Fixed Point Theorem With C-Class Functions in Partial Metric Spaces

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ABSTRACT

The aim of this paper is to prove a fixed point theorem using C-class function and ϕ, ψ altering distance functions in partial metric spaces.

General Terms

Primary 47H10; Secondary 54H25

Keywords

Fixed point theorem, coincidence point, metric space, C-class function

1. INTRODUCTION AND MATHEMATICAL PRELIMINARIES

The concept of partial metric spaces were introduced by Matthews in [10] as a part of the study of denotational semantics of data flow networks. These spaces are generalizations of usual metric spaces where the self distance for any point need not be equal to zero.
Let X be non-empty set and $\rho : X \times X \to [0, \infty)$ be a function such that for all $x, y, z \in X$:

(i) $x = y \Leftrightarrow \rho(x, x) = \rho(x, y) = \rho(y, y)$, ($T_0$-separation axiom)
(ii) $\rho(x, x) \leq \rho(x, y)$,
(iii) $\rho(x, y) = \rho(y, x)$, (Symmetry)
(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z)$ (Modified Triangular Inequality)

A partial metric space (for Short PMS) is a pair (X, ρ) such that X is a non-empty set and ρ is a partial metric on X. It is clear that if $\rho(x, y) = 0$, then $x = y$. But if $x = y$, $\rho(x, y)$ may not be 0.

THEOREM 1.1 [10]. *Let (X, ρ) be a complete partial metric space and let $T : X \to X$ be a contraction mapping, that is there exists $\lambda \in [0, 1)$ such that $\rho(Tx, Ty) \leq \lambda\rho(x, y)$, for all $x, y \in X$. Then T has a unique filed point $z \in X$. Moreover, $\rho(z, z) = 0$.*

Later on, Abdelijawad [1], Acar [2], [3], Altun [4], Karapinar and Erhan [14], Oltar and Valero [15] gave some generalizations of the result of Matthews.

THEOREM 1.2. *Let (X, ρ) be a complete partial metric space and let $T : X \to X$ be a map such that $\rho(Tx, Ty) \leq \varphi(M(x, y)$, for all $x, y \in X$ where $M(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), 1/2[\rho(x, Ty) + \rho(y, Tx)]\}$ and φ satisfies one of the following:*

(i) *$\varphi : (0, \infty) \to (0, \infty)$ is an upper semicontinuous from the right such that $\varphi(t) < t$ for all $t > 0$ [17].*
(ii) *$\varphi : (0, \infty) \to (0, \infty)$ is a non decreasing function such that $\varphi n(t) \to 0$ as $n \to \infty$ for all $t > 0$ [18].*

Then T has a unique fixed point $z \in X$. Moreover, $\rho(z, z) = 0$.

On the other hand, Dukic et al. [19] proved the following nice fixed point theorem. Before, we introduce the set S of function $\beta : [0, \infty) \to [0, 1)$ satisfying $\beta(t_n) \to 1$ implies $t_n \to 0$.

THEOREM 1.3. *Let (X, ρ) be a complete partial metric space and let $T : X \to X$ be a self-map. Suppose that there exists $\beta \in S$ such that*

$$\rho(Tx, Ty) \leq \beta(\rho(x, y))\rho(x, y)$$

holds for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$, the Picard sequence $\{T^n x\}$ converges to z when $n \to \infty$.

THEOREM 1.4 [2]. *Let (X, ρ) be a complete partial metric space and let $T : X \to X$ be a self-map. Suppose that there exist $\beta \in S$ such that*

$$\rho(Tx, Ty) \leq \beta(M(x, y))$$

hold for all $x, y \in X$, where

$$M(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), 1/2[\rho(x, Ty) + \rho(y, Tx)]\}.$$

Then T has a unique fixed point $z \in X$.

In 2014 the concept of C-class functions (see Definition 1) was introduced by A.H. Ansari in [7] that is pivotal result in fixed point theory, for example see number (1), (2) from Example 2.

DEFINITION 1 [20]. *A mapping $f : [0, \infty)^2 \to R$ is called C-class function if it is continuous and satisfies following axioms:*

(1) *$f(s, t) \leq s$;*
(2) *$f(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.*

Note for some f we have that $f(0, 0) = 0$.

We denote C-class functions as C.

EXAMPLE 2 [20]. *The following functions* $F : [0,\infty)^2 \to R$ *are elements of* C*, for all* $s,t \in [0,\infty)$*:*

(1) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$.
(2) $F(s,t) = ms$, $0 < m < 1$, $F(s,t) = s \Rightarrow s = 0$;
(3) $F(s,t) = s(1+t)r$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
(4) $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
(5) $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,t) = s \Rightarrow s = 0$;

DEFINITION 3 [21]. *A function* $\psi : [0,\infty) \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

(i) ψ *is non-decreasing and continuous,*
(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

DEFINITION 4. *A function* $\psi : R \to R$ *is called total altering distance function if the following properties are satisfied:*

(i) ψ *is non-decreasing and continuous,*
(ii) $\psi(t) = 0$ *if and only if* $t = 0$.

REMARK 5. *We denote* $\psi\inf$ *set total altering distance functions.*

DEFINITION 6. *An ultra altering distance function is a continuous, nondecreasing mapping* $\phi : [0,\infty) \to [0,\infty)$ *such that* $\phi(t) > 0$, $t > 0$ *and* $\phi(0) = 0$.

REMARK 7. *We denote* u *set ultra altering distance functions.*

DEFINITION 8. *A tripled* (ψ, ϕ, F) *where* $\psi \in \Psi$, $\phi \in \Phi_u$ *and* $F \in C$ *is say to be monotone if for any* $x, y \in [0,\infty)$

$$x \le y \Rightarrow F(\psi(x), \phi(x)) \le F(\psi(y), \phi(y)).$$

EXAMPLE 9. *Let* $F(s,t) = s - t$, $\varphi(x) = \sqrt{x}$

$$\psi(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ x^2 & \text{if } x > 1 \end{cases}$$

then (ψ, φ, F) *is monotone.*

EXAMPLE 10. *Let* $F(s,t) = s - t$, $\varphi(x) = x^2$

$$\psi(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ x^2 & \text{if } x > 1 \end{cases}$$

then (ψ, φ, F) *is not monotone.*

LEMMA 11 [22]. *If* $\{x^n\}$ *with* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ *is not a Cauchy sequence in* (X, p)*, and two sequences* $\{m(k)\}$ *and* $\{n(k)\}$ *of positive integers such that* $n(k) > m(k) > k$*, then the following four sequences*

$$p(x_{m(k)}, x_{n(k)+1}),\ p(x_{m(k)}, x_{n(k)}),$$
$$p(x_{m(k)-1}, x_{n(k)+1}),\ p(x_{m(k)-1}, x_{n(k)})$$

tend to $\varepsilon > 0$*, when* $k \to \infty$.

2. MAIN RESULT

THEOREM 2.1. *(X, ρ) be a complete partial metric space and let $f : X \to X$ be a self-map. Suppose that there exist $F \in C$ such that*

$$\psi(\rho(f_x, f_y) \le F(\psi(M(x,y)), \varphi(M(x,y))) \tag{1}$$

holds for all $x, y \in X$*, where* $\psi \in \Psi_{\inf}$, $\varphi \in \Phi$, $F \in C$*, such that* (ψ, φ, F) *is monotone and*

$$M(x,y) = \max\left\{\rho(x,y), \rho(x,f_x), \rho(y,f_y), \frac{1}{2}[\rho(x,f_y) + \rho(y,f_x)]\right\} \tag{2}$$

Then f *has a unique fixed point* $z \in X$.

PROOF. Suppose x_0 is an arbitrary point of X and define the sequence $\{x^n\}$ in X such that

$$x_n = f_{x_{n-1}} = f^n(x_0) \quad \text{for every } n \in N.$$

If $x_n = x_{n+1}$ for some $n \in N$, then x_n is a fixed point of f and the existence part of the proof is finished. Suppose that $x_n \ne x_{n+1}$ for every $n \in N$.
Then by (1), we have

$$\begin{aligned}\psi(\rho(x_n, x_{n+1})) &= \psi(\rho(f_{x_{n-1}}, f_{x_n})) \\ &\le F(\psi(M(x_{n-1}, x_n), \varphi(M(x_{n-1}, x_n))\end{aligned}$$

on the other hand, since

$$\begin{aligned}&\frac{1}{2}[\rho(x_{n-1}, f_{x_n}) + \rho(x_n, f_{x_{n-1}})] \\ &= \frac{1}{2}[\rho(x_{n-1}, x_{n+1}) + \rho(x_n, x_n)] \\ &\le \frac{1}{2}[\rho(x_{n-1}, x_n) + \rho(x_n, x_{n+1})] \\ &\le \max\{\rho(x_{n-1}, x_n), \rho(x_n, x_{n+1})\} \\ &= \rho(x_{n-1}, x_n).\end{aligned}$$

Then

$$\begin{aligned}&M(x_{n-1}, x_n) \\ &= \max\Big\{\rho(x_{n-1}, x_n), \rho(x_{n-1}, f_{x_{n-1}}), \rho(x_n, f_{x_n}), \\ &\qquad \frac{1}{2}[\rho(x_{n-1}, f_{x_n}) + \rho(x_n, f_{x_{n-1}})]\Big\} \\ &= \max\Big\{\rho(x_{n-1}, x_n), \rho(x_{n-1}, x_n), \rho(x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[\rho(x_{n-1}, x_{n+1}) + \rho(x_n, x_n)]\Big\} \\ &= \rho(x_{n-1}, x_n).\end{aligned}$$

So,

$$\begin{aligned}&\psi(\rho(x_n, x_{n+1})) \le F(\psi(\rho(x_{n-1}, x_n)), \varphi(\rho(x_{n-1}, x_n))) \\ &\qquad \le \psi(\rho(x_{n-1}, x_n)) \\ \Rightarrow\ &\rho(x_n, x_{n+1}) \le \rho(x_{n-1}, x_n).\end{aligned} \tag{3}$$

thus $\{\rho(x_n, x_{n+1})\}$ is a non-increasing sequence of non-negative real numbers. Hence $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = \gamma \ge 0$ for certain $\gamma \in [0,\infty)$.

Now, we will prove that $\gamma = 0$. In the contrary case, from (3) with letting $n \to \infty$

$$\psi(r) = F(\psi(r), \varphi(r))$$

So, $\psi(r) = 0$, or $\varphi(r) = 0$. Consequently, $r = 0$.
This contradicts that $r = 0$.
Therefore, $\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0$.
Now, we show that $\{x^n\}$ is a Cauchy sequence in X i.e.
We prove that $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$.
In the contrary, we suppose that the sequence $\{x^n\}$ is not a Cauchy sequence in (X, ρ), then sequences in Lemma 11 tend to $\varepsilon > 0$, when $k \to \infty$.
So we can see that

$$\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \varepsilon.$$

So from (1)

$$\begin{aligned}\psi(\rho(x_{m(k)}, x_{n(k)})) &= \psi(\rho(f_{x_{m(k)-1}}, f_{x_{n(k)-1}}))\\ &\le F(\psi(M(x_{m(k)-1}, x_{n(k)-1})),\\ &\qquad \varphi(M(x_{m(k)-1}, x_{n(k)-1})))\end{aligned}$$

with letting $k \to \infty$,

$$\psi(\varepsilon) \le F(\psi(\varepsilon), \varphi(\varepsilon))$$

So, $\psi(\varepsilon) = 0$, or $\varphi(\varepsilon) = 0$.
Consequently, $\varepsilon = 0$, this contradicts our assumption that

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = \varepsilon > 0.$$

Therefore,

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = 0.$$

This means that $\{x^n\}$ is a Cauchy sequence in a complete partial metric space (X, ρ) and consequently, there exists $z \in X$ such that

$$0 = \lim_{n,m\to\infty} \rho(x_n, x_m) = \lim_{n\to\infty} \rho(x_n, z) = \rho(z, z).$$

Now, we will prove that z is a fixed point of f. For this assume $\rho(z, f_z) > 0$.
Then, we have

$$\rho(z, f_z) \le \rho(z, f_{x_n}) + \rho(f_{x_n}, f_z) - \rho(f_{x_n}, f_{x_n}),$$

This implies that

$$\begin{aligned}&\psi(\rho(z, f_z) - \rho(z, f_{x_n}) + \rho(f_{x_n}, f_{x_n}))\\ &\quad \le \psi(\rho(f_{x_n}, f_z))\\ &\quad \le F(\psi(M(x_n, z)), \varphi(M(x_n, z)))\end{aligned}$$

where

$$\begin{aligned}M(x_n, z) &= \max\Big\{\rho(x_n, z), \rho(x_n, f_{x_n}), \rho(z, f_z),\\ &\qquad \frac{1}{2}[\rho(x_n, f_z) + \rho(z, f_{x_n})]\Big\}\\ &= \max\Big\{\rho(x_n, z), \rho(x_n, x_{n+1}), \rho(z, f_z),\\ &\qquad \frac{1}{2}[\rho(x_n, f_z) + \rho(z, x_{n+1})]\Big\}\\ &= \max\Big\{\rho(z, z), \rho(z, z),\\ &\qquad \rho(z, f_z), [\rho(z, f_z) + \rho(z, z)]/2\Big\}\\ &= \rho(z, f_z).\end{aligned}$$

And so with letting $n \to \infty$

$$\psi(\rho(z, f_z)) \le F(\psi(\rho(z, f_z)), \varphi(\rho(z, f_z)))$$

So, $\psi(\rho(z, f_z)) = 0$, or $\varphi(\rho(z, f_z)) = 0$.
Consequently, $\rho(z, f_z) = 0$, this contradicts the inequality $\rho(z, f_z) > 0$, so that $\rho(z, f_z) = 0$. Thus $z = f_z$.
Suppose that z and w are fixed points of f then, if $z \ne w$, we have by (1)

$$\begin{aligned}\psi(\rho(z, w)) &= \psi(\rho(f_z, fw))\\ &\le F(\psi(M(z, w)), \varphi(M(z, w)))\end{aligned}$$

where

$$\begin{aligned}M(z, w) &= \max\Big\{\rho(z, w), \rho(z, f_z), \rho(w, fw),\\ &\qquad \frac{1}{2}[\rho(z, fw) + \rho(w, f_z)]\Big\}\\ &= \max\Big\{\rho(z, w), \rho(z, z), \rho(w, w),\\ &\qquad \frac{1}{2}[\rho(z, w) + \rho(w, z)]\Big\}\\ &= \rho(z, w).\end{aligned}$$

Thus

$$\psi(\rho(z, w)) \le F(\psi(\rho(z, w)), \varphi(\rho(z, w)))$$

So, $\psi(\rho(z, w)) = 0$, or $\varphi(\rho(z, w)) = 0$.
Consequently, $\rho(z, w) = 0$, a contradiction.
This proves the uniqueness of the fixed point of f. □

3. EXAMPLE

Let $X = [0, 1]$ and $\rho(x, y) = \max\{x, y\}$. Then (X, ρ) is a complete partial metric space.
Let $F(s, t) = m.s$, $0 < m < 1$

$$\begin{aligned}\phi(x) &= \sqrt{x}, \quad x \in [0, 1]\\ \psi(x) &= x, \quad x \in [0, 1]\\ f(x) &= \frac{x^2}{2}, \quad x \in [0, 1].\end{aligned}$$

Clearly, F is a C-class function.
(ψ, ϕ, F) is a monotone function.

All the conditions of Theorem 2.1 are satisfied. ‘0’ is the unique fixed point of f.

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