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On $\mathcal{I}^s$-Open Sets in Ideal Topological Semigroups

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ABSTRACT

In this paper, we introduce and investigate a new class of $semi^* - \mathcal{I}-$open sets, called $\mathcal{I}^s-$open sets in ideal topological semigroups. This class is consider strong form of $\beta^*_{\mathcal{I}}-$*open sets* and weak form of of $semi^* - \mathcal{I}-$open sets and $\beta - \mathcal{I}-$open sets. The interior, closer and frontier operators are studied with the relative property via $\mathcal{I}^s-$open sets.

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Keywords

open set;ideal topological space, topological semigroup.

1. INTRODUCTION

The notion of ideal topological spaces is introduced by Kuratowski, [9]. Many researcher studid about the ideal topological spaces. An idea $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following conditions:
1- if $A \in \mathcal{I}$ and $B \in A$ then $B \in \mathcal{I}$,
2- if $A \in \mathcal{I}$ and $B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.
Applications to various fields were further investigated by Jankovic and Hamlett [1], Dontchev [8] and Arenas et al [6]. An ideal topological space is a topological space (X, τ) with an ideal $\mathcal{I}$ on X, and is denoted by $(X_*, \tau, \mathcal{I})$.

Under the notion of ideal topological spaces, several mathematical researcher introduced the new forms of $\mathcal{I}^s-$ open sets such as $\beta - \mathcal{I}-$open sets by Hatir and Noiri in 2002, [4] and $semi^* - \mathcal{I}-$open sets by Ekici and Noiri in 2012, [3], are the weaker forms. Also $\beta^*_{\mathcal{I}}-$open sets is stronge by Ekici in 2011, [2].

This paper is organized as follows: Section 3 introduces the concept of $\mathcal{I}^s-$ open sets in ideal topological semigroups with its relationship among other known sets. Section 4 introduces the concepts of $\mathcal{I}^s-$interior operator, $\mathcal{I}^s-$clouser operator and $\mathcal{I}^s-$frontior operator. Section 5 studies the relative property via $\mathcal{I}^s-$open sets.

2. PRELIMINARIES

For a topological space (X, τ) and $A \subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

THEOREM 2.1. [7] For a topological space (X, τ) and $A, B \subseteq X$, if B is an open set in X then $Cl(A) \cap B \subseteq Cl(A \cap B)$.

THEOREM 2.2. [7] For a topological space (X, τ),

(1) $Cl(X - A) = X - Int(A)$ for all $A \subseteq X$.
(2) $Int(X - A) = X - Cl(A)$ for all $A \subseteq X$.

In the ideal topological space $(X, \tau, \mathcal{I})$, $A^*(\mathcal{I})$ is defined by:

$A^*(\mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I}$ for each open neighborhood U of $x\}$

is called the local function of A with respect to $\mathcal{I}$ and τ, [9]. When there is no chance for confusion $A^*(\mathcal{I})$ is denoted by A^*. For every ideal topological space $(X, \tau, \mathcal{I})$, there exists a topology τ^* finer than τ, generated by the base

$$\beta(\mathcal{I}, \tau) = \{U - I : U \in \tau \text{ and } I \in \mathcal{I}\}.$$

Observe additionally that $Cl^*(A) = A \cup A^*$, [10] defines a Kuratowski closure operator for τ^*. $Int^*(A)$ will denote the interior of A in (X, τ^*). If $\mathcal{I}$ is an ideal on topological space (X, τ), then $(X, \tau, \mathcal{I})$ is called an ideal topological space.

THEOREM 2.3. [5] Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then for $A, B \subseteq X$, the following properties hold:

(1) $A \subseteq B$ implies that $A^* \subseteq B^*$;
(2) $G \in \tau$ implies that $G \cap A^* \subseteq (G \cap A)^*$;
(3) $A^* = Cl(A^*) \subseteq Cl(A)$;
(4) $(A \cup B)^* = A^* \cup B^*$;
(5) $(A^*)^* \subseteq A^*$.

DEFINITION 2.4. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called:

(1) semi$-\mathcal{I}-$open set, [4] if $A \subseteq Cl^*(Int(A))$;
(2) $\alpha^*_{\mathcal{I}}-$open set, [4] if $A \subseteq Int(Cl^*(Int(A)))$;
(3) $\beta - \mathcal{I}-$open set, [4] if $A \subseteq Cl(Int(Cl^*(A)))$;
(4) $semi^* - \mathcal{I}-$open set, [3] if $A \subseteq Cl(Int^*(A))$.

DEFINITION 2.5. [2] A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is called $\beta^*_{\mathcal{I}}-$*open set* if $A \subseteq Cl(Int^*(Cl(A)))$.

By topological semigroup (X_*, τ), we mean a topological space (X, τ) which is space with associated multiplication $* : X \times X \to X$ such that $*$ is continuous function from the product space $X \times X$ into X. By ideal topological semigroup $(X_*, \tau, \mathcal{I})$, we mean an ideal topological space $(X, \tau, \mathcal{I})$ with associated multiplication $* : X \times X \to X$ such that $*$ is continuous function from the product space $X \times X$ into X. A pair $(Y, \circ)$ is called a $\mathcal{I}-$*subspace* of ideal topological semigroup $(X_\circ, \tau, \mathcal{I})$ if Y is a subspace of X and the continuous function $\circ$ takes the product $Y \times Y$ into Y

and $\circ(x,y) = *(x,y)$ for all $x, y \in Y$. We denote the operation of any $\mathcal{I}-$subspace with the same symbol used for the operation on the ideal topological semigroup under consideration. For any ideal topological space $(X, \tau, \mathcal{I})$, we mean by $(X_\pi, \tau, \mathcal{I})$ the ideal topological semigroup with operration $\pi : X \times X \longrightarrow X$, where $\pi(x,y) = x$ or $\pi(x,y) = y$ for all $x, y \in X$.

3. $\mathcal{I}^S$-OPEN SETS

DEFINITION 3.1. A subset A of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$ is said to be $\mathcal{I}^s-$*open set* if $A \subseteq Cl[Int^*(Cl^*(A))]$. The complement of $\mathcal{I}^s-$open set is said to be $\mathcal{I}^s-$closed set.

For an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, The set of all $\mathcal{I}^s-$open sets in X denoted by $\mathcal{I}^s O(X_*, \tau)$ and the set of all $\mathcal{I}^s-$closed sets in X denoted by $\mathcal{I}^s C(X_*, \tau)$.

EXAMPLE 3.2. In an ideal topological semigroup $(X_\pi, \tau, \mathcal{I})$, where $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.
Note that

$$\mathcal{I}^s O(X, \tau) = P(X) \text{ and } \mathcal{I}^s C(X, \tau) = P(X).$$

THEOREM 3.3. Every $\beta - \mathcal{I}-$open set is $\mathcal{I}^s-$open set.

PROOF. Let A be $\beta - \mathcal{I}-$open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. Then $A \subseteq Cl(Int(Cl^*(A)))$. Since $\tau \subseteq \tau^*$ then

$$A \subseteq Cl(Int(Cl^*(A))) \subseteq Cl[Int^*(Cl^*(A))].$$

That is, A is a $\mathcal{I}^s-$open set. □

The converse of the last theorem need not be true.

EXAMPLE 3.4. In an ideal topological semigroup $(X_\pi, \tau, \mathcal{I})$, where $X = \{a, b, c\}$,

$$\tau = \{\emptyset, X, \{a, b\}\} \text{ and } \mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

the set $\{a\}$ is a $\mathcal{I}^s-$open set but it is not $\mathcal{I} - \beta-$open.

THEOREM 3.5. Every $semi^* - \mathcal{I}-$open set is $\mathcal{I}^s-$open set.

PROOF. Let A be $semi^* - \mathcal{I}-$open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. Then $A \subseteq Cl(Int^*(A))$. Since $\tau \subseteq \tau^*$, then

$$A \subseteq Cl(Int^*(A)) \subseteq Cl[Int^*(Cl^*(A))].$$

That is, A is a $\mathcal{I}^s-$open set. □

The converse of the last theorem need not be true.

EXAMPLE 3.6. In an ideal topological semigroup $(X_\pi, \tau, \mathcal{I})$, where $X = \{a, b, c\}$,

$$\tau = \{\emptyset, X\} \text{ and } \mathcal{I} = \{\emptyset, \{a\}\}$$

the set $\{b\}$ is a $\mathcal{I}^s-$open set but it is not $semi^* - \mathcal{I}-$open.

THEOREM 3.7. Every $\mathcal{I}^s-$open set is a $\beta^*_\mathcal{I}-$open set.

PROOF. Let A be a $\mathcal{I}^s-$open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. Then $A \subseteq Cl[Int^*(Cl^*(A))]$. Since $\tau \subseteq \tau^*$, then

$$A \subseteq Cl[Int^*(Cl^*(A))] \subseteq Cl[Int^*(Cl(A))].$$

That is, A is a $\beta^*_\mathcal{I}-$open set. □

The converse of the last theorem need not be true.

EXAMPLE 3.8. In an ideal topological semigroup $(X_\pi, \tau, \mathcal{I})$, where $X = \{a, b, c\}$,

$$\tau = \{\emptyset, X\} \text{ and } \mathcal{I} = \{\emptyset, \{a\}\}$$

the set $\{a\}$ is a $\beta^*_\mathcal{I}-$open set but it is not $\mathcal{I}^s-$open.

From Theorems (3.3), (3.5) and (3.7), we have the following relation for $\mathcal{I}^s$-open set with the other known sets.

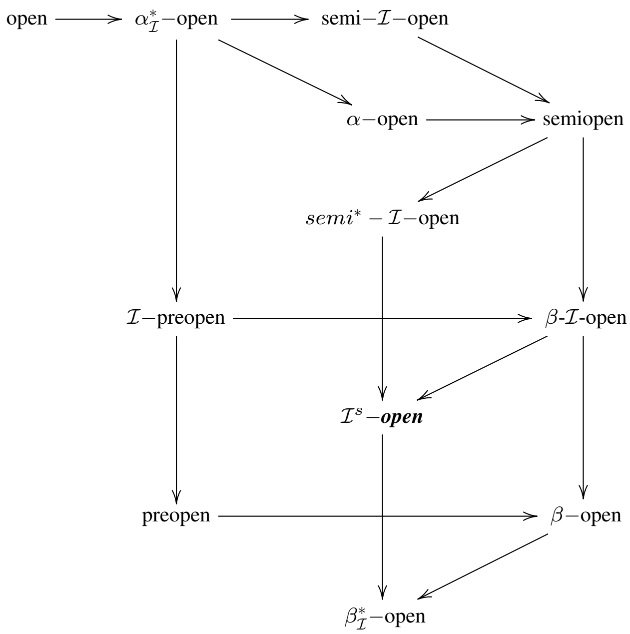

Fig. 1. Relation for sets 1

THEOREM 3.9. A subset A of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$ is $\mathcal{I}^s-$closed set if and only if $Int[Cl^*(Int^*(A))] \subseteq A$.

PROOF. A is a $\mathcal{I}^s-$closed set in $(X_*, \tau, \mathcal{I})$ if and only if $X - A$ is a $\mathcal{I}^s-$ open set in X if and only if

$$(X - A) \subseteq Cl[Int^*(Cl^*(X - A))],$$

if and only if by using Theorem (2.2),

$$\begin{aligned}(X - A) &\subseteq Cl[Int^*(Cl^*(X - A))]\\ &= Cl[Int^*(X - Int^*(A))]\\ &= Cl[X - Cl^*(Int^*(A))]\\ &= X - Int[Cl^*(Int^*(A))]\\ &= X - Int[Cl^*(Int^*(A))].\end{aligned}$$

if and only if $Int[Cl^*(Int^*(A))] \subseteq A$. □

THEOREM 3.10. Let $(X_*, \tau, \mathcal{I})$ be a ideal topological semigroup. If A_λ is $\mathcal{I}^s-$open set for each $\lambda \in \Delta$ then $\cup_{\lambda \in \Delta} A_\lambda$ is $\mathcal{I}^s-$open set, where Δ is an index set.

PROOF. Since A_λ is $\mathcal{I}^s-$open set for each $\lambda \in \Delta$ then $A_\lambda \subseteq Cl[Int^*(Cl^*(A_\lambda))]$ for each $\lambda \in \Delta$. Then

$$\begin{aligned}
\cup_{\lambda\in\Delta} A_\lambda &\subseteq \cup_{\lambda\in\Delta} Cl[Int^*(Cl^*(A_\lambda))] \\
&\subseteq Cl[\cup_{\lambda\in\Delta} Int^*(Cl^*(A_\lambda))] \\
&\subseteq Cl[Int^*(\cup_{\lambda\in\Delta} Cl^*(A_\lambda))] \\
&\subseteq Cl[Int^*(\cup_{\lambda\in\Delta}(A_\lambda \cup (A^*_\lambda))] \\
&\subseteq Cl[Int^*((\cup_{\lambda\in\Delta} A_\lambda) \cup (\cup_{\lambda\in\Delta} A^*_\lambda))] \\
&\subseteq Cl[Int^*(\cup_{\lambda\in\Delta} A_\lambda) \cup (\cup_{\lambda\in\Delta}(A^*_\lambda))] \\
&= Cl[Int^*(Cl^*(\cup_{\lambda\in\Delta} A_\lambda))].
\end{aligned}$$

Hence $\cup_{\lambda\in\Delta} A_\lambda$ is $\mathcal{I}^s-$open set. □

The intersection of two $\mathcal{I}^s-$open sets need not be $\mathcal{I}^s-$open set. In Example (3.6), the sets $A = \{a, b\}$ and $B = \{a, c\}$ are $\mathcal{I}^s-$open sets but $A \cap B = \{a\}$ is not $\mathcal{I}^s-$open set.

THEOREM 3.11. Let $(X_*, \tau, \mathcal{I})$ be an ideal topological semigroup. If U is an open set in (X_*, τ) and A is $\mathcal{I}^s-$open set, then $U \cap A$ is $\mathcal{I}^s-$open set.

PROOF. Since A is $\mathcal{I}^s-$open set then $A \subseteq Cl[Int^*(Cl^*(A))]$. Then by Theorems (2.1),

$$\begin{aligned}
U \cap A &\subseteq U \cap Cl[Int^*(Cl^*(A))] \subseteq Cl[U \cap Int^*(Cl^*(A))] \\
&= Cl[Int^*(U) \cap Int^*(Cl^*(A))] \\
&= Cl[Int^*(U \cap Cl^*(A))] \\
&\subseteq Cl[Int^*(Cl^*(U \cap A))]
\end{aligned}$$

Hence $U \cap A$ is $\mathcal{I}^s-$open set. □

4. $\mathcal{I}^S$-OPERATORS

For an ideal topological semigroup $(X_*, \tau, \mathcal{I})$ and a subset A of X, the $\mathcal{I}^s-$closure set of A is defined as the intersection of all $\mathcal{I}^s-$closed sets containing A and is denoted by ${}_{\mathcal{I}^s}Cl(A)$. The $\mathcal{I}^s-$interior set of A is defined as the union of all $\mathcal{I}^s-$open sets of X contained in A and is denoted by ${}_{\mathcal{I}^s}Int(A)$. From Theorem (3.10), ${}_{\mathcal{I}^s}Cl(A)$ is a $\mathcal{I}^s-$closed subsets of X and ${}_{\mathcal{I}^s}Int(A)$ is $\mathcal{I}^s-$open subsets of X.

REMARK 4.1. For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, it is clear from the definition of ${}_{\mathcal{I}^s}Cl(A)$ and ${}_{\mathcal{I}^s}Int(A)$ that $A \subseteq {}_{\mathcal{I}^s}Cl(A)$ and ${}_{\mathcal{I}^s}Int(A) \subseteq A$.

THEOREM 4.2. For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, ${}_{\mathcal{I}^s}Cl(A) = A$ if and only if A is a $\mathcal{I}^s-$closed set.

PROOF. Let ${}_{\mathcal{I}^s}Cl(A) = A$. Then from definition of ${}_{\mathcal{I}^s}Cl(A)$ and Theorem (3.10), ${}_{\mathcal{I}^s}Cl(A)$ is a $\mathcal{I}^s$-closed set and so A is a $\mathcal{I}^s-$closed set. Conversely, we have $A \subseteq {}_{\mathcal{I}^s}Cl(A)$ by Remark above. Since A is a $\mathcal{I}^s-$closed set, then it is clear from the definition of ${}_{\mathcal{I}^s}Cl(A)$, ${}_{\mathcal{I}^s}Cl(A) \subseteq A$. Hence $A = {}_{\mathcal{I}^s}Cl(A)$. □

THEOREM 4.3. For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, ${}_{\mathcal{I}^s}Int(A) = A$ if and only if A is a $\mathcal{I}^s-$open set.

PROOF. Let ${}_{\mathcal{I}^s}Int(A) = A$. Then from definiton of ${}_{\mathcal{I}^s}Int(A)$ and Theorem (3.10), ${}_{\mathcal{I}^s}Int(A)$ is a $\mathcal{I}^s-$ open and so A is a $\mathcal{I}^s-$ open set. Conversely,since A is $\mathcal{I}^s-$ open set and ${}_{\mathcal{I}^s}Int(A) \subseteq A$ by Temark (4.1). Then ${}_{\mathcal{I}^s}Int(A) = A$. □

THEOREM 4.4. For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, $x \in {}_{\mathcal{I}^s}Cl(A)$ if and only if for all $\mathcal{I}^s-$open set U containing x, $U \cap A \neq \emptyset$.

PROOF. Let $x \in {}_{\mathcal{I}^s}Cl(A)$ and U be a $\mathcal{I}^s-$open set containing x. If $U \cap A = \emptyset$ then $A \subseteq X - U$. Since $X - U$ is a $\mathcal{I}^s-$closed set containing A, then ${}_{\mathcal{I}^s}Cl(A) \subseteq X - U$ and so $x \in {}_{\mathcal{I}^s}Cl(A) \subseteq X - U$. Hence this is contradiction, because $x \in U$. Therefore $U \cap A \neq \emptyset$. Conversely, Let $x \notin {}_{\mathcal{I}^s}Cl(A)$. Then $X - {}_{\mathcal{I}^s}Cl(A)$ is a $\mathcal{I}^s-$open set containing x. Hence by hypothesis, $[X - {}_{\mathcal{I}^s}Cl(A)] \cap A \neq \emptyset$. But this is contradiction, because $X - {}_{\mathcal{I}^s}Cl(A) \subseteq X - A$ by Remark (4.1). □

THEOREM 4.5. For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, $x \in {}_{\mathcal{I}^s}Int(A)$ if and only if there is $\mathcal{I}^s-$open set U such that $x \in U \subseteq A$.

PROOF. Let $x \in {}_{\mathcal{I}^s}Int(A)$ and take $U = {}_{\mathcal{I}^s}Int(A)$. Then by Theorem (3.10) and definition of ${}_{\mathcal{I}^s}Int(A)$ we get that U is a $\mathcal{I}^s-$open set and by Remark (4.1), $x \in U \subseteq A$. Conversely, Let there is $\mathcal{I}^s-$open set U such that $x \in U \subseteq A$. Then by definition of ${}_{\mathcal{I}^s}Int(A)$, $x \in U \subseteq {}_{\mathcal{I}^s}Int(A)$. □

THEOREM 4.6. For a subsets $A, B \subseteq X$ of an ideal topological semigroup $(X, \tau, \mathcal{I})$, the following hold:

(1) If $A \subseteq B$ then ${}_{\mathcal{I}^s}Cl(A) \subseteq {}_{\mathcal{I}^s}Cl(B)$;
(2) ${}_{\mathcal{I}^s}Cl(A) \cup {}_{\mathcal{I}^s}Cl(B) \subseteq {}_{\mathcal{I}^s}Cl(A \cup B)$;
(3) ${}_{\mathcal{I}^s}Cl(A \cap B) \subseteq {}_{\mathcal{I}^s}Cl(A) \cap {}_{\mathcal{I}^s}Cl(B)$;
(4) ${}_{\mathcal{I}^s}Cl(A) \subseteq Cl(A)$.

PROOF. 1. Let $x \in {}_{\mathcal{I}^s}Cl(A)$. Then by Theorem (4.4), for all $\mathcal{I}^s-$open set U containing x, $U \cap A \neq \emptyset$. Since $A \subseteq B$, then $U \cap B \neq \emptyset$. Hence $x \in {}_{\mathcal{I}^s}Cl(B)$. That is, ${}_{\mathcal{I}^s}Cl(A) \subseteq {}_{\mathcal{I}^s}Cl(B)$.
2. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, then by part (1), we get ${}_{\mathcal{I}^s}Cl(A) \subseteq {}_{\mathcal{I}^s}Cl(A \cup B)$ and ${}_{\mathcal{I}^s}Cl(B) \subseteq {}_{\mathcal{I}^s}Cl(A \cup B)$. Then ${}_{\mathcal{I}^s}Cl(A) \cup {}_{\mathcal{I}^s}Cl(B) \subseteq {}_{\mathcal{I}^s}Cl(A \cup B)$.
3. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then by part (1), we get ${}_{\mathcal{I}^s}Cl(A \cap B) \subseteq {}_{\mathcal{I}^s}Cl(A)$ and ${}_{\mathcal{I}^s}Cl(A \cap B) \subseteq {}_{\mathcal{I}^s}Cl(B)$. Then ${}_{\mathcal{I}^s}Cl(A \cap B) \subseteq {}_{\mathcal{I}^s}Cl(A) \cap {}_{\mathcal{I}^s}Cl(B)$.
4. It is clear from Theorem (4.4) and from every open set U is $\mathcal{I}^s-$open set. □

In the last theorem ${}_{\mathcal{I}^s}Cl(A \cup B) \neq {}_{\mathcal{I}^s}Cl(A) \cup {}_{\mathcal{I}^s}Cl(B)$.

EXAMPLE 4.7. In Example (3.8), the sets $A = \{a\}$ and $B = \{b\}$ are $\mathcal{I}^s-$closed sets in $(X_\pi, \tau, \mathcal{I})$. Then

$${}_{\mathcal{I}^s}Cl(A) \cup {}_{\mathcal{I}^s}Cl(B) = A \cup B = \{a, b\}$$

and

$${}_{\mathcal{I}^s}Cl(A \cup B) = {}_{\mathcal{I}^s}Cl(\{a, b\}) = X.$$

THEOREM 4.8. For a subsets $A, B \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, the following hold:

(1) If $A \subseteq B$ then ${}_{\mathcal{I}^s}Int(A) \subseteq {}_{\mathcal{I}^s}Int(B)$;
(2) ${}_{\mathcal{I}^s}Int(A) \cup {}_{\mathcal{I}^s}Int(B) \subseteq {}_{\mathcal{I}^s}Int(A \cup B)$;
(3) ${}_{\mathcal{I}^s}Int(A \cap B) \subseteq {}_{\mathcal{I}^s}Int(A) \cap {}_{\mathcal{I}^s}Int(B)$;
(4) $Int(A) \subseteq {}_{\mathcal{I}^s}Int(A)$.

PROOF. Similar for Theorem (4.6). □

In the last theorem ${}_{\mathcal{I}^s}Int(A \cap B) \neq {}_{\mathcal{I}^s}Int(A) \cap {}_{\mathcal{I}^s}Int(B)$.

EXAMPLE 4.9. In Example (3.8), the sets $A = \{b, c\}$ and $B = \{a, c\}$ are $\mathcal{I}^s-$open sets in $(X, \tau, \mathcal{I})$. Then

$${}_{\mathcal{I}^s}Int(A) \cap {}_{\mathcal{I}^s}Int(B) = A \cap B = \{c\}$$

and

$${}_{\mathcal{I}^s}Int(A \cap B) = {}_{\mathcal{I}^s}Int(\{c\}) = \emptyset.$$

THEOREM 4.10. For a subset $A \subseteq X$ of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, the following hold:

(1) ${}_{\mathcal{I}^s}Int(X - A) = X - {}_{\mathcal{I}^s}Cl(A)$;
(2) ${}_{\mathcal{I}^s}Cl(X - A) = X - \mathcal{I}^s Int(A)$.

PROOF. 1. Since $A \subseteq {}_{\mathcal{I}^s}Cl(A)$ then $X - {}_{\mathcal{I}^s}Cl(A) \subseteq (X - A)$. Since $X - {}_{\mathcal{I}^s}Cl(A)$ is a $\mathcal{I}^s-$open set in $(X_*, \tau, \mathcal{I})$ then

$$X - {}_{\mathcal{I}^s}Cl(A) = {}_{\mathcal{I}^s}Int[X - {}_{\mathcal{I}^s}Cl(A)] \subseteq {}_{\mathcal{I}^s}Int(X - A).$$

For the other side, let $x \in {}_{\mathcal{I}^s}Int(X - A)$. Then there is $\mathcal{I}^s-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\mathcal{I}^s-$closed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathcal{I}^s}Cl(A)$, that is, $x \in X - {}_{\mathcal{I}^s}Cl(A)$.
2. Since ${}_{\mathcal{I}^s}Int(A) \subseteq A$ then $X - A \subseteq X - {}_{\mathcal{I}^s}Int(A)$. Since $X - {}_{\mathcal{I}^s}Int(A)$ is a $\mathcal{I}^s-$closed set in $(X_*, \tau, \mathcal{I})$ then

$${}_{\mathcal{I}^s}Cl(X - A) \subseteq {}_{\mathcal{I}^s}Cl[X - {}_{\mathcal{I}^s}Int(A)] = X - {}_{\mathcal{I}^s}Int(A).$$

For the other side, let $x \in {}_{\mathcal{I}^s}Int(X - A)$. Then there is $\mathcal{I}^s-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\mathcal{I}^s-$closed set containing A and $x \notin X - U$. Hence $x \notin {}_{\mathcal{I}^s}Cl(A)$, that is, $x \in X - {}_{\mathcal{I}^s}Cl(A)$. □

THEOREM 4.11. For a subset $A \subseteq X$ of an ideal topological space $(X_*, \tau, \mathcal{I})$, the following hold:

(1) If I is an open set X then ${}_{\mathcal{I}^s}Cl(A) \cap I \subseteq {}_{\mathcal{I}^s}Cl(A \cap I)$.
(2) If I is an closed set X then ${}_{\mathcal{I}^s}Int(A \cup I) \subseteq {}_{\mathcal{I}^s}Int(A) \cup I$.

PROOF. 1. Let $x \in {}_{\mathcal{I}^s}Cl(A) \cap I$. Then $x \in {}_{\mathcal{I}^s}Cl(A)$ and $x \in I$. Let V be any $\mathcal{I}^s-$open set in $(X_*, \tau, \mathcal{I})$ containing x. By Theorem (3.11), $V \cap I$ is $\mathcal{I}^s-$open set containing x. Since $x \in {}_{\mathcal{I}^s}Cl(A)$ then by Theorem (4.4), $(V \cap I) \cap A \neq \emptyset$. This implies, $V \cap (I \cap A) \neq \emptyset$. Hence by Theorem (4.4), $x \in {}_{\mathcal{I}^s}Cl(A \cap I)$. That is, ${}_{\mathcal{I}^s}Cl(A) \cap I \subseteq {}_{\mathcal{I}^s}Cl(A \cap I)$.
1. Since I is a closed set X then by the part (1) and Theorem (4.10),

$$\begin{aligned} X - [{}_{\mathcal{I}^s}Int(A) \cup I] &= [X - {}_{\mathcal{I}^s}Int(A)] \cap [X - I] \\ &= [{}_{\mathcal{I}^s}Cl(X - A)] \cap [X - I] \\ &\subseteq {}_{\mathcal{I}^s}Cl[(X - A) \cap (X - I)] \\ &\subseteq {}_{\mathcal{I}^s}Cl(X - A) \cap {}_{\mathcal{I}^s}Cl(X - I) \\ &= {}_{\mathcal{I}^s}Cl(X - A) \cap (X - I) \\ &= (X - {}_{\mathcal{I}^s}Int(A)) \cap (X - I) \\ &= (X - {}_{\mathcal{I}^s}Int(A)) \cup I. \end{aligned}$$

Hence ${}_{\mathcal{I}^s}Int(A \cup I) \subseteq {}_{\mathcal{I}^s}Int(A) \cup I$. □

For a subset A of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$, the set

$${}_{\mathcal{I}^s}\Gamma(A) = {}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A)$$

is called $\mathcal{I}^s$*-frontier set* of A in $(X_*, \tau, \mathcal{I})$.

THEOREM 4.12. For a subset $A \subseteq X$ of an ideal topological space $(X_*, \tau, \mathcal{I})$, the following hold:

(1) ${}_{\mathcal{I}^s}Cl(A) = {}_{\mathcal{I}^s}\Gamma(A) \cup {}_{\mathcal{I}^s}Int(A)$;
(2) ${}_{\mathcal{I}^s}\Gamma(A) \cap {}_{\mathcal{I}^s}Int(A) = \emptyset$;
(3) ${}_{\mathcal{I}^s}\Gamma(A) = {}_{\mathcal{I}^s}Cl(A) \cap {}_{\mathcal{I}^s}Cl(X - A)$.

PROOF. 1. ${}_{\mathcal{I}^s}\Gamma(A) \cup {}_{\mathcal{I}^s}Int(A) = ({}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A)) \cup {}_{\mathcal{I}^s}Int(A)$

$$\begin{aligned} &= [{}_{\mathcal{I}^s}Cl(A) \cap (X - {}_{\mathcal{I}^s}Int(A))] \cup {}_{\mathcal{I}^s}Int(A) \\ &= [{}_{\mathcal{I}^s}Cl(A) \cup {}_{\mathcal{I}^s}Int(A)] \cap [(X - {}_{\mathcal{I}^s}Int(A)) \cup {}_{\mathcal{I}^s}Int(A)] \\ &= {}_{\mathcal{I}^s}Cl(A) \cap X = {}_{\mathcal{I}^s}Cl(A). \end{aligned}$$

2. From the definition of ${}_{\mathcal{I}^s}\Gamma(A)$.
3. By Theorem (4.10),

$$\begin{aligned} {}_{\mathcal{I}^s}\Gamma(A) &= {}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A) = {}_{\mathcal{I}^s}Cl(A) \cap (X - {}_{\mathcal{I}^s}Int(A)) \\ &= {}_{\mathcal{I}^s}Cl(A) \cap {}_{\mathcal{I}^s}Cl(X - A). \end{aligned}$$

□

COROLLARY 4.13. For a subset $A \subseteq X$ of an ideal topological space $(X_*, \tau, \mathcal{I})$, ${}_{\mathcal{I}^s}\Gamma(A)$ is $\mathcal{I}^s-$closed set in $(X_*, \tau, \mathcal{I})$.

PROOF. By Theorem (3.10) and the part (3) of the last Theorem. □

THEOREM 4.14. For a subset $A \subseteq X$ of an ideal topological space $(X_*, \tau, \mathcal{I})$, the following hold:

(1) A is a $\mathcal{I}^s-$open if and only if ${}_{\mathcal{I}^s}\Gamma(A) \cap A = \emptyset$;
(2) A is a $\mathcal{I}^s-$closed if and only if ${}_{\mathcal{I}^s}\Gamma(A) \subseteq A$;
(3) A is both $\mathcal{I}^s-$ open and $\mathcal{I}^s-$closed if and only if ${}_{\mathcal{I}^s}\Gamma(A) = \emptyset$.

PROOF. 1. Let A be a $\mathcal{I}^s-$open set. Then ${}_{\mathcal{I}^s}Int(A) = A$. Then by Theorem (4.12),

$${}_{\mathcal{I}^s}\Gamma(A) \cap A = {}_{\mathcal{I}^s}\Gamma(A) \cap {}_{\mathcal{I}^s}Int(A) = \emptyset.$$

Conversely, suppose that ${}_{\mathcal{I}^s}\Gamma(A) \cap A = \emptyset$. Then

$$\begin{aligned} A - {}_{\mathcal{I}^s}Int(A)) &= [A \cap {}_{\mathcal{I}^s}Cl(A)] \cap [A - {}_{\mathcal{I}^s}Int(A))] \\ &= A \cap ({}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A)) \\ &= A \cap {}_{\mathcal{I}^s}\Gamma(A) = \emptyset. \end{aligned}$$

That is, ${}_{\mathcal{I}^s}Int(A) = A$. Hence A is a $\mathcal{I}^s-$open set.
2. Let A be a $\mathcal{I}^s-$closed set. Then ${}_{\mathcal{I}^s}Cl(A) = A$. Then

$${}_{\mathcal{I}^s}\Gamma(A) = {}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A) = A - {}_{\mathcal{I}^s}Int(A) \subseteq A.$$

Conversely, suppose that ${}_{\mathcal{I}^s}\Gamma(A) \subseteq A$. Then by Theorem (4.12),

$${}_{\mathcal{I}^s}Cl(A) = {}_{\mathcal{I}^s}Int(A)) \cup {}_{\mathcal{I}^s}\Gamma(A) \subseteq {}_{\mathcal{I}^s}Int(A)) \cup A \subseteq A.$$

That is, ${}_{\mathcal{I}^s}Cl(A) = A$. Hence A is $\mathcal{I}^s-$closed set.
3. Let A be both $\mathcal{I}^s-$closed set and $\mathcal{I}^s-$open set. Then ${}_{\mathcal{I}^s}Cl(A) = A = {}_{\mathcal{I}^s}Int(A)$. Therefore,

$${}_{\mathcal{I}^s}\Gamma(A) = {}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A) = A - A = \emptyset.$$

Conversely, suppose that ${}_{\mathcal{I}^s}\Gamma(A) = \emptyset$. Then ${}_{\mathcal{I}^s}Cl(A) - {}_{\mathcal{I}^s}Int(A) = \emptyset$. Since

$${}_{\mathcal{I}^s}Int(A) \subseteq {}_{\mathcal{I}^s}Cl(A).$$

Then

$${}_{\mathcal{I}^s}Cl(A) = {}_{\mathcal{I}^s}Int(A).$$

Since

$${}_{\mathcal{I}^s}Int(A) \subseteq A \subseteq {}_{\mathcal{I}^s}Cl(A). Then\ {}_{\mathcal{I}^s}Cl(A) = A = {}_{\mathcal{I}^s}Int(A).$$

That is, ${}_{\mathcal{I}^s}Cl(A) = A$. Hence A is both $\mathcal{I}^s-$ closed set and $\mathcal{I}^s-$open set. □

5. RELATIVE PROPERTY

By bitopological semigroup we mean a triple (X_*, τ, ρ) consists two topo logical semigroups (X_*, τ) and (X_*, ρ). A subset $A \subseteq X$ is said to be $\tau\rho$*-open set* in a bitopological semigroup (X_*, τ, ρ) if $A \subseteq {}_{\tau}Cl[{}_{\rho}Int({}_{\rho}Cl(A))]$. The complement of$\tau\rho-$open set is said to be $\tau\rho-$*closed set*.

THEOREM 5.1. A subset $A \subseteq X$ is a $\mathcal{I}^s-$open set in ideal topological semigroup $(X_*, \tau, \mathcal{I})$, if and only if it is a $\tau\tau^*-$open set in bitopological semigroup (X_*, τ, τ^*).

PROOF. It is clear from the definitions and $Cl^*(A) = {}_{\mathcal{I}}Cl(A)$. $\square$

THEOREM 5.2. A subset A of a bitopological semigroup (X_*, τ, ρ) is $\tau\rho-$closed set, if and only if ${}_{\tau}Int[{}_{\rho}Cl({}_{\rho}Int(A))] \subseteq A$.

PROOF. A is a $\tau\rho-$closed set in X, if and only if $X - A$ is a $\tau\rho-$open set in X, if and only if

$$(X - A) \subseteq {}_{\tau}Cl[{}_{\rho}Int({}_{\rho}Cl(X - A))],$$

if and only if by using Theorem (4.10).

$$\begin{aligned}(X - A) &\subseteq {}_{\tau}Cl[{}_{\rho}Int({}_{\rho}Cl(X - A))] \\ &= {}_{\tau}Cl[{}_{\rho}Int({}_{\rho}Cl(X - A))] \\ &= {}_{\tau}Cl[{}_{\rho}Int(X - {}_{\rho}Int(A))] \\ &= {}_{\tau}Cl[X_{-\rho}Cl({}_{\rho}Int(A))] \\ &= X - {}_{\tau}Int[{}_{\rho}Cl({}_{\rho}Int(A))] \\ &= X - {}_{\tau}Int[{}_{\rho}Cl({}_{\rho}Int(A))],\end{aligned}$$

if and only if ${}_{\tau}Int[{}_{\rho}Cl({}_{\rho}Int(A))] \subseteq A$. $\square$

THEOREM 5.3. Let Y be an open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. If A is a $\mathcal{I}^s-$open set in $(X_*, \tau, \mathcal{I})$ then $A \cap Y$ is a $\tau|_Y\tau^*|_Y-$open set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$.

PROOF. Since A is $\mathcal{I}^s-$open set in $(X_*, \tau, \mathcal{I})$, then $A \subseteq Cl[Int^*(Cl^*(A))]$. By Theorems (2.1)and (4.10), we obtain

$$\begin{aligned}A \cap Y &\subseteq Cl[Int^*(Cl^*(A))] \cap Y \\ &= Cl[Int^*(Cl^*(A))] \cap Y \cap Y \\ &\subseteq Cl[Int^*(Cl^*(A)) \cap Y] \cap Y \\ &= Cl|_Y[Int^*(Cl^*(A)) \cap Y] \\ &= Cl|_Y[Int^*(Cl^*(A)) \cap Int^*(Y)] \\ &= Cl|_Y[Int^*(Cl^*(A) \cap Y)] \\ &= Cl|_Y[Int^*(Cl^*(A) \cap Y \cap Y)] \\ &\subseteq Cl|_Y[Int^*(Cl^*(A \cap Y) \cap Y)] \\ &\subseteq Cl|_Y[Int^*|_Y(Cl^*(A \cap Y) \cap Y)] \\ &= Cl|_Y[Int^*|_Y(Cl^*(A \cap Y) \cap Y)] \\ &= Cl|_Y[Int^*|_Y(Cl^*|_Y(A \cap Y))].\end{aligned}$$

Hence $A \cap Y$ is a $\tau|_Y\tau^*|_Y-$open set in $(Y, \tau|_Y, \tau^*|_Y)$. $\square$

COROLLARY 5.4. Let Y be an open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. If A is a $\mathcal{I}^s-$closed set in $(X_*, \tau, \mathcal{I})$ then $A \cap Y$ is a $\tau|_Y\tau^*|_Y-$closed set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$.

PROOF. Let A be a $\mathcal{I}^s-$closed set in $(X_*, \tau, \mathcal{I})$. Then $X - A$ is a $\mathcal{I}^s-$open set in $(X_*, \tau, \mathcal{I})$. By the last Theorem, $Y - A = (X - A) \cap Y$ is a $\tau|_Y\tau^*|_Y-$open set in $(Y, \tau|_Y, \tau^*|_Y)$. Hence

$$Y - (Y - A) = Y - (Y \cap (X - A)) = Y \cap [(X - Y) \cup A] = A \cap Y$$

is a $\tau|_Y\tau^*|_Y-$closed set in $(Y, \tau|_Y, \tau^*|_Y)$. $\square$

THEOREM 5.5. Let Y be an open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. If A is a $\tau|_Y\tau^*|_Y-$open set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$, then A is $\mathcal{I}^s-$open set I in $(X_*, \tau, \mathcal{I})$.

PROOF. Since A is a $\tau|_Y\tau^*|_Y$-open set in $(Y, \tau|_Y, \tau^*|_Y)$ then $A \subseteq Cl|_Y[Int^*|_Y(Cl^*|_Y(A))]$. Then by Theorems (2.1), (4.10),

$$\begin{aligned}A &\subseteq Cl|_Y[Int^*|_Y(Cl^*|_Y(A))] = Cl[Int^*|_Y(Cl^*|_Y(A))] \cap Y \\ &\subseteq Cl[Int^*|_Y(Cl^*|_Y(A)) \cap Y] = Cl[Int^*|_Y(Cl^*|_Y(A))] \\ &= Cl[Int^*(Cl^*|_Y(A))] = Cl[Int^*(Cl^*|(A) \cap Y)] \\ &\subseteq Cl[Int^*(Cl^*(A \cap Y))] = Cl[Int^*(Cl^*(A))] \\ &= Cl[Int^*(Cl^*(A))].\end{aligned}$$

Hence A is a $\mathcal{I}^s-$open set I in $(X_*, \tau, \mathcal{I})$. $\square$

COROLLARY 5.6. Let Y be an open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$. If A is a $\tau|_Y\tau^*|_Y-$closed set in bitopological space $(Y, \tau|_Y, \tau^*|_Y)$, then A is $\mathcal{I}^s-$closed set I in $(X_*, \tau, \mathcal{I})$.

THEOREM 5.7. Let Y be an open subset of an ideal topological semigroup $(X_*, \tau, \mathcal{I})$ and A be a subset of Y. Then ${}_{\mathcal{I}^s}Cl|_Y(A) = {}_{\mathcal{I}^s}Cl(A) \cap Y$.

PROOF. Let $x \in {}_{\mathcal{I}^s}Cl|_Y(A)$ and I be a $\mathcal{I}^s$-open set in X containing x. By Theorem (5.3), $I \cap Y$ is a $\tau\tau^*-$open set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$ containing x and since $x \in {}_{\mathcal{I}^s}Cl|_Y(A)$, then $I \cap A = (I \cap Y) \cap A \neq \emptyset$. Hence by Theorem (4.4), $x \in {}_{\mathcal{I}^s}Cl(A)$ and since $x \in Y$, this implies $x \in {}_{\mathcal{I}^s}Cl(A) \cap Y$. That is, ${}_{\mathcal{I}^s}Cl|_Y(A) \subseteq {}_{\mathcal{I}^s}Cl(A) \cap Y$. On the other side, let $x \in {}_{\mathcal{I}^s}Cl(A) \cap Y$ and O be a $\tau\tau^*-$open set in bitopological semigroup $(Y, \tau|_Y, \tau^*|_Y)$ containing x. By Corollary (5.6), O is $\mathcal{I}^s-$open set in $(X_*, \tau, \mathcal{I})$. Since $x \in {}_{\mathcal{I}^s}Cl(A)$, then $O \cap A \neq \emptyset$. That is, $x \in {}_{\mathcal{I}^s}Cl|_Y(A)$. Hence ${}_{\mathcal{I}^s}Cl(A) \cap Y \subseteq {}_{\mathcal{I}^s}Cl|_Y(A)$. $\square$

THEOREM 5.8. Let (X_*, τ, τ^*) and $(Y, \rho, \tau^{*'})$ be two ideal topological semigroups. A subset $A \times B \subseteq X_* \times Y$ is $(\tau \times \rho)(\tau^* \times \rho_{\tau^{*'}})-$open set in bitopological semigroup $(X_* \times Y, \tau \times \rho, \tau^* \times \rho_{\tau^{*'}})$ if and only if A is a $\mathcal{I}^s-$open set in (X_*, τ, τ^*) and B is a $\mathcal{I}^s-$open set in $(Y, \rho, \tau^{*'})$.

PROOF. It is clear that,

$$\begin{aligned}&{}_{\tau\times\rho}Cl[{}_{\tau^*\times\rho_{\tau^{*'}}}Int({}_{\tau^*\times\rho_{\tau^{*'}}}Cl(A \times B))] \\ =\ &{}_{\tau\times\rho}Cl[{}_{\tau^*\times\rho_{\tau^{*'}}}Int({}_{\tau^*}Cl(A) \times {}_{\rho_{\tau^{*'}}}Cl(B))] \\ =\ &{}_{\tau\times\rho}Cl[{}_{\tau^*}Int({}_{\tau^*}Cl(A)) \times {}_{\rho_{\tau^{*'}}}Int({}_{\rho_{\tau^{*'}}}Cl(B))] \\ =\ &{}_{\tau}Cl[{}_{\tau^*}Int({}_{\tau^*}Cl(A))] \times {}_{\rho}Cl[{}_{\rho_{\tau^{*'}}}Int({}_{\rho_{\tau^{*'}}}Cl(B))].\end{aligned}$$

We have $A \times B \subseteq C \times D$, if and only if $A \subseteq C$ and $B \subseteq D$. $\square$

6. REFERENCES

[1] D. Jankovic, and T. Hamlett, New topologies from old via ideals, Amer. Math. Monthly, 97(4), (1990), 295-310.

[2] E, Ekici, On $\mathcal{AC}_{\mathcal{I}}-$sets, $\mathcal{BC}_{\mathcal{I}}-$sets, $\beta^*_{\mathcal{I}}-$sets and decompositions of continuity in ideal topological spaces, Creat. Math. Inform; 20(1), (2011), 47-54.

[3] E. Ekici and T. Noiri, *-hyporconnected ideal topological space, Analele Stiintifice Ale Universitatii Al. I. Cuza Din Iasi-Serie Nona-Matematica, 1, (2012), 121-129.

[4] E. Hatir and T. Noiri, On decompositions of continuity via idealization, Acta Math. Hungar, 96, (2002), 341-349.

[5] E. Hatir, A note on $\delta\alpha - I-$open sets and semi$^* - I-$oprn sets, Math. Commun. 16, (2011), 433-445.

[6] F. Arenas, J. Dontchev and M. Puertas, Idealization of some weak separation axioms, Acta. Math. Hungar, 89(1), (2000), 47-53.

[7] F. Helen, Introdution to general topology, Boston, Universety of Massachutts (1968).

[8] J. Dontchev, Strong B-sets and another decomposition of continuity, Acta Math. Hungar, 75(3), (1997), 259-265.

[9] K. Kuratowski, Topology, Academic Press, New York, (1), (1966).

[10] R. Vaidyanathaswamy, The localization theory in set topology,Proc. Indian Acad. Sci. Sect. A; 20, (1944), 51-61.