A Four-step Collocation Procedure by means of Perturbation term with Application to Third-order Ordinary Differential Equation

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ABSTRACT
In this paper, we developed a new method within an interval of four for numerical solution of third-order ordinary differential equations. Interpolation and collocation approach was used by choosing interpolation points at $s=3$steps points using power series, while collocation points at $r=(k-1)$step points. The method adopts a combination of powers series and perturbation terms gotten from the Legendre polynomials, giving rise to a polynomial of degree $r+s-2$and $r+s$equations. All the analysis on the derived method shows that it is stable has order of accuracy p=2, convergent and the region is absolutely stable. Numerical examples were provided to test the performance of the new method. The developed method was used to solve problems ranging from linear, non-linear and non-stiff Problem to test the applicability of the new method. Results obtained when compared with existing methods in the literature shows that the method is accurate, efficient and computational reliable.

Keywords
Four-step, Collocation Procedure, Perturbation term, Legendre Polynomial, Interpolation, Application to Third-order ODEs, Zero Stability, Direct solution, Consistence, Convergent, Absolutely stable

1. INTRODUCTION
Numerical analysis is the fraction of mathematics which provides convenient methods for obtaining the solution of mathematical problems and to dig out useful information from available solutions which are not expressed in tractable forms. Such problems begin for the most part, from real world applications of algebra, geometry, calculus and they include variables which change consistently. There are three fundamental ways for focusing on the numerical analysis; it can develop techniques for the computational problems that emerge from utilization of science, it can be the provision and examination of algorithms for fundamental calculations that are common to numerous applications, and it can be theoretical work on questions that are very important to the accomplishment of algorithms.

Many real life problems in sciences, engineering biology and social sciences are model of third order ordinary differential equations. Interestedly, some differential equations arising from the modelling of physical phenomena, often do not have analytic solutions, hence the development of numerical method to obtain approximation solutions become necessary. [1]. this paper focuses on application of third order ordinary differential equations of the form.

$$y'''(x)=f(x,y,y',y''),$$
$$y(x_0)=y_0, y'_{(x_1)}=y_1, y''_{(x_2)}=y_2 \qquad (1.1)$$

In the past, equation (1.1) is usually solved by method of reducing it to its equivalent system of first order ordinary differential equations and thereafter appropriate numerical method for first order ODEs would be applied to solve the systems. On the other hand, the reduction of higher order ordinary differential equations, to a system of first order has serious problems, which included: consumption of human effort, computational burden and non economization of computer time, as discussed by [2, 3, 4 ,5, 6, 7]. But to cater for the setbacks encountered in reduction method and also bring about improvement on numerical methods [8, 9, 10, 11, 12, 13, 14] developed block methods for solving higher order ODEs directly in which the accuracy is better than, when it is reduced to system of first order ordinary differential equations.

A variety of Linear multi-step method for solving equation (1.1) directly have been developed by some researchers such as [15, 16, 17, 18, 19, 20]. They develops a block method for the solution of third order ordinary differential equations.

In the light of this, we want to derive a block method for 4 steps linear multi-step method using power series as the interpolation equation and power series with Legendre polynomial as the perturbation term as the collocation equation to solve equation (1.1) without reduction to first order ODEs.

2. DERIVATION OF THE METHOD
In this section, we derive discrete method to solve (1.1) at a sequence of nodal points $x_n=x_0+nh$, where $h>0$ is the step length or grid size defined by $h=x_{n+1}-x_n$ and $y(x)$denotes the true solution to (1.1) while the approximate solution is denoted by the point series.

$$y_{(x)}=c_0x_n^0+c_1x_n^1+c_2x_n^2+\ldots+c_kx_n^k \qquad (2.1)$$

The proposed method depends on the perturbed collocation method with respect to the power series with the Legendre polynomials as the perturbation term. Interpolation and collocation procedures are used by choosing interpolation point at $s=3$grid points and collocation points at

$r=(k-1)$step points. We have a polynomial of degree $r+s-2$ and $(r+s)$ equations.

In the first place, we consider the approximation solution of (1.1) in the power series.

$$p_i(x)=x^i,\ i=0,1,...,k$$

Hence (2.1) becomes k

$$y_k(x)=c_i p_i(x)=\sum_{i=0}^{k} c_i x^i \tag{2.2}$$

With the third derivatives as

$$y_k'''(x)=c_i p_i'''(x)=\sum_{i=0}^{R} i(i-1)(i-2)c_i x^{i-3} \tag{2.3}$$

Combining equation (1.1) and (2.3), with the perturbation term, we have

$$\sum_{i=n} c_i p_i'''(x)=f(x,y,y',y'')+\lambda L_k(x_{n+i}),\ i=2(1)k \tag{2.4}$$

Where $L_k(x)$ is the Legendre polynomial of degree k valid in $x_n \le x \le x_{n+k}$. λ is a perturbed parameter.

In particular, we shall be dealing with case k = 4 (four step points), where equation (2.2) and (2.4) are the interpolation and collocation equations respectively.

The well-known Legendre polynomials are generated using the Rodrigues formula

$$P_n(x)=\frac{1}{2^n n!}\frac{1}{dx^n}\left[\left(x^2-1\right)^n\right], \text{ where}$$

$L_0(x)=1, L_1(x)=x$. The rest are computed using the recurrence formula.

$$L_{i+1}(x)=\frac{2i+1}{i+1}xL_i(x)-\frac{i}{i+1}L_{i-1}(x),\ i=1,2,...$$

giving: $L_2(x)=\frac{1}{2}\left(3x^2-1\right), L_3(x)=\frac{1}{2}\left(5x^3-3x\right),$

$$L_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right), L_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right) \text{etc} \tag{2.5}$$

In order to use these polynomials in the interval $[x_n, x_{n+k}]$, we define the shifted Legendre polynomials by introducing the change of variables.

$$x=\frac{2\bar{x}-(x_{n+k}+x_n)}{(x_{n+k}-x_n)} \text{ [21]} \tag{2.6}$$

Interpolating (2.2) at s grid points and collocating (2.4) at k-1 grid points respectively leads to the following systems of equation.

$$\sum_{i=0}^{s} c_i p_i(x)=y_{x+s}, s=0,1,2 \tag{2.7}$$

$$\sum_{i=0}^{k} c_i p_i'''(x)=f_{n+j}+\lambda L_k(x_{n+j}),\ j=2(1)k \tag{2.8}$$

Now we take the polynomial $L_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$ to obtain value for $L_4(x_{n+3})$ and $L_4(x_{n+4})$ to be $3/8$, $-\frac{37}{128}$ and $\frac{443}{8}$ respectively

In addition, from (2.3), $c_0 p_0'''(x)=0, c_1 p_1'''(x)=0, c_2 p_2'''(x)=0, c_3 p_3'''(x)=6c_3$ and $c_4 p_4'''(x)=24c_4 x_{n+4}$

Then (2.8) will reduce to the form:

$$0+0+0+6c_3+24c_4x_{n+4}=f(x,y,y',y'')+\lambda L_4(x_{n+i}), i=2,3 and\ 4 \tag{2.9}$$

We now collocate equation (2.9) at $x_{n+i}, i=2$ and 3 and interpolate equation (2.1) at $x_{n+i}, i=0,1,2$ to produce a system of 6 equations with $c_i, i=0,1,2,3,4 and\ \lambda$ which in matrix from is

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & 0 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & 0 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & 0 \\ 0 & 0 & 0 & 6 & 24_{x+2} & -3/8 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 37/128 \\ 0 & 0 & 0 & 6 & 24x_{n+4} & -443/8 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+1} \\ y_{n+2} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \quad (2.10)$$

Equation (2.10) is solved by Gaussian elimination method to obtain the value of the unknown parameters, $c_i, (i = 0,1,2,3,4)$ and λ, which are substituted into (2.1) to yield a continuous implicit four steps method in the form of a continuous linear multistep method describe by the formula

$$y_{(x)} = \alpha_o y_n + \alpha_1 y_{n+2} + \alpha_2 y_{n+2} + h^3 \sum_{j=2}^{4} \beta_j (x) f_{n+j}, j = 2(1)k \quad (2.11)$$

Where

$$\alpha_0(t) = \frac{1t^2}{2} + 2 - \frac{3}{2}t + 1$$

$$\alpha_1(t) = -t^2 + 2t$$

$$\alpha_2(t) = \frac{t^2}{2} - \frac{t}{2}$$

$$\beta_2(t) = -\frac{1}{169488}\left(7051h^3t^4 - 84464h^3t^3 + 204035h^3t^2 + 126622h^3t + 3072\right)$$

$$\beta_3(t) = \frac{8}{10593}\left(55h^3t^4 - 437h^3t^3 + 926h^3t^2 - 544h^3t + 48\right)$$

$$\beta_4(t) = \frac{1}{169488}\left(11h^3t^4 - 280h^3t^3 + 763h^3t^2 - 494h^3t - 3072\right) \quad (2.12)$$

are the continuous functions of t with $t = \frac{x_n - x_{n+3}}{h}$, as the transformation equation. Using (2.12) for $x = x_{n+3}$ and x_{n+4}, at t = 0 and 1 respectively, equation (2.11) reduces to

$$y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n = \frac{h^3}{132}\left[-f_{n+4} - 64f_{n+3} + 197f_{n+2}\right]$$

$$y_{n+4} - 6y_{n+2} + 8y_{n+1} - 3y_n = \frac{h^3}{7062}\left[-203f_{n+4} - 6656f_{n+3} + 35107f_{n+2}\right] \quad (2.13)$$

Differentiating (2.12) yields

$$\alpha_0'(t) = t - \frac{3}{2}$$

$$\alpha_1'(t) = -2t + 2$$

$$\alpha_2'(t) = t - \frac{1}{2}$$

$$\beta_2'(t) = -\frac{h^3}{169488}\left(28204t^3 - 253392t^2 + 408070t\right)$$

$$\beta_3'(t) = \frac{8h^3}{10593}\left(220t^3 - 1311t^2 + 1852t - 544\right)$$

$$\beta_4'(t) = \frac{h^3}{169488}\left(44t^3 - 840t^2 + 1526t - 494\right) \quad (2.14)$$

Evaluating (2.14) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}$ *and* x_{n+4}, where $t = -3, -2, -1, 0$ *and* 1, (2.11) yield the following discrete methods respectively.

$$84744hy'_n + 42372y_{n+2} - 169488y_{n+1} + 127116y_n = h^3\left[-247f_{n+4} - 34816f_{n+3} + 63311f_{n+2}\right]$$
$$-42372hy'_{n+1} + 21186y_{n+2} - 21186y_n = h^3\left[-59f_{n+4} - 6944f_{n+3} + 14065f_{n+2}\right]$$
$$28248hy'_{n+2} - 42372y_{n+2} + 56496y_{n+1} - 14124y_n = h^3\left[-75f_{n+4} - 6912f_{n+3} + 16403f_{n+2}\right]$$
$$1926hy'_{n+3} - 4815y_{n+2} + 7704y_{n+1} - 2889y_n = h^3\left[-26f_{n+4} - 1232f_{n+3} + 4789f_{n+2}\right] \quad (2.15)$$
$$84744hy'_{n+4} - 296604y_{n+2} + 508464y_{n+1} - 211860y_n = h^3\left[-2507f_{n+4} - 2048f_{n+3} + 371779f_{n+2}\right]$$

Furthermore, differentiating (2.12) twice, we have

$$\alpha_0''(t) = 1,\ \alpha_1''(t) = -2,\ \alpha_2''(t) = 1,$$
$$\beta_2''(t) = -\frac{h^3}{169488}\left[84612t^2 - 506784t + 408070\right]$$
$$\beta_3''(t) = \frac{8h^3}{10593}\left[660t^2 - 2622t + 1852\right]$$
$$\beta_4''(t) = \frac{h^3}{169488}\left[132t^2 - 1680t + 1526\right] \quad (2.16)$$

Evaluating (2.16) at $x = x_n, x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4}, where $t = -3, -2, -1, 0$ and 1, (2.11) yield the following discrete method respectively,

$$-84744h^2y''_n + 84744y_{n+2} - 169488y_{n+1} + 84744y_n = h^3\left[-763f_{n+4} - 118528f_{n+3} + 204035f_{n+2}\right]$$
$$7704h^2y''_{n+1} - 7704y_{n+2} + 15408y_{n+1} - 7704y_n = h^3\left[-f_{n+4} - 640f_{n+3} + 641f_{n+2}\right] \quad (2.17)$$
$$84744h^2y''_{n+2} - 84744y_{n+2} + 169488y_{n+1} - 84744y_n = h^3\left[-653f_{n+4} - 48128f_{n+3} + 133525f_{n+2}\right]$$
$$84744h^2y''_{n+3} - 84744y_{n+2} + 169488y_{n+1} - 84744y_n = h^3\left[-1163f_{n+4} - 4736f_{n+3} + 175387f_{n+2}\right]$$
$$84744h^2y''_{n+4} - 84744y_{n+2} + 169488y_{n+1} - 84744y_n = h^3\left[-1541f_{n+4} + 123136f_{n+3} + 132637f_{n+2}\right]$$

Now we obtained the modified block formulae from (2.13), (2.15) and 2.17 as

$$\begin{pmatrix}
396 & -396 & 132 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
56496 & 42372 & 0 & 7062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-169488 & 42372 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 21186 & 0 & 0 & -42372h & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
56496 & -42372 & 0 & 0 & 0 & 28248h & 0 & 0 & 0 & 0 & 0 & 0 \\
7704 & -4815 & 0 & 0 & 0 & 0 & 1926h & 0 & 0 & 0 & 0 & 0 \\
508464 & -296604 & 0 & 0 & 0 & 0 & 0 & 84744h & 0 & 0 & 0 & 0 \\
-169488 & 84744 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
15408 & -7704 & 0 & 0 & 0 & 0 & 0 & 0 & 7704h^2 & 0 & 0 & 0 \\
169488 & -84744 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84744h^2 & 0 & 0 \\
169488 & -84744 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84744h^2 & 0 \\
169488 & -84744 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 84744h^2
\end{pmatrix}
\begin{pmatrix}
y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y'_{n+1} \\ y'_{n+2} \\ y'_{n+3} \\ y'_{n+4} \\ y''_{n+1} \\ y''_{n+2} \\ y''_{n+3} \\ y''_{n+4}
\end{pmatrix}$$

$$= \begin{pmatrix} -132 & 0 & 0 \\ -21186 & 0 & 0 \\ 127116 & 84774h & 0 \\ -21186 & 0 & 0 \\ -14124 & 0 & 0 \\ -2889 & 0 & 0 \\ -211860 & 0 & 0 \\ 84744 & 0 & -84744h^2 \\ -7704 & 0 & 0 \\ 84744 & 0 & 0 \\ -84744 & 0 & 0 \\ -84744 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_n \\ y_n^{'} \\ y_n^{''} \end{pmatrix} + h^3 \begin{pmatrix} 197 & -64 & -1 \\ 35107 & -6656 & -203 \\ 63311 & -34816 & -247 \\ 14065 & -6944 & -59 \\ 16403 & -6912 & -75 \\ 4789 & -1232 & -26 \\ 371779 & -2048 & -2507 \\ 204035 & -118528 & -763 \\ 641 & -640 & -1 \\ 133525 & -48128 & -653 \\ 175387 & -4736 & -1163 \\ 132637 & 123136 & -1541 \end{pmatrix} \begin{pmatrix} f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \quad (2.18)$$

Taking the normalized version of 2.18, we obtained the block solution

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+1}^{'} \\ y_{n+2}^{'} \\ y_{n+3}^{'} \\ y_{n+4}^{'} \\ y_{n+1}^{''} \\ y_{n+2}^{''} \\ y_{n+3}^{''} \\ y_{n+4}^{''} \end{pmatrix} = \begin{pmatrix} 1 & h & \frac{h^2}{2} \\ 1 & 2h & 2h^2 \\ 1 & 3h & \frac{9h^2}{2} \\ 1 & 4h & 8h^2 \\ 0 & 1 & h \\ 0 & 1 & 2h \\ 0 & 1 & 3h \\ 0 & 1 & 4h \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_n \\ y_n^{'} \\ y_n^{''} \end{pmatrix} + \begin{pmatrix} \frac{77413h^3}{169488} & -\frac{3056h^3}{10593} & -\frac{269h^3}{169488} \\ \frac{3909h^3}{1177} & -\frac{6976h^3}{3531} & -\frac{43h^3}{3531} \\ \frac{189933h^3}{18832} & -\frac{6528h^3}{1177} & -\frac{741h^3}{18832} \\ \frac{225040h^3}{10593} & \frac{111104h^3}{10593} & -\frac{944h^3}{14593} \\ \frac{56297h^2}{42372} & -\frac{8728h^2}{10593} & -\frac{199h^2}{42372} \\ \frac{49246h^2}{10593} & -\frac{27872h^2}{10593} & -\frac{188h^2}{10593} \\ \frac{42195h^2}{4708} & -\frac{5208h^2}{1177} & -\frac{177h^2}{4708} \\ \frac{140576h^2}{10593} & -\frac{55168h^2}{10593} & -\frac{664h^2}{10593} \\ \frac{11727h}{4708} & -\frac{1744h}{1177} & -\frac{43h}{4708} \\ \frac{14065h}{3531} & -\frac{6944h}{3531} & -\frac{59h}{3531} \\ \frac{197h}{44} & -\frac{16h}{11} & -\frac{h}{44} \\ \frac{4676h}{1177} & \frac{64h}{1177} & -\frac{32h}{1177} \end{pmatrix} \begin{pmatrix} f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{pmatrix} \quad (2.19)$$

To simultaneously obtain values for $y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+1}^{'}, y_{n+2}^{'}, y_{n+3}^{'}, y_{n+4}^{'}, y_{n+1}^{''}, y_{n+2}^{''}, y_{n+3}^{''}$ and $y_{n+4}^{''}$ Equation (2.19) can be written explicitly as:

$$y_{n+1} = y_n + hy_n^{'} + \frac{h^2}{2} y_n^{''} + \frac{h^3}{169488}\left[77413 f_{n+2} - 48896 f_{n+3} - 269 f_{n+4}\right]$$

$$y_{n+2} = y_n + 2hy_n^{'} + 2h^2 y_n^{''} + \frac{h^3}{3531}\left[11727 f_{n+2} - 6976 f_{n+3} - 43 f_{n+4}\right]$$

$$y_{n+3} = y_n + 3hy_n^{'} + \frac{9h^2}{2} y_n^{''} + \frac{h^3}{18832}\left[189933 f_{n+2} - 104448 f_{n+3} - 741 f_{n+4}\right]$$

$$y_{n+4} = y_n + 4hy_n^{'} + 8h^2 y_n^{''} + \frac{h^3}{10593}\left[225040 f_{n+2} - 111104 f_{n+3} - 944 f_{n+4}\right]$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{42372}\left[56297 f_{n+2} - 34912 f_{n+3} - 199 f_{n+4}\right]$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{10593}\left[49246 f_{n+2} - 27872 f_{n+3} - 188 f_{n+4}\right]$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{4708}\left[42195 f_{n+2} - 20832 f_{n+3} - 177 f_{n+4}\right]$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{10593}\left[140576 f_{n+2} - 55168 f_{n+3} - 664 f_{n+4}\right] \quad (2.20)$$

$$y''_{n+1} = y''_n + \frac{h}{4708}\left[11727 f_{n+2} - 6976 f_{n+3} - 43 f_{n+4}\right]$$

$$y''_{n+2} = y''_n + \frac{h}{3531}\left[14065 f_{n+2} - 6944 f_{n+3} - 59 f_{n+4}\right]$$

$$y''_{n+3} = y''_n + \frac{h}{44}\left[197 f_{n+2} - 64 f_{n+3} - f_{n+4}\right]$$

$$y''_{n+4} = y''_n + \frac{h}{1177}\left[4676 f_{n+2} + 64 f_{n+3} - 32 f_{n+4}\right]$$

3 ANALYSIS OF THE METHOD

Basic properties of the method are analysed to establish its validity. These properties help to show the nature of convergence of the method. These properties include order and error constant, consistency and zero stability. All these put together reveal the nature of convergence of the method. Also the regions of absolute stability of the method have also been established in this section. However, brief introductions of these properties are made for a better understanding of the section.

3.1 Order and error constant of the method

Let the linear difference operator L associated with the continuous multi-step method (2.11) be defined as

$$L\left[y(x), h\right] = \sum_{j=0} \left\{\alpha_j y(x_n + jh) - h^3 \beta . y'''(x_n + jh)\right\}; j = 0, 1, 2 - k. \text{ [5]} \quad (3.1)$$

Where $y(x)$ is an arbitrary test function that is continuously differentiable in the interval [a,b], and α_0 and β_0 are both non – zero.

Expanding $y(x_n + jh)$ and $y'''(x_n + jh)$, $j = 0, 1, 2, 3, ..., k$ in Taylor's series about x_n and collecting like terms in h and y gives.

$$L\left[y(x).h\right] = C_0 y(x) + C_1 hy'(x) + C_2 h^2 y^2(x) + ... + Cph^p y^p(x) + ...$$

Definition 1

The difference operator L and the associated implicit multi-step method (2.11) are said to be of order p ,if in (3.2)

$$c_0 = c_1 = c_2 = ...c_p = c_{p+1} = c_{p+2} = 0, c_{p+3} \neq 0 \quad (3.2)$$

Then c_{p+3} is called the error constant and it implies that the local truncation error is given by

$$t_{n+k} = c_{p+3} h^{p+3} y^{(p+3)}_{(x_n)} + 0\left(h^{p+4}\right) \text{ [5]} \quad (3.3)$$

3.2 Order and error constant of the new method

From equation (2.13)

$$y_{n+4} - 6y_{n+2} + 8y_{n+1} - 3y_n = \frac{h^3}{7062}\left[-203 f_{n+4} - 6656 f_{n+3} + 35107 f_{n+2}\right]$$

Can be rewritten in the form;

$$y_{n+4} - 6y_{n+2} + 8y_{n+1} - 3y_n - h^3\left[\frac{35017}{7062} f_{n+2} - \frac{3328}{3531} f_{n+3} - \frac{203}{7062} f_{n+4}\right] \quad (3.4)$$

Expanding (3.4) in Taylor series form, we have

$$\sum_{j=0}^{\infty} \frac{(4)^j (h)^j}{j!} y_n^j - 6\sum_{j=0}^{\infty} \frac{(2)^j (h)^j}{j!} y_n^j + 8\sum_{j=0}^{\infty} \frac{(1)^j (h)^j}{j!} y_n^j - 3y_n - \sum_{j=0}^{\infty} \frac{(h)^{j+2}}{(j+2)} y_n^{j+2}\left[.\frac{35107}{7062}(2)^{j-2} - \frac{3328}{3531}(3)^{j-2} - \frac{203}{7062}(4)^{j-2}\right] = 0$$

On evaluation

$$c_0 = 1 - 6 + 8 - 3 = 0$$

$$c_1 = \frac{(4)^1}{1!} - \frac{1}{1!}\left(6(2)^1 - 8(1)^1 + 3(0)^1\right) = 0$$

$$c_2 = \frac{(4)^2}{2!} - \frac{1}{2!}\left(6(2)^2 - 8(1)^2 + 3(0)^2\right) = 0$$

$$c_3 = \left[\frac{(4)^3}{3!} - \frac{1}{3!}\left(6(2)^3 - 8(1)^3 + 3(0)^3\right)\right] - \frac{1}{0!}\left(\frac{35107}{7062}(2)^0 - \frac{3328}{3531}(3)^0 - \frac{203}{7062}(4)^0\right) = 0$$

$$c_4 = \left[\frac{(4)^4}{4!} - \frac{1}{4!}\left(6(2)^4 - 8(1)^4 + 3(0)^4\right)\right] - \frac{1}{1!}\left(\frac{35107}{7062}(2)^1 - \frac{3328}{3531}(3)^1 - \frac{203}{7062}(4)^1\right) = 0$$

$$c_5 = \left[\frac{(4)^5}{5!} - \frac{1}{5!}\left(6(2)^5 - 8(1)^5 + 3(0)^5\right)\right] - \frac{1}{2!}\left(\frac{35107}{7062}(2)^2 - \frac{3328}{3531}(3)^2 - \frac{203}{7062}(4)^2\right) = \frac{5398}{3531}$$

Hence the main method is of order p = 2, with error constant $c_{p+3} = \frac{5398}{3531}$

3.3 Order and error constant of the new block method:

Using part of the block in (2.20) i.e.

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{169488}\left[77413 f_{n+2} - 48896 f_{n+3} - 269 f_{n+4}\right]$$

$$y_{n+2} = y_n + 2hy_n' + 2h^2 y_n'' + \frac{h^3}{3531}\left[11727 f_{n+2} - 6976 f_{n+3} - 43 f_{n+4}\right]$$

$$y_{n+3} = y_n + 3hy_n' + \frac{9h^2}{2}y_n'' + \frac{h^3}{18832}\left[189933 f_{n+2} - 104448 f_{n+3} - 741 f_{n+4}\right]$$

$$y_{n+4} = y_n + 4hy_n' + 8h^2 y_n'' + \frac{h^3}{10593}\left[225040 f_{n+2} - 111104 f_{n+3} - 944 f_{n+4}\right]$$

as

$$y_{n+1} - y_n - hy_n' - \frac{h^2}{2}y_n'' - h^3\left[\frac{77413}{169488} f_{n+2} - \frac{3056}{10593} f_{n+3} - \frac{269}{169488} f_{n+4}\right] = 0$$

$$y_{n+2} - y_n - 2hy_n' - 2h^2 y_n'' - h^3\left[\frac{3909}{1177} f_{n+2} - \frac{6976}{3531} f_{n+3} - \frac{43}{3531} f_{n+4}\right] = 0 \qquad (3.5)$$

$$y_{n+3} - y_n - 3hy_n' - \frac{9h^2}{2}y_n'' - h^3\left[\frac{189933}{18832} f_{n+2} - \frac{6528}{1177} f_{n+3} - \frac{741}{18832} f_{n+4}\right] = 0$$

$$y_{n+4} - y_n - 4hy_n' - 8h^2 y_n'' - h^3\left[\frac{225040}{10593} f_{n+2} - \frac{111104}{10593} f_{n+3} - \frac{944}{10593} f_{n+4}\right] = 0$$

And using Taylor's series expansion on (3.5) and collecting terms in h and y lead to the following:

$$c_n = \frac{(1)^n .1}{n!} - \frac{1}{(n-3)!}\left(\frac{77413}{169488}(2)^{n-3} - \frac{3056}{10593}(3)^{n-3} - \frac{269}{169488}(4)^{n-3}\right)$$

$$c_n = \frac{(2)^n .1}{n!} - \frac{1}{(n-3)!}\left(\frac{3909}{1177}(2)^{n-3} - \frac{6976}{3531}(3)^{n-3} - \frac{43}{3531}(4)^{n-3}\right)$$

$$c_n = \frac{(3)^n .1}{n!} - \frac{1}{(n-3)!}\left(\frac{189933}{18832}(2)^{n-3} - \frac{6528}{1177}(3)^{n-3} - \frac{741}{18832}(4)^{n-3}\right)$$

$$c_n = \frac{(4)^n .1}{n!} - \frac{1}{(n-3)!}\left(\frac{225040}{10593}(2)^{n-3} - \frac{111104}{10593}(3)^{n-3} - \frac{944}{10593}(4)^{n-3}\right)$$

On evaluating at n = 0, 1, 2, 3 and 4, $c_0 = c_1 = c_2 = c_3 = c_4 = 0$

(3.6)

$$C_{p+3} = \left[\frac{85963}{211860}, \frac{46118}{17655}, \frac{83883}{11770}, \frac{739168}{52965}\right]^T$$

Hence the method is of order p = 2, with error constant

3.4 Consistency

Given a continuous implicit multi-step method (2.11), the first and second characteristics polynomials are defined as:

$$\rho(z) = \sum_{J=0}^{k} \alpha_j Z^j \tag{3.7}$$

$$\sigma(z) = \sum_{J=0}^{k} \beta_J Z^J \tag{3.8}$$

Where Z is the principle root, $\alpha_k \neq 0$ and $\alpha_0^3 + \beta_0^3 \neq 0$

Definition 2

The continuous implicit multi-step method (2.11) is said to be consistent if it satisfies the following conditions

i. The order $P \geq 1$

ii. $\sum_{j=0}^{k} \alpha_j = 0$

iii. $\rho(l) = \rho'(l)$

iv. $\rho'''(l) = 3!\sigma(l)$ [5]

Remark:

Condition (i) is sufficient for the associated block method to be consistent i.e. $P \geq 0$ [22, 25]

Recall the main method; (2.13)

$$y_{n+4} - 6y_{n+2} + 8y_{n+1} - 3y_n = \frac{h^3}{7062}\left[-203 f_{n+4} - 6656 f_{n+3} + 35107 f_{n+2}\right]$$

The first and second characteristics polynomial of the method are given by:

$$\rho(z) = z^4 - 6z^2 + 8z - 3$$

And

$$\sigma(z) = \frac{-203z^4 - 6656z^3 + 35107z^2}{7062}$$ respectively

By definition 2, the method (2.13) is consistent, since it satisfies the following:

i. The order of the method is $p = 2 \geq 1$

ii. $\alpha_0 = -3, \alpha_1 = 8, \alpha_2 = -6$ and $\alpha_3 = 1$

Thus $\sum_{j=0}^{3} \alpha_j, j = 0,1,2,3, \sum_{j=0}^{3} \alpha_j = -3 + 8 - 6 + 1 = 0$

iii. $\rho(z) = z^4 - 6z^2 + 8z - 3$

$\rho(1) = (1)^4 - 6(1)^2 + (1) - 3 = 0$

$\rho^1(z) = 4z^3 - 12z + 8$

$\rho^1(1) = 4(1)^3 - 12(1) + 8 = 0$

$\therefore \rho(1) = \rho^1(1) = 0$

iv. $\rho'''(1) = 3!\sigma(1)$

Recall $\rho'(z) = 4z^3 - 12z + 8$

$\rho''(z) = 12z^2 - 12$

$$\rho'''(z) = 24z$$
$$\therefore \rho'''(1) = 24(1) = 24$$

Recall $\sigma(z) = \dfrac{-203z^4 - 6656z^3 + 35107z^2}{7062}$

Hence $\sigma(1) = \dfrac{-203(1)^4 - 6656(1)^3 + 35107(1)^2}{7062} = 4$

$$\therefore 3!\sigma(1) = 6 \times 4 = 24$$
$$\therefore \rho'''(1) = 3!\sigma(1) = 24$$

3.5 Zero Stability

Definition 3

The continuous implicit multi-step method (2.11) is said to be zero-stable if no root of the first characteristics polynomial $\rho(z)$ has modulus greater than one, and if every root of modulus one has multiplicity not greater than three. [23].

Definition 4

The implicit block method (2.19) is said to be zero stable if the roots $Z_s, S = 1,...,n$ of the first characteristics polynomial $\overline{P}(z)$, defined by

$$\overline{P}(z) = \det\left(Z\overline{A} - \overline{E}\right) \tag{3.9}$$

Satisfies $|Z_s| \le 1$ and every root with $|Z_s| = 1$ has multiplicity not exceeding three in the limit as $h \to 0$

3.6 Zero stability of the block method

From (2.19), using definition as $h \to 0$

$\overline{p}(z) = \det\left[z\overline{A} - \overline{E}\right]$ gives $z^9(z-1)$, which when solved gives; $z_2 = z_3 = z_4 = ... = z_9 = 0$ and $z_1 = 1$

Hence the block method is stable.

3.7 Zero stability of the main method

Recall the first characteristics polynomial of (2.13) given by

$$p(z) = z^4 - 6z^2 + 8z - 3 \tag{3.10}$$

Equating (3.10) to zero and solving for Z, gives $(Z-1)(Z-1)(Z-1)(Z+3) = 0$

$$\therefore Z_1 = Z_2 = Z_3 = 1$$

The roots of Z of (3.10) for $|Z| = 1$ is simple, hence the method is zero stable as $h \to 0$ as defined by (3) and by the stability of the block method (2.19)

3.8 Convergence

The convergence of the continuous implicit multi-step method (2.11) is considered in the light of the basic properties, in conjunction with the fundamental theorem of Dahlquist, [24], for linear multistep method. In what follows, we state Dahlquist's theorem without proof.

Theorem 3.1: Dahlquist theorem [5]

The necessary and sufficient condition for a linear multi-step method to be convergent is for it to be consistent and zero stable.

Remark

The numerical method derived here are considered to be convergent by theorem 3.1 as $h \to 0$. Following theorem (3.1), the method (2.13) is convergence since it satisfies the necessary and sufficient conditions of consistency and zero stability.

Region of Absolute Stability of the method

Definition 5

If the first and second characteristics polynomials of linear multi-step the method are ρ and σ respectively, then the polynomial equation can be written as

$$\pi(r, \overline{h}) \Rightarrow \rho(r) - \overline{h}\,\sigma(r) = 0 \tag{3.11}$$

Where $\overline{h} = (\lambda h)^3$, then $\pi(r, \overline{h})$ is called the stability polynomial of the method defined by p and σ and $\overline{h} = (\lambda h)^3$ is the test equation. [12,13]

So, to get the graph of the stability region,

we make $\overline{h}$ the subject of the formular from (3.11) to get

$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} \tag{3.12}$$

which is plotted in MATLAB environment to produce the required absolute stability region of the method that will be plotted in a graph.

Using definition 5 and expressing the first and second characteristics polynomials of equation (2.13) as $\rho(r) = r^4 - 6r^2 + 8r - 3$, and $\sigma(r) = \frac{-203r^4 - 6656r^3 + 35107r^2}{7062}$ and using the boundary locus method, where

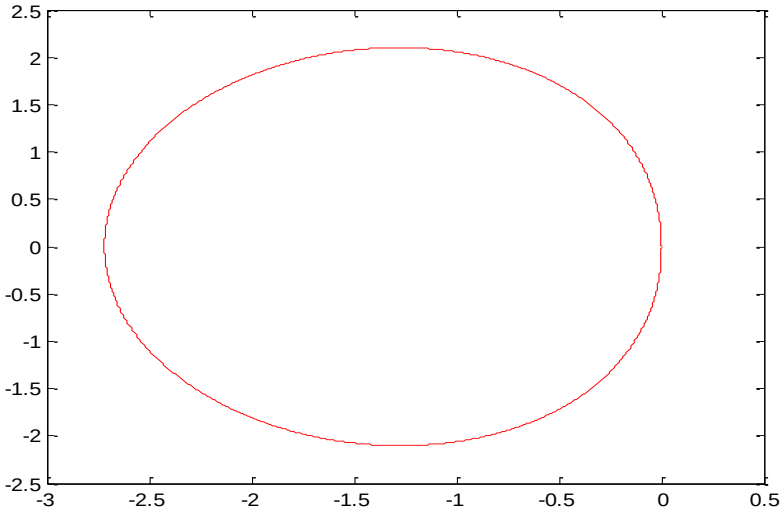
$$\bar{h}(r) = \frac{\rho(r)}{\sigma(r)} = \frac{7062\left(r^4 - 6r^2 + 8r - 3\right)}{-203r^4 - 6656r^3 + 35107r^2}$$

And $\bar{h} = (\lambda h)^3$ and setting $r = e^{i\theta}$, where $\ell^{i\theta} = \cos\theta + i\sin\theta$, we have

$$\bar{h}(\theta) = \frac{7062\left[\left(\cos 4\theta - 6\cos 2\theta + 8\cos\theta - 3\right) + i\left(\sin 4\theta - 6\sin 2\theta + 8\sin\theta - 3\right)\right]}{\left(-203\cos 4\theta - 6656\cos 3\theta + 35107\cos 2\theta\right) + i\left(-203\sin 4\theta - 6656\sin 3\theta + 35107\sin 2\theta\right)}$$ This is simplified to the form

$$x(\theta) + iy(\theta)$$

And plotted in MATLAB environment to produce the required absolute stability region of the method as shown below



4 NUMERICAL EXAMPLES

In order to study the efficiency of the developed method, we present some numerical examples with the following four problems. The continuous implicit multi-step method 4SM was applied to solve the following test problems.

Problem one (Non-Stiff problem)

$y''' = 3\sin x, y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1,$

Exact solution: $y(x) = 3\cos x + \left(\frac{x^2}{2}\right) - 2$

Source: [15]

Problem two (Non-linear problem)

$y''' = y'\left(2xy'' + y'\right), y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0$

$h = 0.1$. Exact solution: $y(x) = 1 + \frac{1}{2} In\left[\frac{2+x}{2-x}\right]$

Source: [16]

Problem three (Linear Problem)

$y''' = x - 4y', y(0) = 0, y'(0) = 0, y''(0) = 1$

$h = 0.1$. Exact solution: $y(x) = -\frac{3}{16}\cos(2x) + \frac{3}{16} + \frac{x^2}{8}$

Source: [17]

Problem four (Stiff problem)

$y''' - y'' + y' - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1$

h =0.01. Exact solution: $y(x) = \cos x$

Source: [18]

Table 1: Showing the exact solution and the computed results from the proposed method for problem one and its comparism with predictor- corrector method of order eight problem in [15]

x-value	Exact solution	4SM	Error in 4SM	Error in predictor/ correction method of order 8. [15]
0.1	0.990012495834	0.990012495345	4.8906e-010	4.1723e-009
0.2	0.960199733523	0.960199730257	3.2663e-009	9.5785e-008
0.3	0.911009467376	0.911009457080	1.0296e-008	3.9916e-007
0.4	0.843182982008	0.843182958499	2.3509e-008	1.0369e-006
0.5	0.757747685671	0.757747640907	4.4764e-008	2.1285e-006
0.6	0.656006844729	0.656006768881	7.5847e-008	3.7895e-006
0.7	0.539526561853	0.539526443415	1.1844e-007	6.1301e-006
0.8	0.410120128041	0.410119953933	1.7411e-007	9.2537e-006
0.9	0.269829904811	0.269829660522	2.4429e-007	1.3257e-005
1.0	0.120906917604	0.120906587327	3.3028e-007	1.822e-005

It can be observed in table 1 that the four step block multi-step method of order p=2 is more accurate than predictor-corrector method of order p=8 in [15].

Table 2: Showing the exact solution and the computed results from the proposed method for problem two and its comparism with non linear problem in [16]

x-value	Exact solution	4SM	Error in 4SM	Error in [16]
0.1	1.050041729278	1.050041729298	2.0348e-011	1.9315e-008
0.2	1.100335347731	1.100335347299	4.3140e-010	5.6083e-007
0.3	1.151140435936	1.151140430348	5.5879e-009	3.7551e-006
0.4	1.202732554054	1.202732525793	2.8261e-008	1.3403e-005
0.5	1.255412811882	1.255412714163	9.7719e-008	3.2591e-005
0.6	1.309519604203	1.309519334228	2.6997e-007	5.8165e-005
0.7	1.365443754271	1.365443105333	6.4894e-007	7.1524e-005
0.8	1.423648030193	1.423647514442	1.4158e-006	2.5648e-005
0.9	1.484700278594	1.484697384407	2.8942e-006	1.7092e-004
1.0	1.549306144334	1.549300493471	5.6508e-006	6.7064e-004

It can be observed in table 2, that the four step block multistep method of order p=2 is more accurate than the non-linear problem in [16].

Table 3: Showing the exact solution and the computed results from the proposed method for problem three and its comparism with problem in [17]

x-value	Exact solution	4SM	Error in 4SM	Error in [17]
0.1	0.004987516654	0.004987513772	2.8818e-009	1.6655e-008
0.2	0.019801063624	0.019801030731	3.2893e-008	3.8096e-007
0.3	0.043999572204	0.043999452664	1.1954e-007	1.5665e-006
0.4	0.076867401997	0.076867204907	2.8709e-007	3.9866e-006
0.5	0.117443317649	0.117442763670	5.5398e-007	7.9597e-006
0.6	0.164557921035	0.164556991288	9.2975e-007	1.3680e-005
0.7	0.216881160706	0.216879745800	1.4149e-006	2.1196e-005
0.8	0.272974910431	0.272972910201	1.9995e-006	3.0396e-005
0.9	0.331350392754	0.331347729201	2.6636e-006	4.1009e-005
1.0	0.390527531852	0.390524154206	3.3776e-006	5.2605e-005

It could be observed in table 3, that the four step block multi-step method of order p=2 is better off than that in [17]

Table 4: Showing the exact solution and the computed results from the proposed method for problem four and its comparism with problem in [18]

x-value	Exact solution	4SM	Error in 4SM	Error in [18]
0.01	0.9999500004166	0.9999500004161	5.1470e-013	6.7200e-007
0.02	0.9998000066665	0.9998000066633	3.1950e-012	1.3441e-006
0.03	0.9995500337489	0.999550037404	8.5684e-012	2.0170e-006
0.04	0.9992001066609	0.9992001066443	1.6662e-011	2.6884e-006
0.05	0.9987502603949	0.9987502603668	2.8100e-011	3.3594e-006
0.06	0.9982005399352	0.9982005398906	4.4536e-011	
0.07	0.9975510002523	0.9975510001867	6.6520e-011	
0.08	0.9968017063026	0.9968017062085	9.4104e-011	
0.09	0.9959273301190	0.9959527328862	1.2574e-010	
0.10	0.9950041652780	0.9950041651140	1.6393e-010	

It could be observed in table 4, that the four-step block multi-step method of order p=2 is more accurate than that in [18]

5 CONCLUSION

In this study, we developed a continuous implicit multi-step method with application to third- order ordinary differential equations. The method is consistent, convergent and zero stable. The method derived, efficiently solved third order initial value problem as displaced in tables 1 - 4. In terms of accuracy, our method performs better than the existing methods compared with in the literature.

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