Clique Dominating Sets of Direct Product Graph of Cayley Graphs with Arithmetic Graphs

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ABSTRACT
The basic ideas of graph theory are introduced in the 18th century by the great mathematician Leonard Euler. Since then relatively in a short period, major developments of graph theory has occurred and inspired to a larger degree and it has become the source of interest to many researchers. To some extent this may be due to the ever growing importance of computer science and its connection with graph theory.

Product of graphs are introduced in Graph Theory very recently and developing rapidly. In this paper, we consider direct product graphs of Cayley graphs with Arithmetic graphs and discuss Clique domination parameter of these graphs.

Keywords
Euler totient Cayley graph, Arithmetic graph, direct product graph, Clique dominating set.

Subject Classification: 68R10

1. INTRODUCTION
Theory of domination in graphs introduced by Ore [1] and Berge [2] is an emerging area of research in graph theory today. It was initiated as a problem in the game of chess around 1850. It is about placing of minimum number of queens on an n x n chess board so as to cover or dominate every square by at least one queen. The solution to these problems are nothing but domination in graphs, whose vertices are the queens of the chess board and vertices u, v are adjacent if a queen moves from u to v in one move.

Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Computer Science is one of the many fields in which graph products are becoming common place.

The direct product was introduced by Alfred North Whitehead and Bertrand Russell in their Principia Mathematica [3] is also called as the tensor product, categorical product, cardinal product, relational product, Kronecker product, weak direct product or conjunction.

Direct Product Graph $G_1 \times G_2$
Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, \dots, u_l\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$ respectively. Then the direct product of these two graphs denoted by $G_1 \times G_2$ is defined to be a graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 such that any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \times G_2$ are adjacent if u_1u_2 is an edge of G_1 and v_1v_2 is an edge of G_2.

In this paper we study the dominating clique of the direct product graph of Euler totient Cayley graph G_1 and Arithmetic V_n graph G_2.

It is easy to see that the dominating clique exists only for connected graphs.

2. EULER TOTIENT CAYLEY GRAPH $G(Z_n, \varphi)$
For any positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all integers less than n and relatively prime to n. That is $S = \{r/1 \le r < n$ and $GCD(r, n) = 1\}$. Then $|S| = \varphi(n)$, where φ is the Euler totient function. We can see that S is a symmetric subset of the group $(Z_n, \oplus)$.

The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2, \dots, n-1\}$ and the edge set is $E = \{(x, y)/x - y \in S$ or $y - x \in S\}$. This graph is denoted by $G(Z_n, \varphi)$.

Some properties of Euler totient Cayley graphs and enumeration of Hamilton cycles and triangles can be found in Madhavi [4]. The Euler totient Cayley graph $G(Z_n, \varphi)$ is a complete graph if n is a prime and it is $\varphi(n)$ - regular.

The clique domination parameter of these graphs are studied by the authors [5] and the following results are required and they are presented without proofs.

Theorem 2.1: If n is a prime, then the clique domination number of $G(Z_n, \varphi)$ is 1.
Theorem 2.2: If $n = p^\alpha$, where $\alpha > 1$, then the clique domination number of $G(Z_n, \varphi)$ is 2.
Theorem 2.3: Suppose $n \ne p^\alpha, \alpha \ge 1$. If n is an even number, then the clique domination number does not exist for the graph $G(Z_n, \varphi)$.
Theorem 2.4: Let n be neither p^α, $\alpha \ge 1$ nor an even number and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots \dots, p_k$ are primes, $p_1 < p_2 < \dots < p_k$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are integers ≥ 1. If λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n, and if $\lambda + 1 \le p_1$, then the clique domination number of $G(Z_n, \varphi)$ exists and equals $\lambda + 1$. Otherwise clique domination number does not exist.

3. ARITHMETIC V_n GRAPH
Vasumathi [6] introduced the concept of Arithmetic V_n graph and studied some of its properties.

Their definition of Arithmetic V_n graph is as follows.

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u and v are adjacent in V_n graph if and only if $GCD(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. In doing so we have deleted the vertex 1 from the graph as the contribution of this isolated vertex is nothing, when domination parameters are enumerated.

The clique domination number of graph is obtained by the authors and the proofs of the following theorems can be found in [5].

Theorem 3.1: If n is a prime, then the clique domination number of $G(V_n)$ is 1.
Theorem 3.2: If n is the product of two distinct primes, then the clique domination number of $G(V_n)$ is 1.
Theorem 3.3: Let $\boldsymbol{n}$ be neither a prime nor $\boldsymbol{p_1 p_2}$, where $\boldsymbol{p_1, p_2}$ are two distinct primes and $\boldsymbol{n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}}$ where $\boldsymbol{\alpha_i \geq 1}$. Then the clique domination number of $\boldsymbol{G(V_n)}$ is $\boldsymbol{k}$, where $\boldsymbol{k}$ is the core of $\boldsymbol{n}$.

4. DIRECT PRODUCT GRAPH OF $G(Z_n, \varphi)$ WITH $G(V_n)$

In this paper the direct product graph of Euler totient Cayley graph with Arithmetic V_n graph is considered. The properties and some domination parameters of these graphs can be found in [7].

Let $G(Z_n, \varphi)$ denote the Euler Totient Cayley graph and $G(V_n)$ denote the Arithmetic V_n graph. Since $G(Z_n, \varphi)$ and $G(V_n)$ are two simple graphs, by the definition of adjacency in the direct product, the graph $G_1 \times G_2$ is a simple graph.

Further the graph $G_1 \times G_2$ is a completely disconnected graph, if n is a prime and the degree of a vertex in $G_1 \times G_2$ is given by

$$\begin{aligned} deg_{G_1 \times G_2}(u_i, v_j) &= deg_{G_1}(u_i) \cdot deg_{G_2}(v_j) \\ &= \varphi(n) \cdot deg_{G_2}(v_j). \end{aligned}$$

5. CLIQUE DOMINATING SETS OF DIRECT PRODUCT GRAPH

In this section minimum clique dominating sets of direct product graph of $G(Z_n, \varphi)$ with $G(V_n)$ are discussed and obtained its clique domination number in various cases.

Clique domination

A dominating set D of vertices in a graph G is called a dominating clique if the induced subgraph $< D >$ is a complete graph. The clique domination number γ_{cl} of G is the minimum cardinality of a clique dominating set.
The results on clique domination of direct product graph $G_1 \times G_2$ are as follows.
Theorem 5.1: If n is a prime, then the clique domination number does not exist for the graph $G_1 \times G_2$.
Proof: Suppose n is a prime. Then $G_1 \times G_2$ is a completely disconnected graph on n vertices. Hence there are no edges between these n vertices. So, all these vertices form a dominating set.

Therefore $\gamma(G_1 \times G_2) = n$.

Obviously, the induced subgraph $< D >$ does not form a complete graph.

Hence clique domination number doesn't exist for the graph $G_1 \times G_2$.

Theorem 5.2: If $n = 2p$, where p is an odd prime or $n = 2^\alpha$, where $\alpha > 1$, then clique domination number does not exist for the graph $G_1 \times G_2$.
Proof: Suppose $n = 2p$, where p is an odd prime or $n = 2^\alpha, \alpha > 1$. Then $G_1 \times G_2$ is a disconnected graph.
Hence the clique domination number does not exist for the given graph $G_1 \times G_2$.
Theorem 5.3: If $n = p^\alpha$, where p is an odd prime and $\alpha > 1$ is an integer, then the clique domination number does not exist for the graph $G_1 \times G_2$.
Proof: Let $n = p^\alpha$, where p is an odd prime and $\alpha > 1$, is an integer.
Consider the graph $G_1 \times G_2$. Let V_1 and V_2 denote the vertex sets of G_1, G_2 respectively.
Then $V(G_1) = \{0,1,2, \dots, p^\alpha - 1\} = V_1$, $V(G_2) = \{p, p^2, \dots, p^\alpha\} = V_2$.

Let $V(G_1 \times G_2) = V_1 \times V_2$.

By Remark 1 of Theorem 2.3.2 of Chapter 2 [5], we know that $D_1 = \{u_{d_1}, u_{d_2}\}$ is a dominating set of G_1 with minimum cardinality 2.

By Theorem 3.3, we know that $D_2 = \{p\}$ is a dominating set and also a dominating clique of G_2 with minimum cardinality.

$$\begin{aligned} \text{Let } D = D_1 \times D_2 &= \{u_{d_1}, u_{d_2}\} \times \{p\} \\ &= \{(u_{d_1}, p), (u_{d_2}, p)\} \end{aligned}$$

Now we verify whether the set $D = \{(u_{d_1}, p), (u_{d_2}, p)\}$ form a dominating set or not in $G_1 \times G_2$.

Consider the set of vertices in $V - D$ which are $\{u_{d_3}, u_{d_4}, u_{d_5}, \dots, u_{d_n}\} \times \{p, p^j\} \cup \{u_{d_1}, u_{d_2}\} \times \{p^j\}$, $j > 1$.
The vertices $\{u_{d_3}, u_{d_4}, \dots, u_{d_n}\} \times \{p^j\}$ in $V - D$ are adjacent to either $\{u_{d_1}, p\}$ or $\{u_{d_2}, p\}$ as D_1, D_2 are dominating sets of G_1, G_2 respectively.
But the vertices $\{u_{d_3}, u_{d_4}, \dots, u_{d_n}\} \times \{p\}$ are not dominated by the vertices in D, because there is no edge between the vertex p to itself.

Thus D does not form a dominating set with cardinality 2.

Now consider another (u_{d_1}, p^j) or (u_{d_2}, p^j), $j \neq 1$.
Let $D = \{(u_{d_1}, p), (u_{d_2}, p), (u_{d_1}, p^j)\}$, $j \neq 1$.

Similarly by the above argument and by the definition of direct product we can see that all the vertices $\{u_{d_3}, u_{d_4}, \dots, u_{d_n}\} \times \{p\}$ are not dominated by the vertex $(u_{d_1}, p^j) \in D$. So, D is not a dominating set of $G_1 \times G_2$. Similar is the case with $D = \{(u_{d_1}, p), (u_{d_2}, p), (u_{d_2}, p^j)\}$, $j \neq 1$.

Therefore if D is a dominating set of $G_1 \times G_2$, then $|D| \geq 4$.

Now we claim that a set of four vertices in $G_1 \times G_2$ forms a dominating set.
Let $D = \{(u_{d_1}, p), (u_{d_2}, p), (u_{d_1}, p^j), (u_{d_2}, p^j)\}$, $j \neq 1$.

Let (u, v) be any vertex in $V - D$ in $G_1 \times G_2$. Then the vertex u in G_1 is adjacent to either u_{d_1} or u_{d_2} in D_1 as D_1 is a dominating set of G_1. The vertex v in G_2 is adjacent to either p or p^j according to $GCD\,(v, p) = p$ or $GCD\,(v, p^j) = p$.

Thus by the definition of direct product, the vertex (u, v) in $V - D$ is adjacent to either (u_{d_1}, p) or (u_{d_1}, p^j); or (u_{d_2}, p) or (u_{d_2}, p^j) in D. Thus D becomes a dominating set of $G_1 \times G_2$. Since $|D| = 4$, it follows that D is a minimal dominating set of $G_1 \times G_2$.

Therefore $\gamma(G_1 \times G_2) = 4$.

But by the definition of direct product the vertices (u_{d_1}, p) and (u_{d_1}, p^j) are not adjacent and also the vertices (u_{d_2}, p) and $(u_{d_2}, p^j))$ are not adjacent. So, the induced subgraph $< D >$ gives a disconnected graph. Hence, we cannot get a complete graph on these vertices.

Similarly, if we construct a dominating set of any cardinality in any manner then also, we cannot get a complete graph on these vertices, because by the definition of direct product and by the definition of edges in G_1 and G_2 respectively.

Hence clique domination number does not exist for the graph $G_1 \times G_2$.

Remark: By replacing p by 2, we can see that the dominating set constructed in the above theorem is also a dominating set for the case $n = 2^\alpha$. In this case the graph becomes disconnected. When p is an odd prime, the graph is not disconnected.

Theorem 5.4: If n is neither a prime nor $2p$ nor power of a prime and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots p_k^{\alpha_k}$, where $p_1, p_2, \dots \dots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \dots \dots, \alpha_k$ are integers ≥ 1, then the clique domination number does not exist for the graph $G_1 \times G_2$.

Proof: Let $n \neq p, n \neq p^\alpha, n \neq 2p$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots \dots \dots p_k^{\alpha_k}$, where $\alpha_i \geq 1$. Consider the graph $G_1 \times G_2$. Let V_1, V_2, V denote the vertex sets of G_1 , G_2 and $G_1 \times G_2$ respectively.

By Remark 4 of Theorem 2.4.3 of Chapter 2 [5], we know that $D_1 = \{u_{d_1}, u_{d_2}, \dots, u_{d_{\lambda+1}}\}$ is a dominating set of G_1 with minimum cardinality $\lambda + 1$. As in the Remark of Theorem 3.3, we know that $D_2 = \{p_k, p_k p_1, \dots, p_k p_{k-1}\}$ is a dominating set of G_2 with cardinality k.

Let $D = D_1 \times D_2$

$= \{u_{d_1}, u_{d_2}, \dots \dots .., u_{d_{\lambda+1}}\} \times \{p_k, p_k p_1, \dots \dots .., p_k p_{k-1}\}$.

Now we claim that D is a dominating set of $G_1 \times G_2$. Let (u, v) be any vertex of $V - D$ in $G_1 \times G_2$. Then the vertex u is adjacent to some vertex u_{d_l} in D_1 and the vertex v is adjacent to either $p_k p_j$ for $1 \leq j \leq k - 1$ or p_k as D_1 and D_2 are dominating sets of G_1 and G_2 respectively. That is the vertex (u, v) is adjacent to $(u_{d_l}, p_k p_j)$ or (u_{d_l}, p_k) in D. Thus $D = D_1 \times D_2$ is a dominating set of $G_1 \times G_2$ with cardinality $(\lambda + 1)k$.

We now verify whether D is a dominating clique or not.

Consider the vertices in

$D = \{(u_{d_1}, p_k), (u_{d_1}, p_k p_1), \dots \dots \dots \dots \dots, (u_{d_1}, p_k p_{k-1})$

. . . . .
. . . . .
. . . . .

$(u_{d_{\lambda+1}}, p_k), (u_{d_{\lambda+1}}, p_k p_1), \dots \dots \dots .., (u_{d_{\lambda+1}}, p_k p_{k-1})\}$.

By the definition of direct product, the vertex (u_{d_i}, p_k) is not adjacent to the vertex $(u_{d_{i+1}}, p_k)$, where $i = 1, 2, \dots .., \lambda$.

So, we cannot get a complete graph on these vertices in D. Similarly if we construct a dominating set of any cardinality in any manner then also, we cannot get a complete graph on these vertices, by the definition of direct product and also by the definition of edges in G_1 and G_2 respectively.

Hence dominating clique does not exist for the graph $G_1 \times G_2$.

6. CONCLUSION

Using Number theory, it is interesting to study the Clique Dominating Sets of Direct Product Graph of Cayley Graphs with Arithmetic Graphs. This work gives the scope for the study of clique dominating sets of product graphs like strong product and lexicographic product of these graphs and the authors have also studied this aspect.

7. ILLUSTRATIONS

$\boldsymbol{n = 13 (Prime\ number)}$

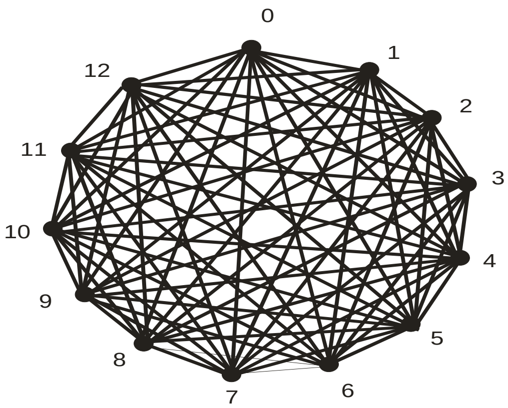


Fig.1

$\boldsymbol{G_1 = G(Z_{13}, \varphi)}$

Fig.2

$\boldsymbol{G_2 = G(V_{13})}$

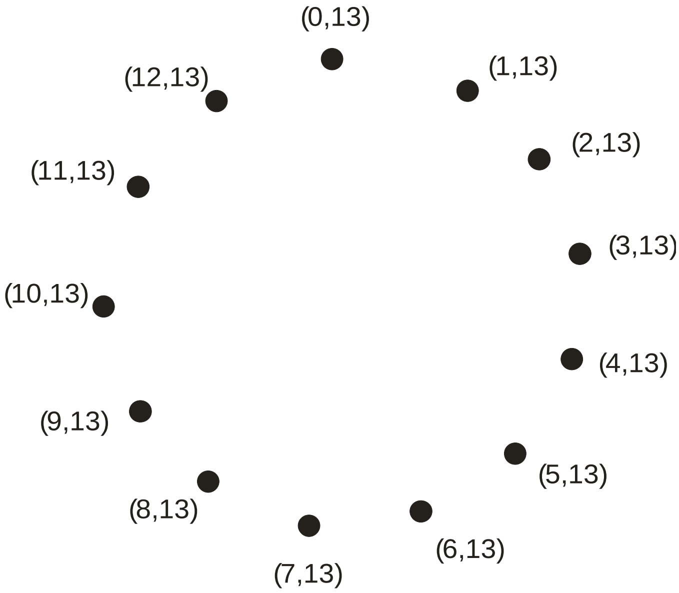


Fig.3
$\boldsymbol{G_1 \times G_2}$
Clique dominating set does not exist

$$\boldsymbol{n = 2 \times 5 = 10}$$

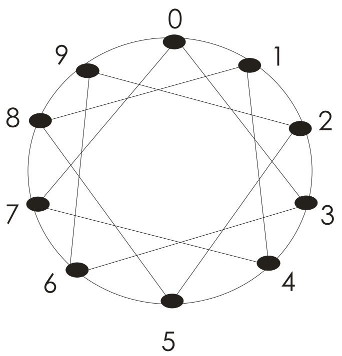


Fig.4
$G_1 = G(Z_{10}, \varphi)$

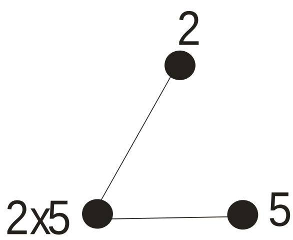


Fig.5
$G_2 = G(V_{10})$

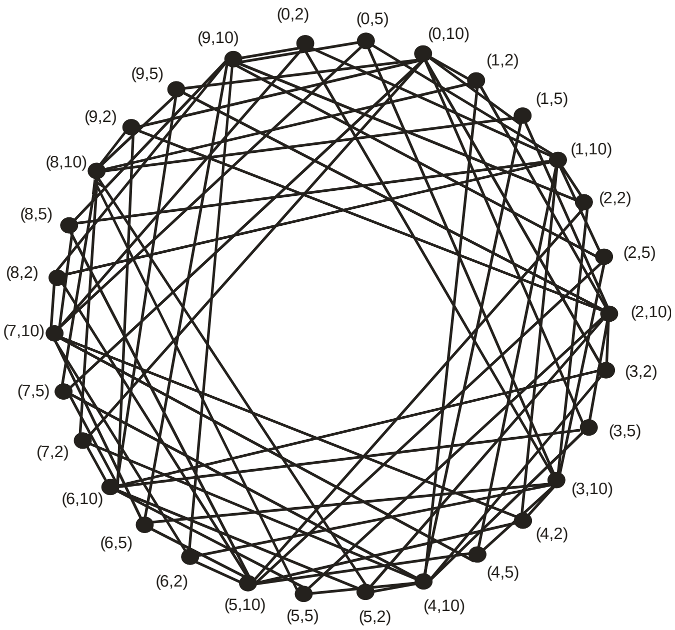


Fig.6

$\boldsymbol{G_1 \times G_2}$

Clique dominating set does not exist

$\boldsymbol{n = 2^3 = 8}$

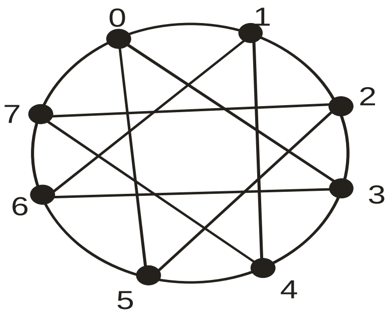


Fig.7

$G_1 = G(Z_8, \varphi)$

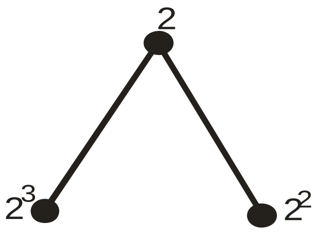


Fig.8

$G_2 = G(V_8)$

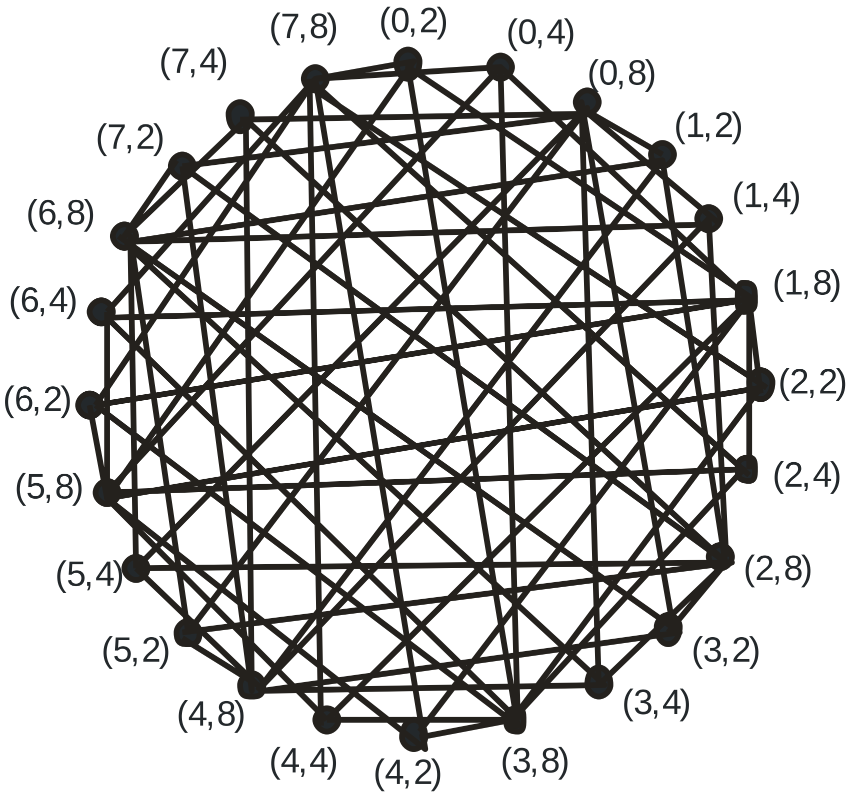
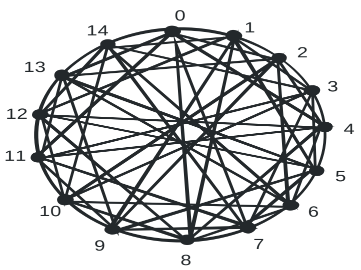


Fig.9

$\boldsymbol{G_1 \times G_2}$

Clique dominating set does not exist

$$\boldsymbol{n = 3 \times 5 = 15}$$



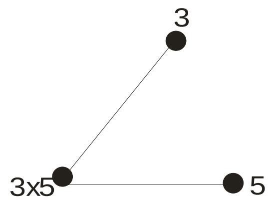


Fig.10

Fig.11

$\boldsymbol{G_1 = G(Z_{15}, \varphi)}$

$\boldsymbol{G_2 = G(V_{15})}$

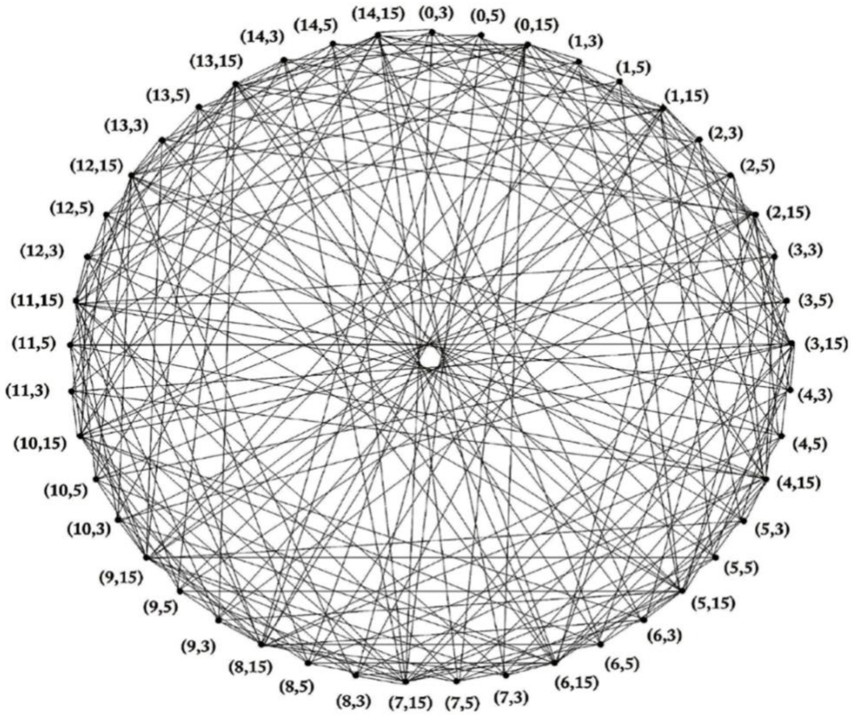


Fig.12
$\boldsymbol{G_1 \times G_2}$
Clique dominating set does not exist

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