On D-Preopen Sets in D-Matric Spaces

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ABSTRACT

The purpose of this paper is to introduce and investigate weak form of D-open sets in D-metric spaces, namely D-preopen sets. The relationships among this form with the other known sets are introduced. We give the notions of the interior operator, the closure operator and frontier operator via D-preopen sets.

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Keywords

Open set, Metric spaces.

1. INTRODUCTION

Metric spaces is one of the most important spaces in mathematics there are various type of generalization of metric spaces, [5]. The axiomatic approach to the metric spaces is given by a french mathematician M. Frechet in year 1812, [7]. In 1984, Dhage, [3], introduced a new notion of a new structure of D-metric space which is a natural generalization of the notion of ordinary metric space to higher dimensional metric spaces, [4]. In 2000, Dhage, [2], introduced some results in D-metric spaces are obtained and the notion of open and closed balls. In 2013, [6], exhibited methods of generating D-metrics from certain types of real valued partial functions on the three dimensional Euclidean space. In 2017, Ali Fora, Massadeh and Bataineh, [1], introdused and a new topological structure of D-closed set.

This paper is organized as follows. Section 2 is devoted to some preliminaries. Section 3 introduces the concept of D-preopen sets by utilizing the D-open balls. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the concepts of the interior operator, the closure operator and frontier operator via D-preopen sets.

2. PRELIMINARIES

DEFINITION 2.1. [7]. Let X be any nonempty set. A function $d : X \times X \rightarrow [0, \infty)$ is called a metric function on X if it satisfies the following three conditions for all $x, y, z \in X$:

(1) (positive property) $d(x, y) \geq 0$ with equality if and only if $x = y$;
(2) (symmetric property) $d(x, y) = d(x, y)$;
(3) (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$.

A pair (X, d), where d is a metric on X is called a metric space. By $O_\varepsilon(x)$, we mean the open ball with center x and radius $\varepsilon > 0$, that is,

$$O_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

By $C_\varepsilon(x)$, we mean the closed ball with center x and radius $\varepsilon > 0$, that is,

$$C_\varepsilon(x) = \{y \in X : d(x, y) \leq \varepsilon\}.$$

For metric space (X, d) and $G \subseteq X$, the set G said to be open set if for any point $x \in G$, there exists $\varepsilon > 0$ such that $O_\varepsilon(x) \subseteq G$. The set G is called closed set in metric space (X, d) if $X - G$ is an open set in metric space (X, D). For the set of real numbers R, we mean by the usual metric space (R, d),

$$d(x, y) = |x - y| \text{ for all } x, y \in R$$

For metric space (X, d) and $G \subseteq X$, the interior operator of G is denoted by $Int(G)$ and the clouser operator of G is denoted by $Cl(G)$.

DEFINITION 2.2. [4]. A nonempty set X, together with a function $D : X \times X \times X \rightarrow [0, \infty)$ is called a D-metric space, denoted by (X, D) if D satisfies the following $x, y, z, u \in X$:

(1) $D(x, y, z) = 0 \rightarrow x = y = z$ (coincidence);
(2) $D(x, y, z) = D(p(x, y, z))$, where p is a permutation of x, y, z (symmetry);
(3) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, u \in X$ (tetrahedral inequality).

By $O^D_\varepsilon(x)$, we mean the D-open ball with center x and radius $\varepsilon > 0$, that is,

$$O^D_\varepsilon(x) = \{y \in X : d(x, y, y) < \varepsilon\}.$$

By $C^D_\varepsilon(x)$, we mean the D-closed ball with center x and radius $\varepsilon > 0$, that is,

$$C^D_\varepsilon(x) = \{y \in X : d(x, y, y) \leq \varepsilon\}.$$

The set $G \subseteq X$ is called D-open set in D-metric space (X, D) if for every $x \in G$, there is $\varepsilon > 0$ such that $O^D_\varepsilon(x) \subseteq G$. The set G is called D-closed set in D-metric space (X, D) if $X - G$ is D-open set in D-metric space (X, D). For D-metric space (X, D) and $G \subseteq X$, the interior set of G is denoted by $Int_D(G)$ and the clouser set of G is denoted by $Cl_D(G)$.

THEOREM 2.3. [2]. Let (R, D) be D-metric space where

$$D(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\}$$

and (R, d) is usual metric space. Then for a fixed $x \in R$, the D-open balls $O_\varepsilon^D(x)$ and $O_\varepsilon^D(x)$ are the sets in given by: $O_\varepsilon^D(x) = (x-\varepsilon, x+\varepsilon)$.

THEOREM 2.4. [2]. Let (R, D) be D-metric space, where

$$D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

and (R, d) is usual metric space. Then for a fixed $x \in R$, the D-open balls $O_\varepsilon^D(x)$ and $O_\varepsilon^D(x)$ are the sets in given by: $O_\varepsilon^D(x) = (x-\varepsilon/2, x+\varepsilon/2)$.

THEOREM 2.5. [2]. Every D-open O_ε^D(x) , $x \in X, \varepsilon > 0$ is a D-open set in X (i.e., it contains a ball of each of its points).

THEOREM 2.6. [1]. Every a finite set in a D-metric space (X, D) must be D-closed set.

THEOREM 2.7. [2]. Every ball $C_\varepsilon^D(x)$ in a D-metric space (X, D) is D-closed set.

THEOREM 2.8. [2]. Arbitrary union and finite intersection of D-open balls $O_\varepsilon^D(x), x \in X$ is D-open set.

THEOREM 2.9. [1]. Let $D : X \times X \times X \times X \to [0, \infty)$ be a D-metric on X having a finite range. Then every subset A of X is D-closed set.

3. D-PREOPEN SETS

DEFINITION 3.1. Let (X, D) be a D-metric space. A subset $G \subseteq X$ is called a D-preopen set in D-metric space (X, D) if for every $x \in G$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$.A subset $G \subseteq X$ is called a D-preclosed set in D-metric space (X, D) if $X - G$ is a D-preopen set in D-metric space (X, D).

The set of all D-preopen sets in X denoted by $D_pO(X, D)$ and the set of all D-preclosed sets in X denoted by $D_pC(X, D)$.

EXAMPLE 3.2. Let (R, D) be D-metric space given by

$$D(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\},$$

where (R, d) is usual metric space on the set of real number R. An open interval $G = (0, 2)$ is D-preopen set in (R, D). For every $x \in G$, take $\delta = min\{|x|, |2-x|\} > 0$. If $y \in O_\delta^D(x)$, then $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$.

EXAMPLE 3.3. In Example(3.2), a closed interval $G = [-1, 1]$ is not D-preopen set, since at x=1, take $y = (2+\delta)/2 \in O_\delta^D(1)$ and $\varepsilon = \delta/2 > 0$. Note that $O_{\delta/2}^D((2+\delta)/2) \cap G = \emptyset$. That is, $G = [-1, 1]$ is not D-preopen set in (R, D).

THEOREM 3.4. Every D-open set is a D-preopen set.

PROOF. Let G be any D-open set in D-metric space (X, D). Let $x \in G$ be arbitrary point. Then there is $\delta > 0$ such that $O_\delta^D(x) \subseteq G$. For every $y \in O_\delta^D(x)$, $y \in O_\varepsilon^D(y)$ and $y \in G$ for every $\varepsilon > 0$. That is, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence G is D-preopen set. □

The converse of above theorem need not be true.

EXAMPLE 3.5. In Example(3.2), the set of rational numbers Q is a D-preopen set but not D-open set in (R, D).

Note that the intersection of two D-preopen sets no need to be D-preopen set. In Example(3.2), the set of rational numbers Q is a D-preopen set but not D-open set in (R, D) and the set $IR \cup \{q\}$ is a D-preopen set in (R, D), where IR is the set of irrational numbers and q is any rational number, but $Q \cap (IR \cup \{q\}) = \{q\}$ is not D-preopen set.

The following theorem shows that the intersection of a D-open set and a D-preopen set is a D-preopen set.

THEOREM 3.6. The intersection of a D-open set and a D-preopen set is a D-preopen set.

PROOF. Let A be D-open set and B be D-preopen set in D-metric space in (X, D). Let $x \in A \cap B$ be arbitrary point. Then $x \in A$ and $x \in B$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $O_{\delta_1}^D(x) \subseteq A$ and for every $y \in O_{\delta_2}^D(x)$, $O_\varepsilon^D(y) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Take $\delta = min\{\delta_1, \delta_2\} > 0$. Then $O_\delta^D(x) \subseteq A$ and for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap B \neq \emptyset$ for every $\varepsilon > 0$. Now for every $y \in O_\delta^D(x)$ and since A is D-open set, then there is $\varepsilon_y > 0$ such that $O_{\varepsilon_y}^D(y) \subseteq A$ and $O_{min\{\varepsilon_y, \varepsilon\}}^D(y) \cap B \neq \emptyset$. Since $O_{min\{\varepsilon_y, \varepsilon\}}^D(y) \cap B \subseteq O_\varepsilon^D(y) \cap A \cap B$, then $O_\varepsilon^D(y) \cap (A \cap B) \neq \emptyset$ for every $\varepsilon > 0$. That is $A \cap B$ is D-preopen set. □

THEOREM 3.7. The union of any family of D-preopen sets is D-preopen set.

PROOF. Let G_λ be a D-preopen subset of D-metric space (X, D) for all $\lambda \in \Delta$. Let $x \in \bigcup_{\lambda \in \Delta} G_\lambda$ be an arbitrary point. Then there is at least $\lambda_0 \in \Delta$ such that $x \in G_{\lambda_0}$. Since G_{λ_0} is a D-preopen then for every $x \in G_{\lambda_0}$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G_{\lambda_0} \neq \emptyset$ for every $\varepsilon > 0$. Since $G_{\lambda_0} \subseteq \bigcup_{\lambda \in \Delta} G_\lambda$, then for every x $\in \bigcup_{\lambda \in \Delta} G_\lambda$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap \bigcup_{\lambda \in \Delta} G_\lambda \neq \emptyset$ for every $\varepsilon > 0$. That is $\bigcup_{\lambda \in \Delta} G_\lambda$ is D-preopen set. □

4. D-PREOPEN OPERATORS

In this section, we define the interior operator, the closure operator and frontier operator via D-preopen sets.

DEFINITION 4.1. Let (X, D) be a D-metric space and $G \subseteq X$. The D_P-closure operator of G is denoted by $Cl_P^D(G)$ and defined by

$$Cl_P^D(G) = \cap\{H \subseteq X : G \subseteq H \text{ and } H \text{ is D-preclosed set}\}.$$

The D_P-interior functor of G is denoted by $Int_P^D(G)$ and defined by

$$Int_P^D(G) = \cup\{H \subseteq X : H \subseteq G \text{ and } H \text{ is D-preopen set}\}.$$

REMARK 4.2.

(1) From Theorem(3.7), $Cl_P^D(G)$ is a D-preclosed set and $Int_P^D(G)$ is D-preopen set in D-metric space (X, D).
(2) For a D-metric space (X, D) and $G \subseteq X$, it is clear from the definition of $Cl_P^D(G)$ and $Int_P^D(G)$ that $G \subseteq Cl_P^D(G)$ and $Int_P^D(G) \subseteq G$.

THEOREM 4.3. For a D-metric space (X, D) and $G \subseteq X$, $Cl_P^D(G) = G$ if and only if G is a D-preclosed set.

PROOF. Let $Cl_P^D(G) = G$. Then from definition of $Cl_P^D(G)$ and Theorem(3.7), $Cl_P^D(G)$ is a D-preclosed set and G is a D-preclosed set. Conversely, we have $G \subseteq Cl_P^D(G)$ by Remark(4.2). Since G is a D-preclosed set, then it is clear from the definition of $Cl_P^D(G)$, $Cl_P^D(G) \subseteq G$. Hence $G = Cl_P^D(G)$. □

THEOREM 4.4. For a D-metric space (X, D) and $G \subseteq X$, and $Int_P^D(G) = G$ if and only if G is a D-preopen set.

PROOF. Let G be D-preopen set. Then for all $x \in G$, we have $x \in G \subseteq G$. That is, $G \subseteq Int_P^D(G)$. Then $G = Int_P^D(G)$ from Remark(4.2). The converse is trivial. □

THEOREM 4.5. For a D-metric space (X, D) and $G \subseteq X$, $x \in Cl_P^D(G)$ if and only if for all D-preopen set M containing x, $M \cap G \neq \emptyset$.

PROOF. Let $x \in Cl_P^D(G)$ and M be any D-preopen set containing x. If $M \cap G = \emptyset$ then $G \subseteq X - M$. Since $X - M$ is a D-preclosed set containing G, then $Cl_P^D(G) \subseteq X - M$ and so $x \in Cl_P^D(G) \subseteq X - M$. Hence this is contradiction, because $x \in M$. Therefore $M \cap G \neq \emptyset$.
Conversely, Let $x \notin Cl_P^D(G)$. Then $X - Cl_P^D(G)$ is a D-preopen set containing x. Hence by hypothesis, $[X - Cl_P^D(G)] \cap G \neq \emptyset$. But this is contradiction, because $X - Cl_P^D(G) \subseteq X - G$. □

THEOREM 4.6. For a D-metric space (X, D) and $G \subseteq X$, $x \in Int_P^D(G)$ if and only if there is D-preopen set M such that $x \in M \subseteq G$.

PROOF. Let $x \in Int_P^D(G)$ and take $M = Int_P^D(G)$. Then by Theorem(4.5) and definition of $Int_P^D(G)$ we get that M is a D-preopen set and by Remark(4.2), $x \in M \subseteq G$. Conversely, let there is D-preopen set M such that $x \in M \subseteq G$ Then by definition of $Int_P^D(G)$, $x \in M \subseteq Int_P^D(G)$. □

THEOREM 4.7. For a D-metric space (X, D) and $G, M \subseteq X$, the following hold:

(1) If $G \subseteq M$ then $Cl_P^D(G) \subseteq Cl_P^D(M)$.
(2) $Cl_P^D(G) \cup Cl_P^D(M) \subseteq Cl_P^D(G \cup M)$.
(3) $Cl_P^D(G \cap M) \subseteq Cl_P^D(G) \cap Cl_P^D(M)$.
(4) $Cl_P^D(G) \subseteq Cl_D(G)$.

PROOF. (1) Let $x \in Cl_P^D(G)$. Then by Theorem(4.5), for all D-preopen set N containing x, $N \cap G \neq \emptyset$. Since $G \subseteq M$ then $N \cap M \neq \emptyset$. Hence $x \in Cl_P^D(M)$. That is, $Cl_P^D(G) \subseteq Cl_P^D(M)$.
(2) Since $G \subseteq G \cup M$ and $M \subseteq G \cup M$, then by part(1), $Cl_P^D(G) \subseteq Cl_P^D(G \cup M)$ and $Cl_P^D(M) \subseteq Cl_P^D(G \cup M)$. Hence $Cl_P^D(G) \cup Cl_P^D(M) \subseteq Cl_P^D(G \cup M)$.
(3) Since $G \cap M \subseteq G$ and $G \cap M \subseteq M$, then by part(1), $Cl_P^D(G \cap M) \subseteq Cl_P^D(G)$ and $Cl_P^D(G \cap M) \subseteq Cl_P^D(M)$. Hence $Cl_P^D(G \cap M) \subseteq Cl_P^D(G) \cap Cl_P^D(M)$.
(4) It is clear from Theorem(4.5) and from every D-open set is D-preopen set.

□

In the above theorem $Cl_P^D(G \cup M) \neq Cl_P^D(G) \cup Cl_P^D(M)$ as it is shown in the following example.

EXAMPLE 4.8. Let (R, D) be D-metric space, where

$$D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

and (R, d) is usual metric space. Let $G = IR$ and $M = Q - [\{q\}]$, where Q is the set of rational numbers, IR is the set of irrational numbers and q is any rational number. Since G and M are D-precloced sets in R. Then $Cl_P^D(G) \cup Cl_P^D(M) = G \cup M = R - \{q\}$. If $R - \{q\}$ is D-precloced set in R then $\{q\}$ is D-preopen set but $\{q\}$ is not D-preopen set and this contradiction. Hence $R - \{q\}$ is not D-precloced set in R. Since $R - \{q\} \subseteq Cl_P^D(R - \{q\})$ then

$$Cl_P^D(G \cup M) = Cl_P^D(R - \{q\}) = R.$$

THEOREM 4.9. For a D-metric space (X, D) and $G, M \subseteq X$, the following hold:

(1) If $G \subseteq M$ then $Int_P^D(G) \subseteq Int_P^D(M)$.
(2) $Int_P^D(G) \cup Int_P^D(M) \subseteq Int_P^D(G \cup M)$.
(3) $Int_P^D(G \cap M) \subseteq Int_P^D(G) \cap Int_P^D(M)$.
(4) $Int_D(G) \subseteq Int_P^D(G)$.

PROOF. (1) Let $x \in Int_P^D(G)$. Then by Theorem(4.6), there is D-preopen set N such that $x \in N \subseteq G$ Since $G \subseteq M$ then $x \in N \subseteq M$. Hence $x \in Int_P^D(M)$. That is, $Int_P^D(G) \subseteq Int_P^D(M)$.
(2) Since $G \subseteq G \cup M$ and $M \subseteq G \cup M$, then by part(1), $Int_P^D(G) \subseteq Int_P^D(G \cup M)$ and $Int_P^D(M) \subseteq Int_P^D(G \cup M)$. Hence $Cl_P^D(G) \cup Int_P^D(M) \subseteq Int_P^D(G \cup M)$.
(3) Since $G \cap M \subseteq G$ and $G \cap M \subseteq M$, then by part(1), $Int_P^D(G \cap M) \subseteq Int_P^D(G)$ and $Int_P^D(G \cap M) \subseteq Int_P^D(M)$. Hence $Int_P^D(G \cap M) \subseteq Int_P^D(G) \cap Int_P^D(M)$.
(4) It is clear from Theorem(4.5) and from every D-open set is D-preopen set.

□

In the above theorem $Int_P^D(G \cap M) \neq Int_P^D(G) \cap Int_P^D(M)$ as it is shown in the following example.

EXAMPLE 4.10. In Example(4.8), take $G = Q \cup \{r\}$ and $M = IR$, where Q is the set of rational numbers, IR is the set of irrational numbers and r is any irrational number. Since G and M are D-preopen sets in R. Then $Int_P^D(G) \cap Int_P^D(M) = G \cap M = (Q \cup \{r\}) \cap IR = \{r\}$. Since $\{r\}$ is not D-preopen set and $Int_P^D(\{r\}) \subseteq \{r\}$ then $Int_P^D(G \cap M) = Int_P^D(\{r\}) = \emptyset$.

THEOREM 4.11. For a D-metric space (X, D) and $G \subseteq X$, the following hold:

(1) $Int_P^D(X - G) = X - Cl_P^D(G)$.
(2) $Cl_P^D(X - G) = X - Int_P^D(G)$.

PROOF. (1) Since $G \subseteq Cl_P^D(G)$, then $X - Cl_P^D(G) \subseteq X - G$. Since $Cl_P^D(G)$ is a D-preclosed set then $X - Cl_P^D(G)$ is a D-preopen set. Then

$$X - Cl_P^D(G) = Int_P^D[X - Cl_P^D(G)] \subseteq Int_P^D(X - G).$$

For the other side, let $x \in Int_P^D(X - G)$. Then there is D-preopen set N such that $x \in N \subseteq X - G$. Then $X - N$ is a D-preclosed set containing G and $x \notin X - N$. Hence $x \notin Cl_P^D(G)$, that is, $x \in X - Cl_P^D(G)$.
(2) Since $Int_P^D(G) \subseteq G$, then $X - G \subseteq X - Int_P^D(G)$. Since $Int_P^D(G)$ is a D-preopen set then $X - Int_P^D(G))$ is a D-preclosed set. Then

$$Cl_P^D(X - G) \subseteq Cl_P^D[X - Int_P^D(G)] = X - Int_P^D(G)$$

. For the other side, let $x \notin Cl_P^D(X - G)$. Then by Theorem(4.5), there is a D-preopen set N containing x such that $N \cap (X - G) = \emptyset$. Then $x \in N \subseteq G$, that is, $x \in Int_P^D(G)$. Hence $x \notin X - Int_P^D(G)$. Therefore $X - Int_P^D(G) \subseteq Cl_P^D(X - G)$.

□

THEOREM 4.12. For a subset $G \subseteq X$ of D-metric space (X, D) the following hold:

(1) If M is a D-open set in X then $Cl_P^D(G) \cap M \subseteq Cl_P^D(G \cap M)$.

(2) If M is a D-closed set in X then $Int_P^D(G \cup M) \subseteq Int_P^D(G) \cup M$.

PROOF. (1) Let $x \in Cl_P^D(G) \cap M$. Then $x \in Cl_P^D(G)$ and $x \in M$. Let V be any D-preopen set in (X, D) containing x. By Theorem(3.6), $V \cap M$ is D-preopen set containing x. Since $x \in Cl_P^D(G)$ then by Theorem(4.5), $(V \cap M) \cap G \neq \emptyset$. This implies, $V \cap (M \cap G) \neq \emptyset$. Hence by Theorem(4.5), $x \in Cl_P^D(G \cap M)$. That is, $Cl_P^D(G) \cap M \subseteq Cl_P^D(G \cap M)$.

(2) Since M is a D-closed set X then by the part(1) and Theorem(4.11),

$$\begin{aligned} X - [Int_P^D(G) \cup M] &= [X - Int_P^D(G)] \cap [X - M] \\ &= [Cl_P^D(X - G)] \cap [X - M] \\ &\subseteq Cl_P^D[(X - G) \cap (X - M)] \\ &= Cl_P^D(X - (G \cup M)) \\ &= X - (Int_P^D(G \cup M)). \end{aligned}$$

Hence $Int_P^D(G \cup M) \subseteq Int_P^D(G) \cup M$.

□

LEMMA 4.13. For a D-metric space (X, D) and $G \subseteq X$, $x \in Cl_D(G)$ if and only if for all $\varepsilon > 0$, $O_\varepsilon^D(x) \cap G \neq \emptyset$.

PROOF. Let $x \in Cl_D(G)$ and $\varepsilon > 0$. If $O_\varepsilon^D(x) \cap G = \emptyset$ then $G \subseteq X - O_\varepsilon^D(x)$. Since $X - O_\varepsilon^D(x)$ is a D-closed set containing G, then $Cl_D(G) \subseteq X - O_\varepsilon^D(x)$ and $x \in Cl_D(G) \subseteq X - O_\varepsilon^D(x)$. Hence this is contradiction, because $x \in O_\varepsilon^D(x)$. Therefore $O_\varepsilon^D(x) \cap G \neq \emptyset$.
Conversely, Let $x \notin Cl_D(G)$. Then $X - Cl_D(G)$ is a D-open set containing x. Then there is $\varepsilon > 0$ such that $O_\varepsilon^D(x) \subseteq X - Cl_D(G)$ Hence by hypothesis, $O_\varepsilon^D(x) \cap G \neq \emptyset$. But this is contradiction, because $O_\varepsilon^D(x) \subseteq X - Cl_D(G) \subseteq X - G$. □

THEOREM 4.14. A subset $G \subseteq X$ of D-metric space (X, D) is a D-preopen set if and only if $G \subseteq Int_D(Cl_D(G))$.

PROOF. Suppose that G is a D-preopen set. Let $x \in G$ be arbitrary point. Then there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. By Lemma(4.13), we get that $O_\delta^D(x) \subseteq Cl_D(G)$. That is, $x \in Int_D(Cl_D(G))$. Hence $G \subseteq Int_D(Cl_D(G))$.
Conversely, Suppose that $G \subseteq Int_D(Cl_D(G))$ and $x \in G$ is arbitrary point. Then $x \in Int_D(Cl_D(G))$. That is, there is $\delta > 0$ such that $O_\delta^D(x) \subseteq Cl_D(G)$. Hence for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. Hence G is a D-preopen set. □

For a subset G of D-metric space (X, D) the D-frontier operator of G is defined by

$$\Gamma_P^D(G) = Cl_P^D(G) - Int_P^D(G).$$

THEOREM 4.15. For a subset $G \subseteq X$ of D-metric space (X, D), the following hold:

(1) $Cl_P^D(G) = \Gamma_P^D(G) \cup Int_P^D(G)$.
(2) $\Gamma_P^D(G) \cap Int_P^D(G) = \emptyset$.
(3) $\Gamma_P^D(G) = Cl_P^D(G) \cap Cl_P^D(X - G)$.

PROOF. Note that

(1) $\Gamma_P^D(G) \cup Int_P^D(G)$

$$\begin{aligned} &= (Cl_P^D(G) - Int_P^D(G)) \cup Int_P^D(G) \\ &= [Cl_P^D(G) \cap (X - Int_P^D(G))] \cup Int_P^D(G) \\ &= [Cl_P^D(G) \cup Int_P^D(G)] \cap [(X - Int_P^D(G)) \cup Int_P^D(G)] \\ &= Cl_P^D(G) \cap X = Cl_P^D(G). \end{aligned}$$

(2) It is clear from the definition of $\Gamma_P^D(G)$.
(3) By Theorem(4.11),

$$\begin{aligned} \Gamma_P^D(G) &= Cl_P^D(G) - Int_P^D(G) = Cl_P^D(G) \\ &\quad \cap (X - Int_P^D(G)) \\ &= Cl_P^D(G) \cap Cl_P^D(X - G). \end{aligned}$$

This is the desired. □

COROLLARY 4.16. For a subset $G \subseteq X$ of D-metric space (X, D), $\Gamma_P^D(G)$ is D-preclosed set in (X, D).

PROOF. By Theorem(4.9) and the part(3) of the last theorem. □

THEOREM 4.17. For a subset $G \subseteq X$ of D-metric space (X, D), the following hold:

(1) G is a D-preopen set if and only if $\Gamma_P^D(G) \cap G = \emptyset$.

(2) G is a D-preclosed set if and only if $\Gamma_P^D(G) \subseteq G$.

(3) G is both D-preopen set and D-preclosed set if and only if $\Gamma_P^D(G) = \emptyset$.

PROOF. (1) Let G be a D-preopen set. Then $Int_P^D(G) = G$. Then by Theorem(4.15),

$$\Gamma_P^D(G) \cap G = \Gamma_P^D(G) \cap Int_P^D(G) = \emptyset$$

Conversely, suppose that $\Gamma_P^D(G) \cap G = \emptyset$. Then

$$\begin{aligned} G - Int_P^D(G) &= [G \cap Cl_P^D(G)] - [G \cap Int_P^D(G)] \\ &= G \cap (Cl_P^D(G) - Int_P^D(G)) \\ &= G \cap \Gamma_P^D(G) = \emptyset. \end{aligned}$$

That is, $Int_P^D(G) = G$. Hence G is a D-preopen set.

(2) Let G be a D-preclosed set. Then $Cl_P^D(G) = G$. Then

$$\Gamma_P^D(G) = Cl_P^D(G) - Int_P^D(G) = G - Int_P^D(G) \subseteq G.$$

Conversely, let $\Gamma_P^D(G) \subseteq G$. Then by Theorem(4.15),

$$Cl_P^D(G) = Int_P^D(G) \cup \Gamma_P^D(G) \subseteq Int_P^D(G) \cup G \subseteq G.$$

That is, $Cl_P^D(G) = G$. Hence G is D-preclosed set.

(3) Let G be both D-preclosed set and D-preopen set. Then $Cl_P^D(G) = G = Int_P^D(G)$. Then

$$\Gamma_P^D(G) = Cl_P^D(G) - Int_P^D(G) = G - G = \emptyset.$$

Conversely, suppose that $\Gamma_P^D(G) = \emptyset$. Then $Cl_P^D(G) - Int_P^D(G) = \emptyset$. Since $Int_P^D(G) \subseteq Cl_P^D(G)$ then $Cl_P^D(G) = Int_P^D(G)$. Since $Int_P^D(G) \subseteq G \subseteq Cl_P^D(G)$ then

$$Cl_P^D(G) = G = Int_P^D(G).$$

That is, $Cl_P^D(G) = G$. Hence G is both D-preclosed set and D-preopen set. □

5. REFERENCES

[1] A. A. Ali Fora, M. O. Massadeh and M. S. Bataineh, A new Structure and Contribution in D-metric Spaces, British Journal of Mathematics and Computer Science, 22(1), (2017), 1-9.

[2] B. C. Dhage, Generalized Metric Space and Topological Structure I, Analele Stiitifice Ale Univ. Cuza Iasi Mat., 46, (2000), 3-24.

[3] B. C. Dhage, (1984), A study of some fixed point theorms, Ph.D.Thesis. Marathwada Univ. Aurangabad, India.

[4] C. D. Bele and U. P. Dolhare, An Extension of Common Fixed Point Theorem in D-Metric Space, International Journal of Mathematics and its Applications, 5, (2017), 13-18.

[5] C. G. Aras, S. Bayramov and M. I. Yazar, Soft D-Metric Spaces, Bol. Soc. Paran. Mat, 38, (2020), 137-147.

[6] I. Ramabhadrasarma and S. Sambasivarao, On D-metric spaces, Journal of Global Research in Mathematical Archives, 12, (2013), 31-39.

[7] J. Muscat, (2016), Function anaylsise, Springer Heidelberg New York, London.