New Generalized Extreme Value Distribution and its Bivariate Extension

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ABSTRACT

In this paper, a new distribution called New Generalized Extreme Value (NGEV) distribution is introduced. Also, the statistical properties of this model are studied, such as, quantiles, moment generating function and moments of order statistics. Moreover, maximum likelihood estimators of it's parameters are discussed. An application of NGEV distribution to a survival times in months of 20 acute myeloid leukemia patients data set is provided. Also, bivariate New Generalized Extreme Value BNGEV distribution.is introduced a Marshall-Olkin type. Marginal and conditional distribution functions are studied. Furthermore, maximum likelihood estimates (MLEs) of the parameters are presented. An application of BNGEV distribution to an UEFA Champion′s League data set is provided and the profiles of the log-likelihood function of parameters of NGEVD and BNGEVD are plotted.

General Terms

Univariate distribution, Bivariate distribution

Keywords

Extreme value distribution, Exponentiated Weibull distribution, Moment generating function, Joint cumulative distribution function, Maximum likelihood estimation

1. INTRODUCTION

Extreme value distributions are the limiting distributions for the minimum or the maximum of a very large collection of random observations from the same arbitrary distribution. In the context of reliability modeling, extreme value distributions for the minimum are frequently encountered. The weibull distribution and the extreme value distribution have a useful mathematical relationship, i.e. the natural log of a weibull random time is an extreme value random observation.
The aim of this paper is to introduce a new generalized extreme value (NGEV) distribution by using the mathematical relationship between weibull distribution and extreme value distribution. Also, we introduce its bivariate and we call it bivariate New Generalized Extreme Value (BNGEV) distribution, whose marginals are NGEV distributions. It is a Marshall-Olkin type. Many authors used this method to introduce a new bivariate distributions, see for example Marshall and Olkin [2], Kundu and Gupta [1], Sarhan and Balakrishnan [3], Sarhan et al. [6], El-Bassiouny et al.[4] and El-Gohary et al. [5].
This article is organized as follows, the new generalized extreme value (NGEV) distribution are proposed in Section 2. Various properties including quantiles, median and moment generating function are investigated in Section 3. Rényi entropy is provided in Section 4. Moments of order statistics are obtained in Section 4. Section 5 is devoted to the maximum likelihood estimates of the parameters and the information matrix of the NGEV distribution. An application of NGEV distribution to a survival times in months of 20 acute myeloid leukemia patients data set is provided and the profiles of the log-likelihood function of parameters of NGED are ploted in section 6. In section 7, we introduced the bivariate case and we call it bivariate New Generalized Extreme Value (BNGEV) distribution. Also, various properties including the joint survival function, the joint cumulative distribution function, the joint probability density function, marginal probability density functions are investigated in Section 8. Section 9 is devoted to the maximum likelihood estimates of the parameters of the BNGEV distribution. In Section 10, an application of the BNGE distribution to a UEFA Champion′s League data set are provided and the profiles of the log-likelihood function of parameters of BNGED are ploted. Finally, the results of this paper are concluded in Section 11.

2. NEW GENERALIZED EXTREME VALUE DISTRIBUTION

In this section, we discuss the new generalized extreme value (NGEV) distribution. This distribution is derived from an exponentiated weibull distribution. Let X be a random variable has Exponentiated Weibull (EW) distribution [9] with parameters α, λ and $k > 0$, then its cdf is given by

$$G(x) = \left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha},\ x \geq 0,\ \alpha, k, \lambda > 0 \qquad (1)$$

and the probability density function (pdf) is given by

$$g(x) = \frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^k}\right)^{\alpha-1},\ x \geq 0,\ \alpha, k, \lambda > 0$$

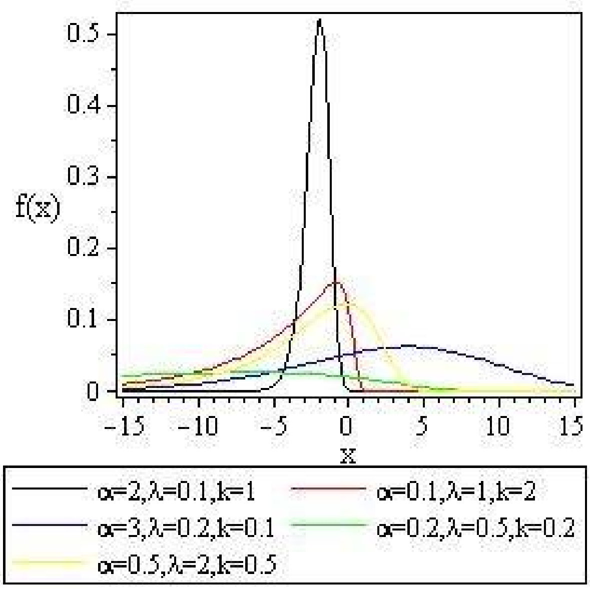

Fig. 1. The pdf of the NGEV distribution at different values of its parameters.

2.1 NGEV Specifications

THEOREM 1. *Let a non-negative random variable Y has the exponentiated weibull distribution, symbolically we write $Y \sim EW(\alpha, \lambda, k)$. Define a new random variable $X = \log Y$, then the random variable X has the new generalized extreme value distribution, symbolically we write $X \sim NGEV(\alpha, \lambda, k)$. The cumulative distribution function and the probability density function of X are respectively given by*

$$F_X(x) = \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha}, \ -\infty < x < \infty, \ \alpha, k, \lambda > 0 \quad (2)$$

and

$$f_X(x) = \alpha k \left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha - 1}. \quad (3)$$

PROOF. Since

$$\begin{aligned} F_X(x) &= P[X \le x] = P[\log Y \le x] \\ &= P[Y \le e^x] = G(e^x) \\ &= \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha}, \ -\infty < x < \infty, \alpha, k, \lambda > 0. \end{aligned}$$

By diffrentiation the cdf of X given in (2) with respect to x, we find the pdf of X given in (3), which complete the proof. ■

Since the cdf of NGEV is in closed form, we can use it to generate simulated data by using the following formula

$$x = \ln\left(\lambda\left(-\ln\left(1 - U^{\frac{1}{\alpha}}\right)\right)^{\frac{1}{k}}\right),$$

where U is a random variable which follows a standard uniform distribution on (0,1) interval.

Figures 1 and 2 illustrate some of the possible shapes of the pdf and cdf of the NGEV distribution.

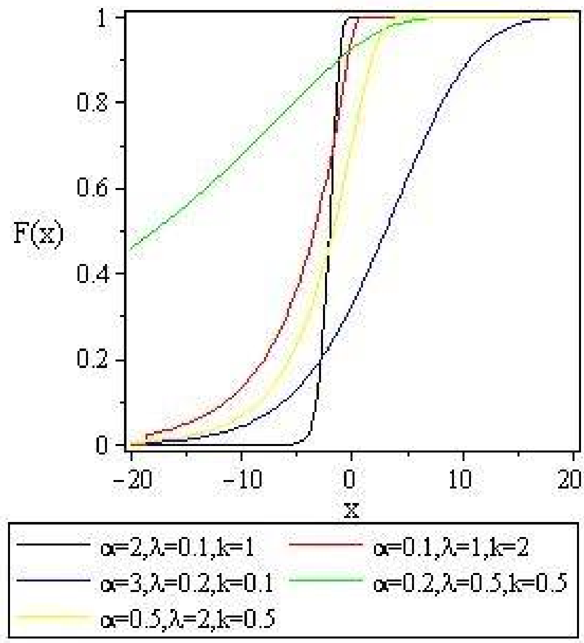

Fig. 2. The cdf of the NGEV distribution at different values of its parameters.

3. PROPERTIES OF THE NGEV DISTRIBUTION

In this section, we discuss some properties of the NGEV distribution.

3.1 Quantile, Median and Mode

The qth quantile, can be computed to the NGEV distribution as

$$x_q = \ln\left(\lambda\left(-\ln\left(1 - q^{\frac{1}{\alpha}}\right)\right)^{\frac{1}{k}}\right), \ 0 < q < 1. \quad (4)$$

The median of the NGEV distribution, i.e. $x_{0.5}$, is given by

$$median(X) = x_{0.5} = \ln\left(\lambda\left(-\ln\left(1 - (0.5)^{\frac{1}{\alpha}}\right)\right)^{\frac{1}{k}}\right). \quad (5)$$

Moreover, the mode of the NGEV distribution can be obtained by deriving its pdf with respect to x and equal it to zero. Thus the mode of the NGEV distribution can be obtained as a solution of the following nonlinear equation

$$(\alpha - 1)\left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k}\left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{-1} - \left(\frac{e^x}{\lambda}\right)^k + 1 = 0. \quad (6)$$

One can't get an explicit solution of (6) in the general case. Numerical methods should be used to solve it.

3.2 Moment generating function of NGEVD

In this subsection, the moment generating function and the rth moment about the origin of the NGEV distribution are computed.

THEOREM 2. *The moment generating function of this NGEV distribution is as follows*

$$M_X(t) = \alpha \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \lambda^t \frac{1}{(i+1)^{\frac{t}{k}+1}} \Gamma\left(\frac{t}{k}+1\right)$$

PROOF.

$$\begin{aligned} M_X(t) &= E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{tx} \alpha k \left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha-1} dx \\ &= \alpha k \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \int_{-\infty}^{\infty} e^{tx} \left(\frac{e^x}{\lambda}\right)^k \\ &\quad \times e^{-(i+1)\left(\frac{e^x}{\lambda}\right)^k} dx \end{aligned}$$

let $(i+1)\left(\frac{e^x}{\lambda}\right)^k = u$, $x = \ln\left(\frac{\lambda}{(i+1)^{1/k}}\right) + \frac{1}{k}\ln(u)$, $dx = \frac{1}{ku} du$, $u = 0 \to \infty$

$$\begin{aligned} M_X(t) &= \alpha \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \lambda^t \frac{1}{(i+1)^{\frac{t}{k}+1}} \int_0^{\infty} e^{-u} u^{\frac{t}{k}} du \\ &= \alpha \sum_{i=0}^{\infty} \binom{\alpha-1}{i} (-1)^i \lambda^t \frac{1}{(i+1)^{\frac{t}{k}+1}} \Gamma\left(\frac{t}{k}+1\right). \end{aligned}$$

■

The rth moment about the origin of the NGEV distribution is given by $E(X^r) = \frac{d^r}{dt^r} M_X(t) \mid_{t=0}$ and by using maple software, we find $M_X(0) = 1$.

4. RÉNYI ENTROPY OF NGEVD

In this section, we compute the rényi entropy which is a measure of the uncertain variation

THEOREM 3. *The Rényi entropies for the NGEV distribution are given by*

$$I_R(\delta) = \frac{1}{1-\delta} \log[I(\delta)],\ \delta \neq 1 \tag{7}$$

where

$$I(\delta) = \alpha^\delta k^{\delta-1} \sum_{i=0}^{\infty} \binom{\alpha\delta-\delta}{i} (-1)^i \left(\frac{1}{\delta+i}\right)^\delta \Gamma(\delta) \tag{8}$$

PROOF.

$$I_R(\delta) = \frac{1}{1-\delta} \log(I(\delta))$$

$$\begin{aligned} I(\delta) &= \int_{\Re} f^\delta(x)\, dx = \int_{-\infty}^{\infty} \alpha^\delta k^\delta \left(\frac{e^x}{\lambda}\right)^{\delta k} e^{-\delta\left(\frac{e^x}{\lambda}\right)^k} \\ &\quad \times \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha\delta-\delta} dx \\ &= \alpha^\delta k^\delta \sum_{i=0}^{\infty} \binom{\alpha\delta-\delta}{i} (-1)^i \int_{-\infty}^{\infty} \left(\frac{e^x}{\lambda}\right)^{\delta k} e^{-(\delta+i)\left(\frac{e^x}{\lambda}\right)^k} dx \end{aligned}$$

let $(\delta+i)\left(\frac{e^x}{\lambda}\right)^k = u$, $x = \ln\left(\frac{\lambda}{(\delta+i)^{1/k}}\right) + \frac{1}{k}\ln(u)$, $dx = \frac{1}{ku} du$, $u = 0 \to \infty$

$$\begin{aligned} I(\delta) &= \alpha^\delta k^{\delta-1} \sum_{i=0}^{\infty} \binom{\alpha\delta-\delta}{i} (-1)^i \left(\frac{1}{\delta+i}\right)^\delta \int_0^{\infty} e^{-u} u^{\delta-1} du \\ &= \alpha^\delta k^{\delta-1} \sum_{i=0}^{\infty} \binom{\alpha\delta-\delta}{i} (-1)^i \left(\frac{1}{\delta+i}\right)^\delta \Gamma(\delta) \end{aligned}$$

■

5. MOMENTS OF ORDER STATISTICS OF NGEVD

Suppose that n random variables $X_1, X_{2,} \ldots, X_n$ are ordered in non-decreasing magnitude and written as $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$. Where the smallest order statistic is denoted by $X_{1:n}$, the second smallest is denoted by $X_{2:n}$, and so on, and the largest order statistic is denoted by $X_{n:n}$ and $X_{r:n}$ is called r^{th} order statistic.
In the definition of order statistics, there is no restriction on whether $X_r^{,}$s are independent or identically distributed. But many well-known results about order statistics are under the classical assumption that $X_r^{,}$s are independent and identically distributed (iid). The pdf of the r^{th} order statistic is

$$\begin{aligned} f_{r:n}(x) &= \frac{n!}{(n-r)!\,(r-1)!} f(x) [F(x)]^{r-1} [1-F(x)]^{n-r} \\ &,\ -\infty < x < \infty. \end{aligned}$$

where $f(x)$ comes from (3) and $F(x)$ comes from (2).
The kth moment about zero of the rth order statistic is given by the following theorem

THEOREM 4. *The kth moment about zero of the rth order statistic for the NGEV distribution is given by*

$$\begin{aligned} \mu_{r:n}^{(k)} &= \frac{n!}{(r-1)!\,(n-r)!} \alpha \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \binom{n-r}{i} \binom{\alpha(r+i)-1}{j} \\ &\quad \times (-1)^{i+j} \frac{1}{j+1} \end{aligned}$$

PROOF.

$$\begin{aligned} \mu_{r:n}^{(k)} &= \frac{n!}{(r-1)!\,(n-r)!} \int_{-\infty}^{\infty} f(x) [F(x)]^{r-1} (1-F(x))^{n-r} dx \\ &= \frac{n!}{(r-1)!\,(n-r)!} \alpha k \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \int_{-\infty}^{\infty} \left(\frac{e^x}{\lambda}\right)^k \\ &\quad \times e^{-\left(\frac{e^x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha(r+i)-1} dx \\ &= \frac{n!}{(r-1)!\,(n-r)!} \alpha k \sum_{i=0}^{n-r} \sum_{j=0}^{\infty} \binom{n-r}{i} \binom{\alpha(r+i)-1}{j} \\ &\quad \times (-1)^{i+j} \int_{-\infty}^{\infty} \left(\frac{e^x}{\lambda}\right)^k e^{-(j+1)\left(\frac{e^x}{\lambda}\right)^k} dx \end{aligned}$$

let $(j+1)\left(\frac{e^x}{\lambda}\right)^k = u$, $x = \ln\left(\frac{\lambda}{(i+1)^{1/k}}\right) + \frac{1}{k}\ln(u)$, $dx = \frac{1}{ku}du$, $u = 0 \to \infty$

$$\mu_{r:n}^{(k)} = \frac{n!}{(r-1)!\,(n-r)!}\alpha\sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\binom{n-r}{i}\binom{\alpha(r+i)-1}{j} \times\frac{(-1)^{i+j}}{j+1}\int_0^{\infty}e^{-u}du$$
$$= \frac{n!}{(r-1)!\,(n-r)!}\alpha\sum_{i=0}^{n-r}\sum_{j=0}^{\infty}\binom{n-r}{i}\binom{\alpha(r+i)-1}{j} \times(-1)^{i+j}\frac{1}{j+1}$$

■

6. ESTIMATION AND INFORMATION MATRIX OF NGEVD

In this section we discussed the estimation of the NGEV parameters by using the method of maximum likelihood based on a complete sample.

6.1 Maximum Likelihood Estimators

Let $x_1, x_2, \ldots, x_n$ be a random sample from NGEV distribution. Then the log-likelihood function of the NGEV may be expressed as

$$L = \log l(x;\alpha,k,\lambda) = n\log\alpha + n\log k + k\sum_{i=1}^{n}\log\left(\frac{e^{x_i}}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right) \quad (9)$$

Differentiating the log-likelihood with respect to α, λ and k, respectively, and setting the result equal to zero, we have

$$\frac{\partial L}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right) \quad (10)$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} + \sum_{i=1}^{n}\log\left(\frac{e^{x_i}}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k\log\left(\frac{e^{x_i}}{\lambda}\right) + (\alpha-1)\sum_{i=1}^{n}\frac{e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\left(\frac{e^{x_i}}{\lambda}\right)^k\log\left(\frac{e^{x_i}}{\lambda}\right)}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}} \quad (11)$$

$$\frac{\partial L}{\partial\lambda} = \frac{-nk}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k - \frac{(\alpha-1)k}{\lambda} \times\sum_{i=1}^{n}\frac{e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\left(\frac{e^{x_i}}{\lambda}\right)^k}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}} \quad (12)$$

The maximum likelihood estimates $\widehat{\alpha}$, $\widehat{\lambda}$ and $\widehat{k}$ of the unknown parameters α, λ and k respectively, are obtained by setting these above equations (10)- (12) to zero and solving them simultaneously.

6.2 Asymptotic Confidence Bounds

In this subsection, we derive the asymptotic confidence intervals of the unknown parameters α, λ and k when $\alpha, \lambda > 0$ and $k > 0$. The simplest large sample approach is to assume that the MLEs(α, λ, k) are approximately multivariate normal with mean (α, λ, k) and covariance matrix I_0^{-1} where I_0^{-1} the inverse of the observed information matrix which is defined by

$$I_0^{-1} = -\begin{bmatrix} \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\lambda} & \frac{\partial^2 L}{\partial\alpha\partial k} \\ \frac{\partial^2 L}{\partial\lambda\partial\alpha} & \frac{\partial^2 L}{\partial\lambda^2} & \frac{\partial^2 L}{\partial\lambda\partial k} \\ \frac{\partial^2 L}{\partial k\partial\alpha} & \frac{\partial^2 L}{\partial k\partial\lambda} & \frac{\partial^2 L}{\partial k^2} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} var(\widehat{\alpha}) & cov(\widehat{\alpha},\widehat{\lambda}) & cov(\widehat{\alpha},\widehat{k}) \\ cov(\widehat{\lambda},\widehat{\alpha}) & var(\widehat{\lambda}) & cov(\widehat{\lambda},\widehat{k}) \\ cov(\widehat{k},\widehat{\alpha}) & cov(\widehat{k},\widehat{\lambda}) & var(\widehat{k}) \end{bmatrix} \quad (13)$$

The second partial derivatives included in I_0^{-1} are given as follows

$\frac{\partial^2 L}{\partial\alpha^2} = \frac{-n}{\alpha^2}$

$\frac{\partial^2 L}{\partial k^2} = \frac{-n}{k^2} - \sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k\left(\log\left(\frac{e^{x_i}}{\lambda}\right)\right)^2 + (\alpha-1)\sum_{i=1}^{n}\frac{\left(\log\left(\frac{e^{x_i}}{\lambda}\right)\right)W_i\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)\left(1-\left(\frac{e^{x_i}}{\lambda}\right)^k\right)-W_i^2}{\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^2}$

$\frac{\partial^2 L}{\partial\lambda^2} = \frac{nk}{\lambda^2} - \frac{k(1-k)}{\lambda^2}\sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k + \frac{(\alpha-1)k}{\lambda^2}\sum_{i=1}^{n}\frac{R_i}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}} - \frac{(\alpha-1)k}{\lambda}\sum_{i=1}^{n}\frac{\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)\frac{k}{\lambda}R_i\left(\left(\frac{e^{x_i}}{\lambda}\right)^k-1\right)+\frac{k}{\lambda}R_i^2}{\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^2}$

$\frac{\partial^2 L}{\partial\alpha\partial k} = \sum_{i=1}^{n}\frac{W_i}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}}$

$\frac{\partial^2 L}{\partial\alpha\partial\lambda} = \sum_{i=1}^{n}\frac{\frac{-k}{\lambda}R_i}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}}$

$\frac{\partial^2 L}{\partial k\partial\lambda} = \frac{-n}{\lambda} + \frac{1}{\lambda}\sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k + \frac{k}{\lambda}\sum_{i=1}^{n}\left(\frac{e^{x_i}}{\lambda}\right)^k\log\left(\frac{e^{x_i}}{\lambda}\right) - \frac{(\alpha-1)k}{\lambda}\sum_{i=1}^{n}\frac{\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)W_i\left[-\left(\frac{e^{x_i}}{\lambda}\right)^k+1\right]-R_i^2\log\left(\frac{e^{x_i}}{\lambda}\right)}{\left(1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^2} - \frac{(\alpha-1)}{\lambda}\sum_{i=1}^{n}\frac{R_i}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}}$

where the functions W_i and R_i are given by

$W_i = \log\left(\frac{e^{x_i}}{\lambda}\right)e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\left(\frac{e^{x_i}}{\lambda}\right)^k$,

$R_i = e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\left(\frac{e^{x_i}}{\lambda}\right)^k$.

The above approach is used to derive the $(1-\delta)\,100\%$ confidence intervals for the parameters α, λ and k as in the following forms

$$\widehat{\alpha}\pm Z_{\frac{\delta}{2}}\sqrt{var(\widehat{\alpha})},\ \widehat{\lambda}\pm Z_{\frac{\delta}{2}}\sqrt{var\left(\widehat{\lambda}\right)},\ \widehat{k}\pm Z_{\frac{\delta}{2}}\sqrt{var\left(\widehat{k}\right)},$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})th$ percentile of the standard normal distribution.

7. APPLICATION OF THE NGEV DISTRIBUTION

Now we use a real data set to show that the NGEV distribution is competitive among other known distributions such as Gumbel type-2 (G type-2) distribution, Exponentiated Fréchet (EF) distribution, Fréchet distribution and lognormal (LN) distribution. The data set in Table 1 shows the survival times in months of 20 acute myeloid leukemia patients reported in Afify et al. [19].

Table1: Survival times in months of 20 acute myeloid leukemia patients

2.226	2.113	3.631	2.473	2.720	2.746	1.972
2.050	2.061	3.915	0.871	1.548	2.808	1.079
1.200	0.726	2.967	1.958	2.265	2.353	

The MLEs of the unknown parameters α, λ and k are given in Table 2. Also, the values of the log-likelihood functions L, the statistics AIC (Akaike Information Criterion), $CAIC$ (Consistent Akaike-Information Criteria) and BIC (Bayesian Information Criterion) are given in Table 3 for the five distributions in order to verify which distribution fits better this real data set.

Table 2: The MLEs and the values 0f L

Model	MLEs	L
G type-2	$\widehat{\theta} = 2.6040, \widehat{\phi} = 2.0651$	-29.08667
EF	$\widehat{\alpha} = 0.3928, \widehat{\phi} = 3.4393$	-31.89448
Fréchet	$\widehat{\phi} = 1.7378$	-35.45610
LN	$\widehat{\mu} = 0.6971, \widehat{\sigma} = 0.4360$	-25.71549
NGEV	$\widehat{\alpha} = 26.389, \widehat{\lambda} = 0.292$ $\widehat{k} = 0.382$	-24.347

Table 3: The values 0f AIC, BIC and $CAIC$

Model	AIC	BIC	$CAIC$
G type-2	62.1733	64.1648	66.1648
EF	67.7889	69.7804	71.7804
Fréchet	72.9122	76.9037	74.9079
LN	55.431	57.4225	59.4225
NGEV	54.694	52.5971	56.194

Since the values of $-L$, AIC, BIC and $CAIC$ (see Table 2 and Table 3) are smaller for the NGEV distribution compared with those values of the other models, then the new distribution seems to be a very competitive model based on this real data set.
Substituting the MLEs of the unknown parameters into (13), we get an estimation of the variance covariance matrix as the following:

$$I_0^{-1} = \begin{bmatrix} 1.559 \times 10^3 & -27.535 & -5.764 \\ -27.535 & 0.513 & 0.111 \\ -5.764 & 0.111 & 0.025 \end{bmatrix}$$

The approximate 95% two sided confidence intervals of the unknown parameters α, λ and k are, respectively,given as [-50.99,103.78], [-1.112,1.696], [0.0721,0.6919].
The profiles of the log-likelihood functions of α, λ and k of NGEV for survival times in months of 20 acute myeloid leukemia patients data are ploted in Fig. 3 and Fig. 4. From the plots of the profiles of the log-likelihood function of α, λ and k , we observe that the likelihood equations have a unique solution.

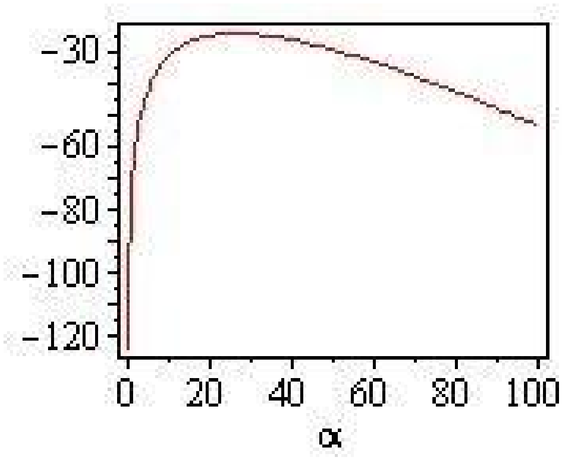

Fig. 3. The profile of the log-likelihood function of α.

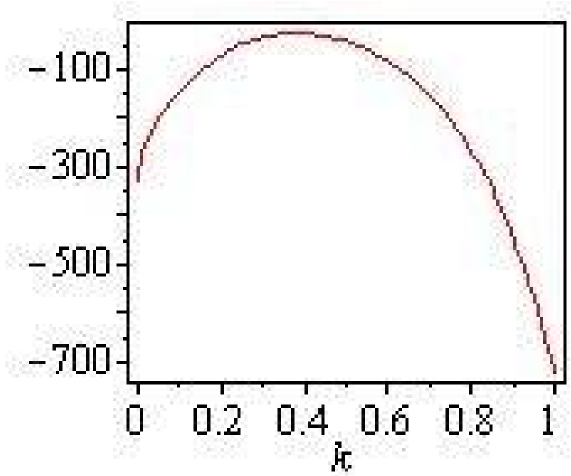

Fig. 4. The profile of the log-likelihood function of k.

8. BIVARIATE NEW GENERALIZED EXTREME VALUE DISTRIBUTION

In this section, we discuss the BNGEV distribution.We start with the joint survival function and derive the corresponding joint probability density function of this distribution

8.1 The Joint Cumulative Distribution Function

Suppose that $U_1 \sim NGEV(\alpha_1, \lambda, k)$, $U_2 \sim NGEV(\alpha_2, \lambda, k)$ and $U_3 \sim NGEV(\alpha_3, \lambda, k)$ are independent random variables. Define $X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$. Then, the bivariate vector $(X_1, X_2) \sim BNGEV(\alpha_1, \alpha_2, \alpha_3, \ \lambda, k)$
In the following lemma, We study the joint cumulative distribution function of the random varibles X_1 and X_2.

LEMMA 5. *The joint cdf of X_1 and X_2 is given by*

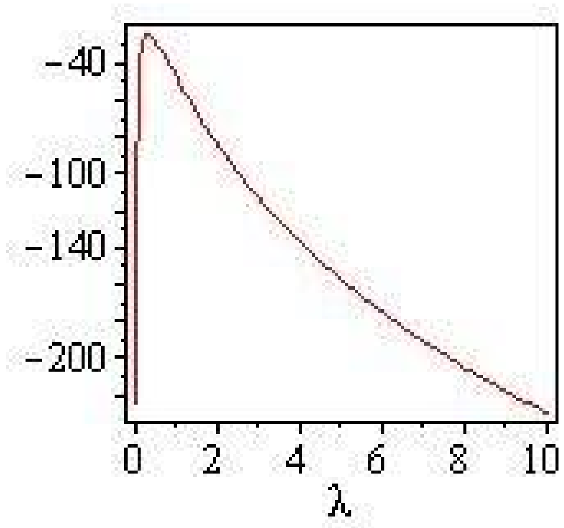

Fig. 5. The profile of the log-likelihood function of λ.

$$\begin{aligned} F_{BNGEV}(x_1,x_2) &= \left(1-e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1}\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_2} \\ &\times\left(1-e^{-\left(\frac{e^{z}}{\lambda}\right)^k}\right)^{\alpha_3}, \end{aligned} \quad (14)$$

where $z=\min(x_1,x_2)$.

PROOF.

$$\begin{aligned} F_{BNGEV}(x_1,x_2) &= P(X_1\le x_1, X_2\le x_2) \\ &= P(\max\{U_1,U_3\}\le x_1, \\ &\quad \max\{U_2,U_3\}\le x_2) \\ &= P(U_1\le x_1, U_2\le x_2, \\ &\quad U_3\le \min(x_1,x_2)). \end{aligned}$$

Where, $U_i\ (i=1,2,3)$ are mutually independent random variables. Then, we obtain

$$\begin{aligned} F_{BNGEV}(x_1,x_2) &= P(U_1\le x_1)P(U_2\le x_2) \\ &\times P(U_3\le \min(x_1,x_2)) \end{aligned}$$

$$\begin{aligned} &= F_{NGEV}(x_1;\alpha_1,\lambda,k)F_{NGEV}(x_2;\alpha_2,\lambda,k) \\ &\times F_{NGEV}(z;\alpha_3,\lambda,k) \\ &= \left(1-e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1}\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_2} \\ &\times\left(1-e^{-\left(\frac{e^{z}}{\lambda}\right)^k}\right)^{\alpha_3}. \end{aligned}$$

■

8.2 The Joint Probability Density Function

In this subsection, we study the joint probability density function of the random varibles X_1 and X_2 in the following theorem.

THEOREM 6. *If the joint cdf of* (X_1,X_2) *is as in (14) then, the joint pdf of* (X_1,X_2) *is given by*

$$f_{BNGEV}(x_1,x_2)=\begin{cases} f_1(x_1,x_2) & if\ x_1<x_2 \\ f_2(x_1,x_2) & if\ x_2<x_1 \\ f_3(x) & if\ x_1=x_2=x \end{cases}$$

where

$$\begin{aligned} f_1(x_1,x_2) &= f_{NGEV}(x_1;\alpha_1+\alpha_3,\lambda,k)f_{NGEV}(x_2;\alpha_2,\lambda,k) \\ &= (\alpha_1+\alpha_3)k^2\left(\frac{e^{x_1}}{\lambda}\right)^k e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\alpha_2\left(\frac{e^{x_2}}{\lambda}\right)^k \\ &\times e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\left(1-e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_3-1} \\ &\times\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_2-1} \end{aligned} \quad (15)$$

$$\begin{aligned} f_2(x_1,x_2) &= f_{NGEV}(x_1;\alpha_1,\lambda,k)f_{NGEV}(x_2;\alpha_2+\alpha_3,\lambda,k) \\ &= \alpha_1k^2\left(\frac{e^{x_1}}{\lambda}\right)^k e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\left(1-e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1-1} \\ &\times(\alpha_2+\alpha_3)\left(\frac{e^{x_2}}{\lambda}\right)^k e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k} \\ &\times\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_2+\alpha_3-1} \end{aligned} \quad (16)$$

$$\begin{aligned} f_3(x) &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}f_{NGEV}(x;\alpha_1+\alpha_2+\alpha_3,\lambda,k) \\ &= \alpha_3k\left(\frac{e^{x}}{\lambda}\right)^k e^{-\left(\frac{e^{x}}{\lambda}\right)^k}\left(1-e^{-\left(\frac{e^{x}}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \end{aligned} \quad (17)$$

PROOF. Let us first assume that $x_1<x_2$. Then, $F_{BNGEV}(x_1,x_2)$ in (14) becomes

$$F_{BNGEV}(x_1,x_2)=\left(1-e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_3}\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_2}.$$

Then, upon differentiating this function w.r.t. x_1 and x_2 we obtain the expression of $f_1(x_1,x_2)$ gives in (15). By the same way we obtain $f_2(x_1,x_2)$ when $x_2<x_1$. But $f_3(x)$ cannot be derived in a similar way. For this reason, we use the following identity to derive $f_3(x)$
$\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f_1(x_1,x_2)\,dx_1dx_2+\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}f_2(x_1,x_2)\,dx_2dx_1+\int_{-\infty}^{\infty}f_3(x,x)\,dx=1$
let
$I_1=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}f_1(x_1,x_2)\,dx_1dx_2$ and $I_2=\int_{-\infty}^{\infty}\int_{-\infty}^{x_1}f_2(x_1,x_2)\,dx_2dx_1$
then
$I_1=\int_{-\infty}^{\infty}\int_{-\infty}^{x_2}(\alpha_1+\alpha_3)k^2\left(\frac{e^{x_1}}{\lambda}\right)^k\left(1-e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_3-1}$
$\times\alpha_2e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\left(\frac{e^{x_2}}{\lambda}\right)^k e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_2-1}dx_1dx_2$

$$=\int_{-\infty}^{\infty}\alpha_2k\left(\frac{e^{x_2}}{\lambda}\right)^k e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\left(1-e^{-\left(\frac{e^{x_2}}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1}dx_2 \quad (18)$$

Similarly

$$I_2 = \int_{-\infty}^{\infty} \alpha_1 k \left(\frac{e^{x_1}}{\lambda}\right)^k e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k} \times \left(1 - e^{-\left(\frac{e^{x_1}}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx_1 \qquad (19)$$

From (18) and (19), we get

$\int_0^{\infty} f_3(x)\,dx = 1 - I_1 - I_2$

$= \int_{-\infty}^{\infty} (\alpha_1+\alpha_2+\alpha_3)\, k \left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k} \times \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx -$

$\int_{-\infty}^{\infty} \alpha_2 k \left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx -$

$\int_{-\infty}^{\infty} \alpha_1 k \left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx$

Then

$f_3(x,x) = \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{NGEV}(x;\alpha_1+\alpha_2+\alpha_3,\lambda,k)$

$= \alpha_3 k \left(\frac{e^x}{\lambda}\right)^k e^{-\left(\frac{e^x}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^x}{\lambda}\right)^k}\right)^{\alpha_1+\alpha_2+\alpha_3-1}$

This completes the proof of the theorem. ■

8.3 Marginal Probability Density Functions

The following theorem gives the marginal probability density functions of X_1 and X_2.

THEOREM 7. *The marginal probability density functions of X_i $(i = 1, 2)$ is given by*

$$\begin{aligned} f_{X_i}(x_i) &= (\alpha_i+\alpha_3)\, k \left(\frac{e^{x_i}}{\lambda}\right)^k e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k} \\ &\quad \times \left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^{\alpha_i+\alpha_3-1} \\ &= f_{NGEV}(x_i;\alpha_i+\alpha_3,\lambda,k), \quad i = 1,2. \end{aligned} \qquad (20)$$

PROOF. The marginal cumulative distribution function of X_i, say $F(x_i)$, as follows:

$$\begin{aligned} F(x_i) &= P(X_i \le x_i) \\ &= P(\max\{U_i, U_3\} \le x_i) \\ &= P(U_i \le x_i, U_3 \le x_i) \end{aligned}$$

since, the random variables U_i $(i = 1, 2)$ and U_3 are mutually independent, then

$$\begin{aligned} F(x_i) &= P(U_i \le x_i)\, P(U_3 \le x_i) \\ &= F_{NGEV}(x_i;\alpha_i+\alpha_3,\lambda,k) \\ &= \left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^{\alpha_i+\alpha_3} \end{aligned} \qquad (21)$$

Differenting w.r.t. x_i we obtain the formula given in (20). ■

8.4 Conditional Probability Density Functions

Given the marginal probability density functions of X_1 and X_2 we can now derive the conditional probability density functions as presented in the following theorem

THEOREM 8. *The conditional probability density functions of X_i, given $X_j = x_j$, $f(x_i|x_j)$, $i, j = 1, 2$; $i \ne j$, is given by*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & if\ x_j < x_i, \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & if\ x_i < x_j, \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & if\ x_i = x_j = x, \end{cases}$$

where

$f^{(1)}_{X_i|X_j}(x_i|x_j) = \left(\alpha_j(\alpha_i+\alpha_3)\, k \left(\frac{e^{x_i}}{\lambda}\right)^k e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^{\alpha_i+\alpha_3-1}\right) \div (\alpha_j+\alpha_3) \left(1 - e^{-\left(\frac{e^{x_j}}{\lambda}\right)^k}\right)^{\alpha_3}$

$f^{(2)}_{X_i|X_j}(x_i|x_j) = \alpha_i k \left(\frac{e^{x_i}}{\lambda}\right)^k e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k} \left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^{\alpha_i-1}$

$f^{(3)}_{X_i|X_j}(x_i|x_j) = \frac{\alpha_3}{\alpha_i+\alpha_3} \left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)^{\alpha_i}.$

PROOF. The proof follows immediately by substituting the joint probability density function of (X_1, X_2) given in (15), (16) and (17) and the marginal probability density function of X_i $(i = 1, 2)$ given in (20), using the relation

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_j}(x_j)}, i = 1, 2. ■$$

9. MAXIMUM LIKELIHOOD ESTIMATORS OF BNGEVD

Kundu and Gupta [1] used the method of maximum likelihood to estimate the unknown parameters of the bivariate generalized exponential distribution. In the same way we use the method of maximum likelihood to estimate the unknown parameters of the BNGVD distribution.

Suppose $((x_{11}, x_{21}), ..., (x_{1n}, x_{2n}))$ is a random sample from BNGVD distribution.Consider the following notation

$I_1 = \{i;\, x_{1i} < x_{2i}\}$, $I_2 = \{i;\, x_{1i} > x_{2i}\}$, $I_3 = \{i;\, x_{1i} = x_{2i} = x_i\}$, $I = I_1 \cup I_2 \cup I_3$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_3| = n_3$, and $n_1 + n_2 + n_3 = n$.

The likelihood function of the sample of size n is given by:

$$l(\alpha_1,\alpha_2,\alpha_3,\lambda,k) =^{n_1}_{i=1} f_1(x_{1i},x_{2i})^{n_2}_{i=1} f_2(x_{1i},x_{2i})^{n_3}_{i=1} f_3(x_i)$$

The log-likelihood function can be expressed as

$L(\alpha_1,\alpha_2,\alpha_3,\lambda,k) = \ln l(\alpha_1,\alpha_2,\alpha_3,\lambda,k)$

$= n_1 \ln(\alpha_1+\alpha_3) + n_1 \ln(k) + k \sum_{i=1}^{n_1} \ln\left(\frac{e^{x_{1i}}}{\lambda}\right) - \sum_{i=1}^{n_1} \left(\frac{e^{x_{1i}}}{\lambda}\right)^k + (\alpha_1+\alpha_3-1) \sum_{i=1}^{n_1} \ln\left(1 - e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\right) +$
$n_1 \ln(\alpha_2) + n_1 \ln(k) + k \sum_{i=1}^{n_1} \ln\left(\frac{e^{x_{2i}}}{\lambda}\right) - \sum_{i=1}^{n_1} \left(\frac{e^{x_{2i}}}{\lambda}\right)^k +$
$(\alpha_2-1) \sum_{i=1}^{n_1} \ln\left(1 - e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\right) + n_2 \ln(\alpha_1) +$
$n_2 \ln(k) + k \sum_{i=1}^{n_2} \ln\left(\frac{e^{x_{1i}}}{\lambda}\right) - \sum_{i=1}^{n_2} \left(\frac{e^{x_{1i}}}{\lambda}\right)^k +$
$(\alpha_1-1) \sum_{i=1}^{n_2} \ln\left(1 - e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\right) + n_2 \ln(\alpha_2+\alpha_3) +$
$n_2 \ln(k) + k \sum_{i=1}^{n_2} \ln\left(\frac{e^{x_{2i}}}{\lambda}\right) - \sum_{i=1}^{n_2} \left(\frac{e^{x_{2i}}}{\lambda}\right)^k +$
$(\alpha_2+\alpha_3-1) \sum_{i=1}^{n_2} \ln\left(1 - e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\right) + n_3 \ln(\alpha_3)$
$+ n_3 \ln(k) + k \sum_{i=1}^{n_3} \ln\left(\frac{e^{x_i}}{\lambda}\right) - \sum_{i=1}^{n_3} \left(\frac{e^{x_i}}{\lambda}\right)^k +$
$(\alpha_1+\alpha_2+\alpha_3-1) \sum_{i=1}^{n_3} \ln\left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right)$

Differentiating the log-likelihood with respect to $\alpha_1, \alpha_2, \alpha_3, \lambda$ and k respectively, and setting the results equal to zero, we have

$$\frac{\partial L}{\partial \alpha_1} = \frac{n_1}{\alpha_1+\alpha_3} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\right) + \frac{n_2}{\alpha_1} + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\right) + \sum_{i=1}^{n_3} \ln\left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right) \quad (22)$$

$$\frac{\partial L}{\partial \alpha_2} = \frac{n_1}{\alpha_2} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\right) + \frac{n_2}{\alpha_2+\alpha_3} + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\right) + \sum_{i=1}^{n_3} \ln\left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right) \quad (23)$$

$$\frac{\partial L}{\partial \alpha_3} = \frac{n_1}{\alpha_1+\alpha_3} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\right) + \frac{n_2}{\alpha_2+\alpha_3} + \frac{n_3}{\alpha_3} + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\right) + \sum_{i=1}^{n_3} \ln\left(1 - e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\right) \quad (24)$$

$$\frac{\partial L}{\partial k} = \frac{n_1}{k} + \sum_{i=1}^{n_1} \ln\left(\frac{e^{x_{1i}}}{\lambda}\right) + (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} \frac{e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{1i}}}{\lambda}\right)}{1-e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}} + \frac{n_1}{k} + \sum_{i=1}^{n_1} \ln\left(\frac{e^{x_{2i}}}{\lambda}\right) + (\alpha_2-1)\sum_{i=1}^{n_1} \frac{e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{2i}}}{\lambda}\right)}{1-e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}} + \frac{n_2}{k} + \sum_{i=1}^{n_2} \ln\left(\frac{e^{x_{1i}}}{\lambda}\right) + \frac{n_2}{k} + \sum_{i=1}^{n_2} \ln\left(\frac{e^{x_{2i}}}{\lambda}\right) + \frac{n_3}{k} + \sum_{i=1}^{n_3} \ln\left(\frac{e^{x_i}}{\lambda}\right) - \sum_{i=1}^{n_1}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{1i}}}{\lambda}\right) - \sum_{i=1}^{n_1}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{2i}}}{\lambda}\right) - \sum_{i=1}^{n_2}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{1i}}}{\lambda}\right) - \sum_{i=1}^{n_2}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{2i}}}{\lambda}\right) + (\alpha_1-1)\sum_{i=1}^{n_2} \frac{e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{1i}}}{\lambda}\right)}{1-e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}} + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \frac{e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k \ln\left(\frac{e^{x_{2i}}}{\lambda}\right)}{1-e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}} + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \frac{e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}\left(\frac{e^{x_i}}{\lambda}\right)^k \ln\left(\frac{e^{x_i}}{\lambda}\right)}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}} - \sum_{i=1}^{n_3}\left(\frac{e^{x_i}}{\lambda}\right)^k \ln\left(\frac{e^{x_i}}{\lambda}\right) \quad (25)$$

$$\frac{\partial L}{\partial \lambda} = \frac{-n_1 k}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n_1}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k - (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} \frac{\frac{k}{\lambda}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}}{1-e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}} - \frac{n_1 k}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n_1}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k - (\alpha_2-1)\sum_{i=1}^{n_1} \frac{\frac{k}{\lambda}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}}{1-e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}} - \frac{n_2 k}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n_2}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k + (\alpha_1-1)\sum_{i=1}^{n_2} \frac{\frac{k}{\lambda}\left(\frac{e^{x_{1i}}}{\lambda}\right)^k e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}}{1-e^{-\left(\frac{e^{x_{1i}}}{\lambda}\right)^k}} - \frac{n_2 k}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n_2}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k - (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \frac{\frac{k}{\lambda}\left(\frac{e^{x_{2i}}}{\lambda}\right)^k e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}}{1-e^{-\left(\frac{e^{x_{2i}}}{\lambda}\right)^k}} - \frac{n_3 k}{\lambda} + \frac{k}{\lambda}\sum_{i=1}^{n_3}\left(\frac{e^{x_i}}{\lambda}\right)^k - (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \frac{\frac{k}{\lambda}\left(\frac{e^{x_i}}{\lambda}\right)^k e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}}{1-e^{-\left(\frac{e^{x_i}}{\lambda}\right)^k}} \quad (26)$$

The maximum likelihood estimates $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$, $\widehat{\lambda}$ and $\widehat{k}$ of the unknown parameters α_1, α_2, α_3, λ and k respectively, are obtained by solving Equations (22) - (26).

10. DATA ANALYSIS OF BNGEVD

In this section, a real data set is used to compare the goodness of fiting of the Marshall-Olkin bivariate exponential (MO) distribution and Bivariate New Generalized Extreme value (BNGEV) distribution. The data set (see Table 4) was first analyzed in [1] and represents the soccer data where at least one goal is scored by the home team and at least one goal is scored directly from apenalty kick, foul kick or any other direct kick(all of them will be called kick goals) by any team that has been considered. It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. Clearly, the variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$, (ii) $X_1 = X_2$, (iii) $X_1 > X_2$.

Table 4 UEFA Champion′s League Data

X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2
26	20	66	62	51	28	42	3
63	18	25	9	76	64	27	47
19	19	41	3	64	15	28	28
66	85	16	75	26	48	2	2
40	40	18	18	16	16		
49	49	22	14	44	13		
8	8	42	42	25	14		
69	71	36	52	55	11		
39	39	34	34	49	49		
82	48	53	39	24	24		
72	72	54	7	44	30		

The required numerical evaluations are carried out using the Package of Mathcad software. Table 5 provides the MLEs of the model parameters.The model selection is carried out using the AIC (Akaike information criterion) and the BIC (Bayesian information criterion).

Table 5: The Maximum likelihood estimates (MLEs)

Model	MLEs
MO	$\widehat{\lambda}_1 = 0.012, \widehat{\lambda}_2 = 0.014$ $\widehat{\lambda}_3 = 0.022$
BNGEV	$\widehat{\alpha}_1 = 3.682, \widehat{\alpha}_2 = 1.398$ $\widehat{\alpha}_3 = 3.386, \widehat{\lambda} = 57.447$ $\widehat{k} = 0.022$

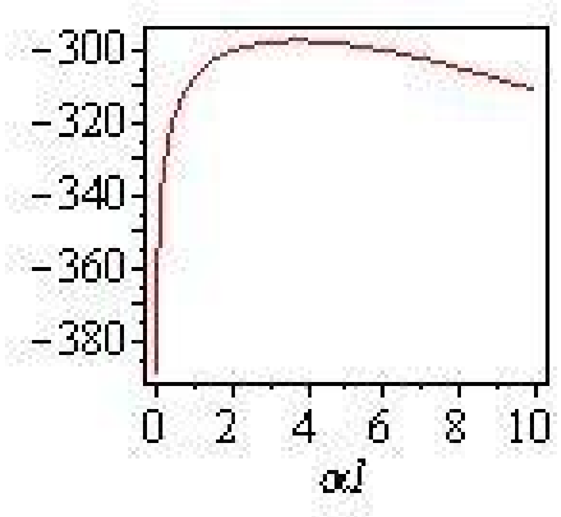

Fig. 6. The profile of the log-likelihood function of α_1

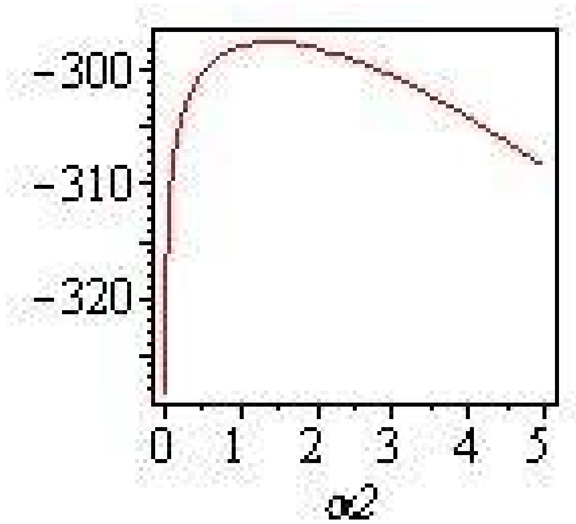

Fig. 7. The profile of the log-likelihood function of α_2

Table 6: The values 0f L, AIC and BIC.

Model	L	AIC	BIC
MO	-339.006	684.012	-344.423
BNGEV	-297.342	604.684	-306.369

Since the values of $-L$, AIC and BIC (see Table 6) are smaller for the BNGEV distribution compared with those values of the other models, then the new distribution seems to be a very competitive model to these data

The profiles of the log-likelihood function of α_1, α_2, α_3, λ and k of BNGEV for UEFA Champion′s League data are ploted in Fig. 5, Fig. 6 and Fig. 7. From the plots of the profiles of the log-likelihood function of α_1, α_2, α_3, λ and k ,we observe that the likelihood equations have a unique solution.

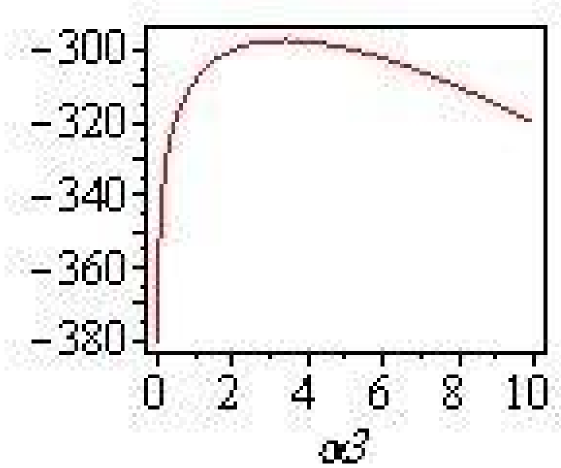

Fig. 8. The profile of the log-likelihood function of α_3

11. CONCLUSIONS

In this paper, we proposed a new generalized extreme value (NGEV) distribution. Some statistical properties of this distribution have been studied and discussed such as quantile, median, moment generating function and moments of order statistics. The maximum likelihood estimators of the parameters are derived. A real data set is analyzed using the new distribution, Gumbel type-2 (G type-2) distribution, Exponentiated Fréchet (EF) distribution, Fréchet distribution and lognormal (LN) distribution. Baised on the comparisons between all these models, we conclude that, the introduced model is highly competitive in the sense of fitting this real data set. Also, bivariate New Generalized Extreme Value (BNGEV) distribution.is introduced. Marginal and conditional distribution functions are studied. Furthermore, maximum likelihood estimates (MLEs) of the parameters are presented. A real data set is analyzed using the new distribution and Marshall-Olkin bivariate exponential (MO) distribution. Baised on the comparisons between all these models, we conclude that, the introduced model is highly competitive in the sense of fitting this real data set.

12. REFERENCES

[1] Kundu, D. and Gupta, R. D. (2009). Bivariate generalized exponential distribution. Journal of Multivariate Analysis, vol. 100, no. 4, 581-593.

[2] Marshall, A. W. Olkin, I. A. (1967). A multivariate exponential distribution. Journal of the American Statistical Association. 62, 30-44.

[3] Sarhan, A. and Balakrishnan, N. (2007). A new class of bivariate distributions and its mixture. Journal of the Multivariate Analysis. 98, 1508-1527.

[4] El-Bassiouny, A. H., El-Damcese, M., Mustafa, A., and Eliwa, M. (2015). Bivariate exponentaited generalized Weibull-Gompertz distribution. arXiv preprint arXiv:1501.02241.

[5] El-Gohary, A., El-Bassiouny, A. H. and El-Morshedy, M. (2016). Bivariate exponentiated modified Weibull extension

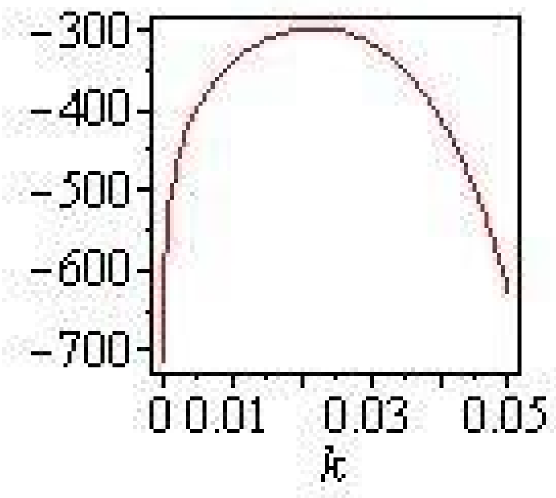

Fig. 9. The profile of the log-likelihood function of k

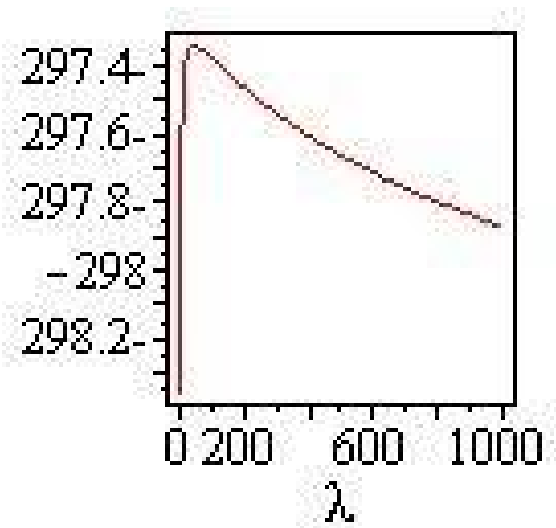

Fig. 10. The profile of the log-likelihood function of λ

distribution. Journal of Statistics Applications & Probability 5, No. 1, 1-12.

[6] Ammar M. Sarhan, David C. Hamilton, Bruce Smith and Debasis Kundu. (2011). The bivariate generalized linear failure rate distribution and its multivariate extension. Computational Statistics and Data Analysis 55, 644-654.

[7] A. Z. Afify, Z. M. Nofal, and N. S. Butt, "Transmuted complementary weibull geometric distribution," Pakistan Journal of Statistics and Operation Research, vol. 10, no. 4, pp. 435–454, 2014.

[8] Meintanis, S. G. (2007). Test of fit for Marshall–Olkin distributions with applications. Journal of Statistical Planning and Inference 137,3954–3963.

[9] Mudholkar GS and Srivastava DK (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Transactions on Reliability, 42, 299–302.