \frac{1}{2^{3k+3k'}} \\ &\times \int_{-1}^{1}\int_{-1}^{1} \frac{\partial^4 f}{\partial u^2 \partial v^2}\left(\frac{u+\hat{n}}{2^k}, \frac{v+\hat{n}'}{2^{k'}}\right) E_m(u)E_{m'}(v)dudv \end{aligned}$$

Where

$$E_m(u) = \frac{(L_{m+2}-L_m)}{(2m+3)(2m+1)} + \frac{(L_{m-2}-L_m)}{(2m-1)(2m+1)}$$

$$\begin{aligned} c_{n,m;n',m'} &= \sqrt{m+\frac{1}{2}}\sqrt{m'+\frac{1}{2}}\; \frac{1}{2^{\frac{5k+5k'}{2}}} \\ &\times \int_{-1}^{1}\int_{-1}^{1} \frac{\partial^4 f}{\partial u^2 \partial v^2}\left(\frac{u+\hat{n}}{2^k}, \frac{v+\hat{n}'}{2^{k'}}\right) E_m(u)E_{m'}(v)dudv \\ &= \frac{1}{\sqrt{(2m+1)(2m'+1)}}\, \frac{1}{2^{\frac{5k+5k'+2}{2}}} \\ &\times \int_{-1}^{1}\int_{-1}^{1} \left[\frac{\partial^4 f}{\partial u^2 \partial v^2}\left(\frac{u+\hat{n}}{2^k}, \frac{v+\hat{n}'}{2^{k'}}\right)\right. \\ &\times A_m(u)A_{m'}(v)dudv] \end{aligned} \quad (7)$$

Where

$$A_m(u) = \frac{(L_{m+2}-L_m)}{(2m+3)} + \frac{(L_{m-2}-L_m)}{(2m-1)}$$

and

$$A_{m'}(v) = \frac{(L_{m'+2}-L_{m'})}{(2m'+3)} + \frac{(L_{m'-2}-L_{m'})}{(2m'-1)}$$

$$\begin{aligned} |c_{n,m,n',m'}|^2 &\le \frac{1}{2^{(5k+5k'+2)}(2m+1)(2m'+1)} \int_{-1}^{1}\int_{-1}^{1} \left|\frac{\partial^4 f}{\partial u^2 \partial v^2}\right|^2 dudv \\ &\times \int_{-1}^{1}\int_{-1}^{1} A_m^2(u)A_{m'}^2(v)dudv \end{aligned}$$

$$\begin{aligned} |c_{n,m,n',m'}|^2 &\le \frac{4M^2}{2^{(5k+5k'+2)}(2m+1)(2m'+1)} \\ &\times \int_{-1}^{1} A_m^2(u)du \int_{-1}^{1} A_{m'}^2(v)du \end{aligned}$$

$$\begin{aligned} Now,\quad \int_{-1}^{1} A_m^2(u)du &= \int_{-1}^{1} \left(\frac{L_{m+2}^2+L_m^2}{(2m+3)^2} + \frac{L_m^2+L_{m-2}^2}{(2m-1)^2}\right) du \\ &\le \frac{8}{(2m-3)^3}, m \ge 2 \end{aligned}$$

(by using orthogonal properties of Legendre polynomials)
Similarly,

$$\int_{-1}^{1} A_{m'}^2(v)dv \le \frac{8}{(2m'-3)^3}, m' \ge 2$$

Hence,

$$\begin{aligned} |c_{n,m:n',m'}|^2 &= \frac{4M^2}{2^{5k+5k'+2}(2m+1)(2m'+1)} \frac{64}{(2m-3)^3(2m'-3)^3} \\ &= \frac{256M^2}{2^{5k+5k'+2}(2m-3)^4(2m'-3)^4} \end{aligned}$$

Now,

$$\begin{aligned} f(x,y) - S_{2^{k-1},M;2^{k'-1},M'} &= \sum_{n=1}^{\infty}\sum_{n'=1}^{\infty}\sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} c_{n,m,n',m'}\psi_{n,m,n',m'} \\ &- \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=0}^{M-1}\sum_{m'=0}^{M'-1} c_{n,m;n',m'}\psi_{n,m;n',m'} \\ &= \left(\sum_{n=1}^{2^{k-1}} + \sum_{n=2^{k-1}+1}^{\infty}\right)\left(\sum_{n'=1}^{2^{k'-1}} + \sum_{n'=2^{k'-1}+1}^{\infty}\right) \\ &\times \left(\sum_{m=0}^{M-1} + \sum_{m=M}^{\infty}\right)\left(\sum_{m'=0}^{M'-1} + \sum_{m'=M,}^{\infty}\right) \\ &\times c_{n,m;n',m'}\psi_{n,m;n',m'} \\ &- \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=0}^{M-1}\sum_{m'=0}^{M'-1} c_{n,m;n',m'}\psi_{n,m;n',m'} \\ &= \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=0}^{M-1}\sum_{m'=M'}^{\infty} c_{n,m;n',m'}\psi_{n,m;n',m'} \\ &+ \sum_{n=1}^{2^{k-1}}\sum_{n'=1}^{2^{k'-1}}\sum_{m=M}^{\infty}\sum_{m'=0}^{M'-1} c_{n,m;n',m'}\psi_{n,m;n',m'} \\ &+ \sum_{n=1}^{2^{k-1}}\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{m=M'}^{\infty} c_{n,m;n',m'}\psi_{n,m;n',m'} \end{aligned} \quad (8)$$

Next,

$$\begin{aligned} \|f - S_{2^{k-1},M;2^{k'-1},M'}\|_2^2 &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty} |\, c_{n,m;n',m'} \,|^2 \\ &+ \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} |\, c_{n,m;n',m'} \,|^2 \\ &+ \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty} |\, c_{n,m;n',m'} \,|^2 \end{aligned}$$

$$\leq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty}\frac{256N_2^2}{2^{5k+5k'+2}(2m'-3)^4(2m-3)^4}$$
$$+\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{M'-1}\frac{256N_2^2}{2^{5k+5k'+2}(2m'-3)^4(2m-3)^4}$$
$$+\cdot\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty}\frac{256N_2^2}{2^{5k+5k'+2}(2m'-3)^4(2m-3)^4}$$
$$\leq \frac{64N_2^2C_2}{2^{4k'+4k+2}}\left(1+\frac{1}{(2M'-3)^3}\right)\left(\frac{1}{(2M-3)^3}\right)$$
$$+\frac{64N_2^2C_2}{2^{4k'+4k+2}}\left(\frac{1}{(2M'-3)^3}\right)\left(1+\frac{1}{(2M-3)^3}\right)$$
$$+\frac{64N_2^2C_2}{2^{4k'+4k+2}}\left(\frac{1}{(2M'-3)^3}\right)\left(\frac{1}{(2M-3)^3}\right)$$
$$=\frac{64N_2^2C_2}{2^{4k'+4k+2}}\times\left[\frac{1}{(2M'-3)^3}+\frac{1}{(2M-3)^3}+\frac{3}{(2M'-3)^3(2M-3)^3}\right]$$
$$,C_2\ being\ a\ suitable\ positive\ constant.$$
$$\leq\frac{64N_2^2C_2}{2^{4k'+4k+2}}\times\left[\frac{1}{(2M'-3)^3}+\frac{1}{(2M-3)^3}+\frac{3}{(2M'-3)^3(2M-3)^3}\right]$$
$$\leq\frac{64N_2^2C_2}{2^{4k'+4k+2}}\times\left[\frac{2}{(2M'-3)^3}+\frac{2}{(2M-3)^3}+\frac{4}{(2M'-3)^5(2M-3)^3}\right]$$
$$=\frac{128N_2^2C_2}{2^{4k'+4k+2}}\times\left(\frac{1}{(2M'-3)^3}+\frac{1}{(2M-3)^3}+\frac{2}{(2M'-3)^3(2M-3)^3}\right)$$
$$\leq\frac{128N_2^2C_2}{2^{4k'+4k+2}}\times\left[\frac{1}{(2M'-3)^3}+\frac{1}{(2M-3)^3}+\frac{2}{(2M'-3)^{\frac{3}{2}}(2M-3)^{\frac{3}{2}}}\right]$$
$$=\frac{128N_2^2C_2}{2^{4k'+4k+2}}\times\left[\frac{1}{(2M'-3)^3}+\frac{1}{(2M-3)^3}+\frac{2}{(2M'-3)^{\frac{3}{2}}(2M-3)^{\frac{3}{2}}}\right]$$
$$=\frac{128N_2^2C_2}{2^{4k'+4k+2}}\left[\frac{1}{(2M'-3)^{\frac{3}{2}}}+\frac{1}{(2M-3)^{\frac{3}{2}}}\right]^2,\quad M\geq 2, M'\geq 2.$$

$$\|f-S_{2^{k-1},M;2^{k'-1},M'}\|_2 = \left[\left(\frac{8\sqrt{2}N_2\sqrt{C_2}}{2^{2k'+2k+1}}\right)\times\left(\frac{1}{(2M'-3)^{\frac{3}{2}}}+\frac{1}{(2M-3)^{\frac{3}{2}}}\right)\right].$$

$$So,\ E^{(1)}_{2^{k-1},M;2^{k'-1},M'} = O\left[\frac{1}{2^{2k'+2k+1}}\left(\frac{1}{(2M'-3)^{\frac{3}{2}}}+\frac{1}{(2M-3)^{\frac{3}{2}}}\right)\right].$$

Thus the Theorem (6.1) is completely established.

7.2 Proof of the Theorem (6.2)

$$\int_{\frac{\hat{n}-1}{2^{k'}}}^{\frac{\hat{n}'+1}{2^{k'}}} f(x,y)L_{m'}(2^{k'}y-\hat{n}')dy = \left(\frac{-1}{2^{2k'}(2m'+1)}\right)\times\int_{-1}^{1}\left[\frac{\partial f}{\partial u}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\times(L_{m'+1}-L_{m'-1})dv\right]$$
$$=\left(\frac{1}{2^{2k'}(2m'+1)}\right)\times\int_{-1}^{1}\left[\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\times\left(\frac{L'_{m'}-L'_{m'-2}}{2m'-1}\right)dv\right]$$
$$-\left(\frac{1}{2^{2k'}(2m'+1)}\right)\times\int_{-1}^{1}\left[\frac{\partial f}{\partial v}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\times\left(\frac{L'_{m'+2}-L'_{m'}}{2m'+3}\right)dv\right]$$

$$=\left(\frac{1}{2^{2k'}(2m'+1)}\right)\left(0-\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\frac{(L_{m'}-L_{m'-2})}{2^{k'}(2m'-1)}dv\right)$$
$$-\frac{1}{2^{2k'}(2m'+1)}\left(0-\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\frac{(L_{m'+2}-L_{m'})}{2^{k'}(2m'+3)}dv\right)$$
$$=-\frac{1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\frac{(L_{m'}-L_{m'-2})}{2^{k'}(2m'-1)}dv$$
$$+\frac{1}{2^{2k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\frac{(L_{m'+2}-L_{m'})}{2^{k'}(2m'+3)}dv$$
$$=\frac{1}{2^{3k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L_{m'+2}-L_{m'})}{(2m'+3)}dv$$
$$-\frac{1}{2^{3k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{(L_{m'}-L_{m'-2})}{(2m'-1)}dv$$
$$=\frac{1}{2^{3k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{L_{m'+2}}{(2m'+3)}dv$$
$$-\frac{1}{2^{3k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{L_{m'}}{(2m'+3)}dv$$

$$-\frac{1}{2^{3k'}(2m'+1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{L_{m'}}{(2m'-1)}dv$$
$$+\frac{1}{2^{3k'}(2m'-1)}\int_{-1}^{1}\frac{\partial^2 f}{\partial v^2}\left(x,\frac{v+\hat{n}'}{2^{k'}}\right)\frac{L_{m'-2}}{(2m'-1)}dv$$

$$
\begin{aligned}
&= \frac{1}{2^{3k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^2 f}{\partial v^2}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L'_{m'+3}-L'_{m'+1}}{(2m'+5)(2m'+3)}\right) dv \\
&- \frac{1}{2^{3k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^2 f}{\partial v^2}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L'_{m'+1}-L'_{m'-1}}{(2m'+1)(2m'+3)}\right) dv \\
&- \frac{1}{2^{3k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^2 f}{\partial v^2}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L'_{m'+1}-L'_{m'-1}}{(2m'+1)(2m'-1)}\right) dv \\
&+ \frac{1}{2^{3k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^2 f}{\partial v^2}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L'_{m'-1}-L'_{m'-3}}{(2m'-3)(2m'-1)}\right) dv \\
&= \frac{-1}{2^{4k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^3 f}{\partial v^3}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L_{m'+3}-L_{m'+1}}{(2m'+5)(2m'+3)}\right) dv \\
&+ \frac{1}{2^{4k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^3 f}{\partial v^3}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L_{m'+1}-L_{m'-1}}{(2m'+1)(2m'+3)}\right) dv \\
&+ \frac{1}{2^{4k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^3 f}{\partial v^3}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L_{m'+1}-L_{m'-1}}{(2m'+1)(2m'-1)}\right) dv \\
&- \frac{1}{2^{4k'}(2m'+1)} \\
&\times \int_{-1}^{1} \frac{\partial^3 f}{\partial v^3}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right)\left(\frac{L_{m'-1}-L_{m'-3}}{(2m'-3)(2m'-1)}\right) dv \\
&= \frac{1}{2^{4k'}(2m'+1)} \int_{-1}^{1} \frac{\partial^3 f}{\partial v^3}\left(x, \frac{v+\hat{n}'}{2^{k'}}\right) E_{m'}(v) dv. \qquad (9)
\end{aligned}
$$

where

$$
\begin{aligned}
E_{m'}(v) &= \frac{L_{m'+1}-L_{m'+3}}{(2m'+3)(2m'+5)} + 2\frac{(L_{m'+1}-L_{m'-1})}{(2m'-1)(2m'+3)} \\
&+ \frac{L_{m'-3}-L_{m'-1}}{(2m'-1)(2m'-3)}
\end{aligned}
$$

Using equation (3),(5) and (9) , we get

$$
\begin{aligned}
c_{n,m;n',m'} &= \left(\frac{1}{2^{\frac{3k+7k'+2}{2}}(2m'+1)^{\frac{1}{2}}(2m+1)^{\frac{1}{2}}}\right) \\
&\times \int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial^4 f}{\partial u \partial v^3}\left(\frac{u+\hat{n}}{2^k}, \frac{v+\hat{n}'}{2^{k'}}\right)\right. \\
&\times \left. P_m(u) E_{m'}(v) du\, dv\right] \qquad (10)
\end{aligned}
$$

Using orthogonal property of Legendre polynomials i.e. $\langle L_{m'}(x), L_{n'}(x)\rangle = \int_{-1}^{1} L_{m'}(x)L_{n'}(x)dx = \begin{cases} 22m'+1 & , m'=n' \\ 0 & , otherwise. \end{cases}$

$$
\begin{aligned}
\int_{-1}^{1} E_{m'}^2(v)dv &= \int_{-1}^{1} \frac{L_{m'+1}^2+L_{m'+3}^2}{(2m'+3)^2(2m'+5)^2}dv \\
&+ \int_{-1}^{1} \frac{4(L_{m'+1}^2+L_{m'-1}^2)}{(2m'-1)^2(2m'+3)^2}dv \\
&+ \int_{-1}^{1} \frac{L_{m'-3}^2+L_{m'-1}^2}{(2m'-1)^2(2m'-3)^2}dv \\
&= \frac{1}{(2m'+3)^2(2m'+5)^2}\left(\frac{2}{2m'+3}+\frac{2}{2m'+7}\right) \\
&+ \frac{4}{(2m'-1)^2(2m'+3)^2}\left(\frac{2}{2m'+3}+\frac{2}{2m'-1}\right) \\
&+ \frac{1}{(2m'-1)^2(2m'-3)^2}\left(\frac{2}{2m'-5}+\frac{2}{2m'-1}\right) \\
&\leq \frac{24}{(2m'-5)^5} \qquad (11)
\end{aligned}
$$

Similarly,

$$
\int_{-1}^{1} P_m^2(u)\, du \leq \frac{4}{(2m-1)}. \qquad (12)
$$

Using Schwarz inequality, equations (11) and (12) in equation (10)

$$
|c_{n,m;n',m'}|^2 \leq \frac{96N_2^2}{2^{3k+7k'}(2m'-5)^6(2m-1)^2}. \qquad (13)
$$

$$
,m' \geq 3, m \geq 1.
$$

Next,

$$
\begin{aligned}
\|f - S_{2^{k-1},M;2^{k'-1},M'}\|_2^2 &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty} |\, c_{n,m;n',m'}\,|^2 \\
&+ \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} |\, c_{n,m;n',m'}\,|^2 \\
&+ \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty} |\, c_{n,m;n',m'}\,|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty} \frac{96N_2^2}{2^{3k+7k'}(2m'-5)^6(2m-1)^2} \\
&+ \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k-1}}\sum_{m'=0}^{M'-1} \frac{96N_2^2}{2^{3k+7k'}(2m'-5)^6(2m-1)^2} \\
&+ .\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=M'}^{\infty} \frac{96N_2^2}{2^{3k+7k'}(2m'-5)^6(2m-1)^2} \\
&\leq \frac{96N_2^2C_2}{2^{6k'+2k+2}}\left(1+\frac{1}{(2M'-5)^5}\right)\left(\frac{1}{(2M-1)}\right)
\end{aligned}
$$

$$+\ \frac{96N_2^2C_2}{2^{6k'+2k+2}}\left(\frac{1}{(2M'-5)^5}\right)\left(1+\frac{1}{(2M-1)}\right)$$
$$+\ \frac{96N_2^2C_2}{2^{6k'+2k+2}}\left(\frac{1}{(2M'-5)^5}\right)\left(\frac{1}{(2M-1)}\right)$$
$$=\ \frac{96N_2^2C_2}{2^{6k'+2k+2}}$$
$$\times\ \left[\frac{1}{(2M'-5)^5}+\frac{1}{(2M-1)}+\frac{3}{(2M'-5)^5(2M-1)}\right]$$
$$,C_2\ being\ a\ suitable\ positive\ constant.$$
$$\leq\ \frac{96N_2^2C_2}{2^{6k'+2k+2}}$$
$$\times\ \left[\frac{1}{(2M'-5)^5}+\frac{1}{(2M-1)}+\frac{3}{(2M'-5)^5(2M-1)}\right]$$
$$\leq\ \frac{96N_2^2C_2}{2^{6k'+2k+2}}$$
$$\times\ \left[\frac{2}{(2M'-5)^5}+\frac{2}{(2M-1)}+\frac{4}{(2M'-5)^=5(2M-1)}\right]$$
$$=\ \frac{192N_2^2C_2}{2^{6k'+2k+2}}$$
$$\times\ \left[\frac{1}{(2M'-5)^5}+\frac{1}{(2M-1)}+\frac{2}{(2M'-5)^5(2M-1)}\right]$$
$$\leq\ \frac{192N_2^2C_2}{2^{6k'+2k+2}}$$
$$\times\ \left[\frac{1}{(2M'-5)^5}+\frac{1}{(2M-1)}+\frac{2}{(2M'-5)^{\frac{5}{2}}(2M-1)^{\frac{1}{2}}}\right]$$
$$=\ \frac{192N_2^2C_2}{2^{6k'+2k+2}}$$
$$\times\ \left[\frac{1}{(2M'-5)^5}+\frac{1}{(2M-1)}+\frac{2}{(2M'-5)^{\frac{5}{2}}(2M-1)^{\frac{1}{2}}}\right]$$
$$=\ \frac{192N_2^2C_2}{2^{6k'+2k+2}}\left[\frac{1}{(2M'-5)^{\frac{5}{2}}}+\frac{1}{(2M-1)^{\frac{1}{2}}}\right]^2,\quad M\geq 1, M'\geq 3.$$

$$\|f-S_{2^{k-1},M;2^{k'-1},M'}\|_2 = \left[\left(\frac{8\sqrt{3}N_2\sqrt{C_2}}{2^{3k'+k+1}}\right)\times\left(\frac{1}{(2M'-5)^{\frac{5}{2}}}+\frac{1}{(2M-1)^{\frac{1}{2}}}\right)\right].$$

So,

$$E^{(2)}_{2^{k-1},M;2^{k'-1},M'}=O\left[\frac{1}{2^{3k'+k+1}}\left[\left(\frac{1}{2M-1}\right)^{\frac{1}{2}}+\left(\frac{1}{2M'-5}\right)^{\frac{5}{2}}\right]\right].$$

Thus the Theorem (6.2) is completely established.

8. CONCLUSIONS

Estimates of the Theorems (6.1) and (6.2) are given by

$$E^{(1)}_{2^{k-1},M;2^{k'-1},M'}(f)=O\left[\frac{1}{2^{2k+2k'+1}}\left[\left(\frac{1}{2M-3}\right)^{\frac{3}{2}}+\left(\frac{1}{2M'-3}\right)^{\frac{3}{2}}\right]\right],$$

for $M\geq 2$ and $M'\geq 2$

$$E^{(2)}_{2^{k-1},M;2^{k'-1},M'}(f)=O\left[\frac{1}{2^{k+3k'+1}}\left[\left(\frac{1}{2M-1}\right)^{\frac{1}{2}}+\left(\frac{1}{2M'-5}\right)^{\frac{5}{2}}\right]\right],$$

for $M\geq 1$ and $M'\geq 3$.

Since $E^{(1)}_{2^{k-1},M;2^{k'-1},M'}(f)\to 0$ and $E^{(2)}_{2^{k-1},M;2^{k'-1},M'}(f)\to 0$ as $k,k',M,M'\to 0$, therefore these estimators are best possible in Wavelet Analysis.

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