On \( \mathcal{T}^g \)-Open Sets in Ideal Topological Semigroups

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ABSTRACT
In this paper, we introduce and investigate a new class of \( \text{semi}^+ - \mathcal{I} \)-open sets, called \( \mathcal{T}^g \)-open sets in ideal topological semigroups. This class is consider strong form of \( \beta \mathcal{I}^g - \mathcal{I} \)-open sets and weak form of of \( \text{semi}^+ - \mathcal{I} \)-open sets and \( \beta - \mathcal{I} \)-open sets. The interior, closer and frontier operators are studied with the relative property via \( \mathcal{T}^g \)-open sets.

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open set;ideal topological space, topological semigroup.

1. INTRODUCTION
The notion of ideal topological spaces is introduced by Kuratowski, [2]. Many researcher studied about the ideal topological spaces. An idea \( \mathcal{I} \) on a topological space \( (X, \tau) \) is a nonempty collection of subsets of \( X \) which satisfies the following conditions:

1. If \( A \in \mathcal{I} \) and \( B \in \mathcal{I} \) then \( A \cup B \in \mathcal{I} \).
2. If \( A \in \mathcal{I} \) then \( A^\circ \in \mathcal{I} \).

Applications to various fields were further investigated by Jankovic and Hamlett [1], Dontchev [3] and Arenas et al [4]. An ideal topological space is the concept of an ideal topological space \( (X, \tau, \mathcal{I}) \) on an ideal space \( X \), and is denoted by \( (X, \tau, \mathcal{I}) \).

Under the notion of ideal topological spaces, several mathematician researcher introduced the new forms of \( \mathcal{T}^g \)-open sets such as \( \beta - \mathcal{I} \)-open sets by Hatir and Noiri in 2002, [4] and \( \text{semi}^+ - \mathcal{I} \)-open sets by Ekici and Noiri in 2012, [5], are the weaker forms. Also \( \beta \mathcal{I}^g \)-open sets is strong by Ekici in 2011, [6].

This paper is organized as follows: Section 3 introduces the concept of \( \mathcal{T}^g \)-open sets in ideal topological semigroups with its relationship among other known sets. Section 4 introduces the concepts of \( \mathcal{T}^g \)-interior operator, \( \mathcal{T}^g \)-cloiser operator and \( \mathcal{T}^g \)-frontier operator. Section 5 studies the relative property via \( \mathcal{T}^g \)-open sets.

2. PRELIMINARY
For a topological space \( (X, \tau) \) and \( A \subseteq X \), throughout this paper, we mean \( \text{Cl}(A) \) and \( \text{Int}(A) \) the closure set and the interior set of \( A \), respectively.

THEOREM 2.1. \( \text{If for a topological space } (X, \tau) \text{ and } A, B \subseteq X, \text{ if } B \text{ is an open set in } X \text{ then } \text{Cl}(A) \cap B \subseteq \text{Cl}(A \cap B). \)

THEOREM 2.2. \( \text{For a topological space } (X, \tau), \)

1. \( \text{Cl}(X - A) = X - \text{Int}(A) \) for all \( A \subseteq X. \)
2. \( \text{Int}(X - A) = X - \text{Cl}(A) \) for all \( A \subseteq X. \)

In the ideal topological space \( (X, \mathcal{I}, \tau) \), \( \mathcal{A}'(\mathcal{I}) \) is defined by:

\( \mathcal{A}'(\mathcal{I}) = \{ x \in X : U \cap A \notin \mathcal{I} \text{ for each open neighborhood } U \text{ of } x \} \)

is called the local function of \( A \) with respect to \( \mathcal{I} \) and \( \tau \). [9]. When there is no chance for confusion \( \mathcal{A}'(\mathcal{I}) \) is denoted by \( \mathcal{A}' \). For every ideal topological space \( (X, \mathcal{I}, \tau) \), there exists a topology \( \tau^* \) finer than \( \tau \), generated by the base

\( \mathcal{B}(\mathcal{I}, \tau) = \{ U - I : U \in \tau \text{ and } I \in \mathcal{I} \}. \)

Observe additionally that \( \text{Cl}^*(A) = A \cup \mathcal{A}' \), [10] defines a Kuratowski closure operator for \( \tau^* \). \( \text{Int}^*(A) \) will denote the interior of \( A \) in \( (X, \tau^*) \). If \( \mathcal{I} \) is an ideal on topological space \( (X, \tau) \), then \( (X, \tau, \mathcal{I}) \) is called an ideal topological space.

THEOREM 2.3. \( \text{Let } (X, \tau, \mathcal{I}) \text{ be an ideal topological space. Then for } A, B \subseteq X, \text{ the following properties hold:} \)

1. \( A \subseteq B \text{ implies that } A^* \subseteq B^*; \)
2. \( G \in \tau \text{ implies that } G \cap A^* \subseteq (G \cap A)^*; \)
3. \( A^* = \text{Cl}(A^*) \subseteq \text{Cl}(A); \)
4. \( (A \cup B)^* = A^* \cup B^*; \)
5. \( (A^*)^* \subseteq A^*. \)

DEFINITION 2.4. A subset \( A \) of an ideal topological space \( (X, \tau, \mathcal{I}) \) is called:

1. \( \text{semi} - \mathcal{I} \)-open set, [4] if \( A \subseteq \text{Cl}^*(\text{Int}(A)); \)
2. \( \alpha^*_\mathcal{I} - \mathcal{I} \)-open set, [6] if \( A \subseteq \text{Int}(\text{Cl}^*(\text{Int}(A))); \)
3. \( \beta - \mathcal{I} \)-open set, [4] if \( A \subseteq \text{Cl}^*(\text{Int}(\text{Cl}^*(A))); \)
4. \( \text{semi} - \mathcal{I} \)-open set, [4] if \( A \subseteq \text{Cl}(\text{Int}(A^*)); \)
5. \( \text{semi} - \mathcal{I} \)-open set, [4] if \( A \subseteq \text{Cl}(\text{Int}(A^*)); \)

DEFINITION 2.5. \( \text{A subset } A \text{ of an ideal topological space } (X, \tau, \mathcal{I}) \text{ is called } \beta^*_\mathcal{I} - \mathcal{I} \text{ open set if } A \subseteq \text{Cl}(\text{Int}(\text{Cl}^*(A))) \).

By topological semigroup \( (X, \tau) \), we mean a topological space \( (X, \tau) \) which is space with associated multiplication \( * : X \times X \to X \) such that \( * \) is continuous function from the product space \( X \times X \) into \( X \). By ideal topological semigroup \( (X, \mathcal{I}, \tau) \), we mean an ideal topological space \( (X, \mathcal{I}, \tau) \) with associated multiplication \( * : X \times X \to X \) such that \( * \) is continuous function from the product space \( X \times X \) into \( X \). A pair \( (Y, \circ) \) is called a \( \mathcal{T} - \text{subspace} \) of ideal topological semigroup \( (X, \mathcal{I}, \tau) \) if \( Y \) is a subspace of \( X \) and the continuous function \( \circ \) takes the product \( Y \times Y \) into \( Y \).
and $\sigma(x,y) = * (x,y)$ for all $x, y \in Y$. We denote the operation of any $I -$ subspace with the same symbol used for the operation on the ideal topological semigroup under consideration. For any ideal topological space $(X, \tau, I)$, we mean by $(X, \tau, I)$ the ideal topological semigroup with operation $\pi: X \times X \rightarrow X$, where $\pi(x, y) = x$ or $\pi(x, y) = y$ for all $x, y \in X$.

3. $I^S$-OPEN SETS

**Definition 3.1.** A subset $A$ of an ideal topological semigroup $(X, \tau, I)$ is said to be $I^s$-open set if $A \subseteq Cl[Int^s(Cl^s(A))]$. The complement of $I^s$-open set is said to be $I^s$-closed set.

For an ideal topological set $(X, \tau, I)$, the set of all $I^s$-open sets in $X$ denoted by $\mathcal{I}^sO(X, \tau)$ and the set of all $I^s$-closed sets in $X$ denoted by $\mathcal{I}^sC(X, \tau)$.

**Example 3.2.** In an ideal topological semigroup $(X, \tau, I)$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $I = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$. Note that

\[ \mathcal{I}^sO(X, \tau) = P(X) \text{ and } \mathcal{I}^sC(X, \tau) = P(X). \]

**Theorem 3.3.** Every $\beta - I -$ open set is $I^s$-open set.

**Proof.** Let $A$ be $\beta - I -$ open subset of an ideal topological semigroup $(X, \tau, I)$. Then $A \subseteq Cl[Int(Cl^s(A))]$. Since $\tau \subseteq \tau^s$ then

\[ A \subseteq Cl[Int(Cl^s(A))] \subseteq Cl[Int(Cl^s(A))]. \]

That is, $A$ is a $I^s$-open set.

The converse of the last theorem need not be true.

**Example 3.4.** In an ideal topological semigroup $(X, \tau, I)$, where $X = \{a, b, c\}$,

\[ \tau = \{\emptyset, X, \{a, b\}\} \text{ and } I = \{\emptyset, \{a\}, \{b\}, \{a,b\}\} \]

the set $\{a\}$ is a $I^s$-open set but it is not $I - \beta -$ open.

**Theorem 3.5.** Every semi$I^s -$ open set is $I^s$-open set.

**Proof.** Let $A$ be semi$I^s -$ open subset of an ideal topological semigroup $(X, \tau, I)$. Then $A \subseteq Cl[Int(Cl^s(A))]$. Since $\tau \subseteq \tau^s$ then

\[ A \subseteq Cl[Int(Cl^s(A))] \subseteq Cl[Int(Cl^s(A))]. \]

That is, $A$ is a $I^s$-open set.

The converse of the last theorem need not be true.

**Example 3.6.** In an ideal topological semigroup $(X, \tau, I)$, where $X = \{a, b, c\}$,

\[ \tau = \{\emptyset, X\} \text{ and } I = \{\emptyset, \{a\}\} \]

the set $\{b\}$ is a $I^s$-open set but it is not semi$I^s -$ open.

**Theorem 3.7.** Every $I^s$-open set is a $\beta_2^I -$ open set.

**Proof.** Let $A$ be a $I^s$-open subset of an ideal topological semigroup $(X, \tau, I)$. Then $A \subseteq Cl[Int(Cl^s(A))]$. Since $\tau \subseteq \tau^s$ then

\[ A \subseteq Cl[Int(Cl^s(A))] \subseteq Cl[Int(Cl^s(A))]. \]

That is, $A$ is a $\beta_2^I -$ open set.

The converse of the last theorem need not be true.

**Example 3.8.** In an ideal topological semigroup $(X, \tau, I)$, where $X = \{a, b, c\}$,

\[ \tau = \{\emptyset, X\} \text{ and } I = \{\emptyset, \{a\}\} \]

the set $\{a\}$ is a $\beta_2^I -$ open set but it is not $I^s -$ open.

From Theorems 3.3, 3.5 and 3.7, we have the following relation for $I^s$-open set with the other known sets.

[Diagram showing the relations between various types of open sets]

**Theorem 3.9.** A subset $A$ of an ideal topological semigroup $(X, \tau, I)$ is $I^s$-closed set if and only if $Int([Cl^s(Cl^s(A))] \subseteq A$.

**Proof.** $A$ is a $I^s$-closed set in $(X, \tau, I)$ if and only if $X - A$ is a $I^s$-open set in $X$ if and only if

\[ (X - A) \subseteq Cl[Int(Cl^s(X - A))]. \]

if and only if by using Theorem 3.7

\[ (X - A) \subseteq Cl[Int([Cl^s([Cl^s(X - A))]) = Cl[Int(X - Int(Cl^s(A))] = X - Cl[Int(Cl^s(A))] = X - Int(Cl^s(A))]. \]

if and only if $Int([Cl^s(Cl^s(A))] \subseteq A$.

**Theorem 3.10.** Let $(X, \tau, I)$ be a ideal topological semigroup. If $A_\lambda$ is $I^s$-open set for each $\lambda \in \Delta$ then $\cup_{\lambda \in \Delta} A_\lambda$ is $I^s$-open set, where $\Delta$ is an index set.
PROOF. Since $A \lambda$ is $T^\ast$-open set for each $\lambda \in \Delta$ then $A \lambda \subseteq Cl(Int^\ast(Cl'(A \lambda)))$ for each $\lambda \in \Delta$. Then
\[
\bigcup_{\lambda \in \Delta} A \lambda \subseteq \bigcup_{\lambda \in \Delta} Cl(Int^\ast(Cl'(A \lambda))) \\
\subseteq Cl\left[\bigcup_{\lambda \in \Delta} Int^\ast(Cl'(A \lambda))\right] \\
\subseteq Cl\left[\bigcup_{\lambda \in \Delta} Int^\ast(Cl'(A \lambda))\right] \\
\subseteq \bigcup_{\lambda \in \Delta} Cl\left[\bigcup_{\lambda \in \Delta} Int^\ast(Cl'(A \lambda))\right] \\
\subseteq Cl\left[\bigcup_{\lambda \in \Delta} Int^\ast(Cl'(A \lambda))\right].
\]
Hence $\bigcup_{\lambda \in \Delta} A \lambda$ is $T^\ast$-open set.

The intersection of two $T^\ast$-open sets need not be $T^\ast$-open set. In Example 5.6, the sets $A = \{a, b\}$ and $B = \{a, c\}$ are $T^\ast$-open sets but $A \cap B = \{a\}$ is not $T^\ast$-open set.

THEOREM 3.11. Let $(X, \tau, \Delta)$ be an ideal topological semigroup. If $U$ is an open set in $(X, \tau, \Delta)$ and $A$ is $T^\ast$-open set, then $U \cap A$ is $T^\ast$-open set.

PROOF. Since $A$ is $T^\ast$-open set then $A \subseteq Cl[Int^\ast(Cl'(A))]$. Then by Theorems 3.10, $U \cap A \subseteq U \cap Cl[Int^\ast(Cl'(A))] \subseteq Cl[U \cap Int^\ast(Cl'(A))] = Cl[Int^\ast(U \cap Int^\ast(Cl'(A)))] = Cl[Int^\ast(U \cap Cl'(A))] \subseteq Cl[Int^\ast(Cl'(U \cap A))]$. Hence $U \cap A$ is $T^\ast$-open set.

4. $T^\ast$-OPERATORS

For an ideal topological semigroup $(X, \tau, \Delta)$ and a subset $A$ of $X$, the $T^\ast$-closure set of $A$ is defined as the intersection of all $T^\ast$-closed sets containing $A$ and denoted by $T^\ast Cl(A)$. The $T^\ast$-interior set of $A$ is defined as the union of all $T^\ast$-open sets of $X$ contained in $A$ and denoted by $T^\ast Int(A)$. From Theorem 3.10, $T^\ast Cl(A)$ is a $T^\ast$-closed subsets of $X$ and $T^\ast Int(A)$ is $T^\ast$-open subsets of $X$.

REMARK 4.1. For a subset $A \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, it is clear from the definition of $T^\ast Cl(A)$ and $T^\ast Int(A)$ that $A \subseteq T^\ast Cl(A)$ and $T^\ast Int(A) \subseteq A$.

THEOREM 4.2. For a subset $A \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, $T^\ast Cl(A) = A$ if and only if $A$ is a $T^\ast$-closed set.

PROOF. Let $T^\ast Cl(A) = A$. Then from definition of $T^\ast Cl(A)$ and Theorem 3.10, $T^\ast Cl(A)$ is a $T^\ast$-closed set and so $A$ is a $T^\ast$-closed set. Conversely, we have $A \subseteq T^\ast Cl(A)$ by Remark above. Since $A$ is a $T^\ast$-closed set, then it is clear from the definition of $T^\ast Cl(A)$, $T^\ast Cl(A) \subseteq A$. Hence $A = T^\ast Cl(A)$.

THEOREM 4.3. For a subset $A \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, $T^\ast Int(A) = A$ if and only if $A$ is a $T^\ast$-open set.

PROOF. Let $T^\ast Int(A) = A$. Then from definition of $T^\ast Int(A)$ and Theorem 3.10, $T^\ast Int(A)$ is a $T^\ast$-open set and so $A$ is a $T^\ast$-open set. Conversely, since $A$ is a $T^\ast$-open set and $T^\ast Int(A) \subseteq A$ by Remark 4.1, then $T^\ast Int(A) = A$.

THEOREM 4.4. For a subset $A \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, $x \in T^\ast Cl(A)$ if and only if for all $T^\ast$-open set $U$ containing $x$, $U \cap A \neq \emptyset$.

PROOF. Let $x \in T^\ast Cl(A)$ and $U$ be a $T^\ast$-open set containing $x$. If $U \cap A = \emptyset$ then $A \subseteq X - U$. Since $X - U$ is a $T^\ast$-closed set containing $A$, then $T^\ast Cl(A) \subseteq X - U$ and so $x \in T^\ast Cl(A) \subseteq X - U$. Hence this is contradiction, because $x \in U$. Therefore $U \cap A \neq \emptyset$. Conversely, let $x \notin T^\ast Cl(A)$. Then $x \notin T^\ast Cl(A)$ is a $T^\ast$-open set containing $x$. Hence by hypothesis, $[X - T^\ast Cl(A)] \cap A \neq \emptyset$.

THEOREM 4.5. For a subset $A \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, $x \in T^\ast Int(A)$ if and only if $x$ is in $T^\ast$-open set such that $x \in U \subseteq A$.

PROOF. Let $x \in T^\ast Int(A)$ and take $U = T^\ast Int(A)$. Then by Theorem 3.10, we have that $U$ is a $T^\ast$-open set and by Remark 4.1, $x \in U \subseteq A$. Conversely, Let there is $T^\ast$-open set $U$ such that $x \in U \subseteq A$. Then by definition of $T^\ast Int(A)$, $x \in U \subseteq T^\ast Int(A)$.

THEOREM 4.6. For a subsets $A, B \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, the following hold:

1) If $A \subseteq B$ then $T^\ast Cl(A) \subseteq T^\ast Cl(B)$;
2) $T^\ast Cl(A) \cup T^\ast Cl(B) \subseteq T^\ast Cl(A \cup B)$;
3) $T^\ast Cl(A \cap B) \subseteq T^\ast Cl(A) \cap T^\ast Cl(B)$;
4) $T^\ast Cl(A) \subseteq T^\ast Cl(B)$.

PROOF. Let $x \in T^\ast Cl(A)$. Then by Theorem 4.4, for all $T^\ast$-open set $U$ containing $x$, $U \cap A \neq \emptyset$. Since $A \subseteq B$, then $U \cap B \neq \emptyset$. Hence $x \in T^\ast Cl(B)$.

2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part (1), we get $T^\ast Cl(A) \subseteq T^\ast Cl(A \cup B)$ and $T^\ast Cl(B) \subseteq T^\ast Cl(A \cup B)$. Then $T^\ast Cl(A) \cup T^\ast Cl(B) \subseteq T^\ast Cl(A \cup B)$.

3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by part (1), we get $T^\ast Cl(A \cap B) \subseteq T^\ast Cl(A)$ and $T^\ast Cl(A \cap B) \subseteq T^\ast Cl(B)$. Then $T^\ast Cl(A \cap B) \subseteq T^\ast Cl(A \cap B)$. Hence $T^\ast Cl(A \cap B) = T^\ast Cl(A \cap B)$.

4. It is clear from Theorem 4.4 and from every open set $U$ is $T^\ast$-open set.

In the last theorem $T^\ast Cl(A \cup B) \neq T^\ast Cl(A) \cup T^\ast Cl(B)$.

EXAMPLE 4.7. In Example 3.3, the sets $A = \{a\}$ and $B = \{b\}$ are $T^\ast$-closed sets in $(X, \tau, \Delta)$. Then $T^\ast Cl(A) \cup T^\ast Cl(B) = A \cup B = \{a, b\}$ and

$T^\ast Cl(A \cup B) = T^\ast Cl((a, b)) = X$.

THEOREM 4.8. For a subsets $A, B \subseteq X$ of an ideal topological semigroup $(X, \tau, \Delta)$, the following hold:

1) If $A \subseteq \emptyset$ then $T^\ast Int(A) \subseteq T^\ast Int(B)$;
2) $T^\ast Int(A) \cup T^\ast Int(B) \subseteq T^\ast Int(A \cup B)$;
3) $T^\ast Int(A \cap B) \subseteq T^\ast Int(A) \cap T^\ast Int(B)$;
4) $T^\ast Int(A) \subseteq T^\ast Int(B)$.

PROOF. Similar for Theorem 4.6.

In the last theorem $T^\ast Int(A \cap B) \neq T^\ast Int(A) \cap T^\ast Int(B)$.

EXAMPLE 4.9. In Example 3.3, the sets $A = \{b, c\}$ and $B = \{a, c\}$ are $T^\ast$-open sets in $(X, \tau, \Delta)$. Then $T^\ast Int(A) \cap T^\ast Int(B) = A \cap B = \{c\}$ and

$T^\ast Int(A \cap B) = T^\ast Int(\{c\}) = \emptyset$. 

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THEOREM 4.10. For a subset $A \subseteq X$ of an ideal topological semigroup $(X, \tau, I)$, the following hold:

1. $\tau \text{Int}(X - A) = X - \tau \text{Cl}(A)$;
2. $\tau \text{Cl}(X - A) = X - \tau \text{Int}(A)$.

**Proof.** 1. Since $A \subseteq \tau \text{Cl}(A)$ then $X - \tau \text{Cl}(A) \subseteq (X - A)$. Since $X - \tau \text{Cl}(A)$ is a $\tau^*$-open set in $(X, \tau, I)$ then $X - \tau \text{Cl}(A) = \tau \text{Int}[X - \tau \text{Cl}(A)] \subseteq \tau \text{Int}(X - A)$.

For the other side, let $x \in \tau \text{Int}(X - A)$. Then there is $\tau^*$-open set $U$ such that $x \in U \subseteq X - A$. Then $X - U$ is a $\tau^*$-closed set containing $A$ and $x \notin X - U$. Hence $x \notin \tau \text{Cl}(A)$, that is, $x \in X - \tau \text{Cl}(A)$. 2. Since $\tau \text{Int}(A)$ is $\tau^*$-closed set in $A$ then $X - A \subseteq X - \tau \text{Int}(A)$. Since $X - \tau \text{Cl}(A)$ is a $\tau^*$-closed set in $(X, \tau, I)$ then $\tau \text{Cl}(X - A) \subseteq \tau \text{Cl}[X - \tau \text{Int}(A)] = X - \tau \text{Int}(A)$.

For the other side, let $x \in \tau \text{Cl}(X - A)$ then there is $\tau^*$-open set $U$ such that $x \in U \subseteq X - A$. Then $X - U$ is a $\tau^*$-closed set containing $x$ and $x \notin X - U$. Hence $x \notin \tau \text{Cl}(A)$, that is, $x \in X - \tau \text{Cl}(A)$. □

**THEOREM 4.11.** For a subset $A \subseteq X$ of an ideal topological space $(X, \tau, I)$, the following hold:

1. If $I$ is an open set in $X$ then $\tau \text{Cl}(A) \cap I \subseteq \tau \text{Cl}(A \cap I)$.
2. If $I$ is a closed set then $\tau \text{Int}(A \cup I) \subseteq \tau \text{Int}(A) \cup I$.

**Proof.** 1. Let $x \in \tau \text{Cl}(A) \cap I$. Then there is $\tau^*$-open set in $(X, \tau, I)$ containing $x$. By Theorem 3.11, $V \cap I$ is $\tau^*$-open set containing $x$. Since $x \in \tau \text{Cl}(A)$ then by Theorem 4.4, $(V \cap I) \cap A \neq \emptyset$. This implies, $V \cap (I \cap A) \neq \emptyset$. Hence by Theorem 4.4, $x \in \tau \text{Cl}(A \cap I)$. That is, $\tau \text{Cl}(A) \cap I \subseteq \tau \text{Cl}(A \cap I)$. 1. Since $I$ is a closed set $X$ then by the part (1) and Theorem 4.10,

$$X - \tau \text{Int}(A \cup I) = X - \tau \text{Int}(A) \cup I.$$  2. From the definition of $\tau \text{Cl}(A)$.

3. By Theorem (4.10),

$$\tau \text{Cl}(A) = \tau \text{Cl}(A - \tau \text{Int}(A)) = \tau \text{Cl}(A) \cap (X - \tau \text{Int}(A)) = \tau \text{Cl}(A) \cap \tau \text{Cl}(X - A).$$

**COROLLARY 4.13.** For a subset $A \subseteq X$ of an ideal topological space $(X, \tau, I)$, $\tau \text{Cl}(A)$ is $\tau^*$-closed set in $(X, \tau, I)$.

**Proof.** By Theorem 4.10 and the part (3) of the last Theorem.

**THEOREM 4.14.** For a subset $A \subseteq X$ of an ideal topological space $(X, \tau, I)$, the following hold:

1. $A$ is a $\tau^*$-open if and only if $\tau \text{Cl}(A) \cap A = \emptyset$;
2. $A$ is a $\tau^*$-closed if and only if $\tau \text{Cl}(A) \subseteq A$;
3. $A$ is both $\tau^*$-open and $\tau^*$-closed if and only if $\tau \text{Cl}(A) = \emptyset$.

**Proof.** 1. Let $A$ be a $\tau^*$-open set. Then $\tau \text{Int}(A) = A$. Then by Theorem 4.13,

$$\tau \text{Cl}(A) \cap A = \tau \text{Cl}(A) \cap \tau \text{Int}(A) = \emptyset.$$ Conversely, suppose that $\tau \text{Cl}(A) \cap A = \emptyset$. Then $A - \tau \text{Int}(A) = [A \cap \tau \text{Cl}(A)] \cap [A - \tau \text{Int}(A)] = A \cap (\tau \text{Cl}(A) - \tau \text{Int}(A)) = A \cap \tau \text{Cl}(A) = \emptyset$.

That is, $\tau \text{Int}(A) = A$. Hence $A$ is a $\tau^*$-open set. 2. Let $A$ be a $\tau^*$-closed set. Then $\tau \text{Cl}(A) = A$. Then

$$\tau \text{Cl}(A) = \tau \text{Cl}(A - \tau \text{Int}(A)) = A - \tau \text{Int}(A).$$ Conversely, suppose that $\tau \text{Cl}(A) \subseteq A$. Then by Theorem 4.12,

$$\tau \text{Cl}(A) = \tau \text{Cl}(A) \cup \tau \text{Gamma}(A) \subseteq \tau \text{Int}(A) \cup A \subseteq A.$$ That is, $\tau \text{Cl}(A) = A$. Hence $A$ is $\tau^*$-closed set. 3. Let $A$ be both $\tau^*$-closed set and $\tau^*$-open set. Then $\tau \text{Cl}(A) = A = \tau \text{Int}(A)$. Therefore,

$$\tau \text{Cl}(A) = \tau \text{Cl}(A) - \tau \text{Int}(A) = A - A = \emptyset.$$ Conversely, suppose that $\tau \text{Gamma}(A) = \emptyset$. Then $\tau \text{Cl}(A) - \tau \text{Int}(A) = \emptyset$. Since

$$\tau \text{Int}(A) \subseteq \tau \text{Cl}(A).$$ Then

$$\tau \text{Cl}(A) = \tau \text{Int}(A).$$ Since

$$\tau \text{Int}(A) \subseteq A \subseteq \tau \text{Cl}(A).$$ Then $\tau \text{Cl}(A) = A = \tau \text{Int}(A)$.

That is, $\tau \text{Cl}(A) = A$. Hence $A$ is both $\tau^*$-closed set and $\tau^*$-open set. □

**5. RELATIVE PROPERTY**

By bitopological semigroup we mean a triple $(X, \tau, \rho)$ consists two topological semigroups $(X, \tau)$ and $(X, \rho)$. A subset $A \subseteq X$ is said to be $\tau \rho$-open set in a bitopological semigroup $(X, \tau, \rho)$ if $A \subseteq \tau \rho \text{Int}(\tau \rho \text{Cl}(A))$. The complement of $\tau \rho$-open set is said to be $\tau \rho$-closed set.
THEOREM 5.1. A subset $A \subseteq X$ is a $T^*$-open set in ideal topological semigroup $(X, \tau, I)$, if and only if it is a $\tau^*\tau$-open set in bitopological semigroup $(X, \tau, \tau)$. 

PROOF. It is clear from the definitions and $Cl'(A) = \tau Cl(A)$.  

THEOREM 5.2. A subset $A$ of a bitopological semigroup $(X, \tau, \rho)$ is $\tau\rho$-closed set, if and only if $\tau Int[Cl'(Int(A))] \subseteq A$. 

PROOF. $A$ is a $\tau\rho$-closed set in $X$, if and only if $X - A$ is a $\tau\rho$-open set in $X$, if and only if 

$$(X - A) \subseteq Cl[(\rho Int(Cl(X - A))],$$

and if only by using Theorem (4.10).

THEOREM 5.3. Let $Y$ be an open subset of an ideal topological semigroup $(X, \tau, I)$. If $A$ is a $T^*$-open set in $(X, \tau, I)$ then $A \cap Y$ is a $\tau|_Y \tau^*\tau$-open set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$. 

PROOF. Since $A$ is $T^*$-open set in $(X, \tau, I)$, then $A \subseteq Cl[Cl'(Cl'(A))]$. By Theorems (2.1) and (4.10), we obtain

$$(A \cap Y) \subseteq Cl[Cl'(Cl'(A))] \cap Y$$

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Hence $A \cap Y$ is a $\tau|_Y \tau^*\tau$-open set in $(Y, \tau|_Y, \tau^*|_Y)$.  

COROLLARY 5.4. Let $Y$ be an open subset of an ideal topological semigroup $(X, \tau, I)$. If $A$ is a $T^*$-closed set in $(X, \tau, I)$ then $A \cap Y$ is a $\tau|_Y \tau^*\tau$-closed set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$. 

PROOF. Let $A$ be a $T^*$-closed set in $(X, \tau, I)$. Then $X - A$ is a $T^*$-open set in $(X, \tau, I)$. By the last theorem, $Y - (X - A) = Y - Y \cap (X - A) = Y \cap (X - Y) \cap A = A \cap Y$ is a $\tau|_Y \tau^*\tau$-closed set in $(Y, \tau|_Y, \tau^*|_Y)$. 

THEOREM 5.5. Let $Y$ be an open subset of an ideal topological semigroup $(X, \tau, I)$. If $A$ is a $\tau|_Y \tau^*\tau$-open set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$, then $A$ is $T^*$-open set $I$ in $(X, \tau, I)$. 

PROOF. Since $A$ is a $\tau|_Y \tau^*\tau$-open set in $(Y, \tau|_Y, \tau^*|_Y)$ then $A \subseteq Cl[Cl'(Cl'(Y))]$. Then by Theorems (2.1), (4.10).

$$(A \cap Y) \subseteq Cl[Cl'(Cl'(Y))] \cap Y$$

Hence $A$ is a $T^*$-open set in $I$ in $(X, \tau, I)$.

COROLLARY 5.6. Let $Y$ be an open subset of an ideal topological semigroup $(X, \tau, I)$. If $A$ is a $\tau|_Y \tau^*\tau$-closed set in bitopological space $(Y, \tau|_Y, \tau^*|_Y)$, then $A$ is $T^*$-closed set in $(Y, \tau, I)$. 

THEOREM 5.7. Let $Y$ be an open subset of an ideal topological semigroup $(X, \tau, I)$ and $A$ be a subset of $Y$. Then $\tau Cl'(A) = \tau Cl(A) \cap Y$. 

PROOF. Let $x \in \tau Cl'(A)$ and $I$ be a $T^*$-open set in $X$ containing $x$. By Theorem (5.1) if $Y \cap \tau Cl'(A)$ is a $\tau^*$-open set in bitopological space $(Y, \tau|_Y, \tau^*|_Y)$ containing $x$ and since $x \in \tau Cl'(A)$, then $I \cap A = (I \cap Y) \cap A \neq \emptyset$. Hence by Theorem (4.4), $x \in \tau Cl(A)$ and since $x \in Y$, this implies $x \in \tau Cl(A) \cap Y$. That is, $\tau Cl'(A) \subseteq \tau Cl(A) \cap Y$. On the other side, let $x \in \tau Cl(A) \cap Y$ and $O$ be a $\tau^*$-open set in bitopological space $(Y, \tau|_Y, \tau^*|_Y)$ containing $x$. By Corollary (5.5), $O$ is $T^*$-open set in $(X, \tau, I)$. Since $x \in \tau Cl(A)$, then $O \cap A \neq \emptyset$. That is, $x \in \tau Cl'(A)$. Hence $\tau Cl(A) \cap Y \subseteq \tau Cl'(A)$. 

THEOREM 5.8. Let $(X, \tau, \rho)$ and $(Y, \rho, \rho')$ be two ideal topological semigroups. A subset $A \times B \subseteq \tau \times \rho \times B \times \rho'$-open set in bitopological semigroup $(X, \tau, \rho) \times (Y, \rho, \rho')$ if and only if $A$ is a $T^*$-open set in $(X, \tau, \rho)$ and $B$ is a $T^*$-open set in $(Y, \rho, \rho')$. 

PROOF. It is clear that, 

$$\tau \rho Cl[\tau \rho \rho' Cl(\tau \rho Cl(A \times B))] = \tau \rho Cl[\tau \rho \rho' Cl(\tau \rho Cl(A \times B))] = \tau \rho Cl[\tau \rho \rho' Cl(\tau \rho Cl(A \times B))] = \tau \rho Cl[\tau \rho \rho' Cl(\tau \rho Cl(A \times B))]$$

We have $A \times B \subseteq C \times D$, if and only if $A \subseteq C$ and $B \subseteq D$.  

6. REFERENCES