

Testing of UBAC(2) Class of Life Distributions based on TTT - Transform

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ABSTRACT

In this paper, a new test statistic for testing exponentiality against used better than aged in increasing concave ordering UBAC(2) is constructed based on total time on test (TTT)-Transform. Critical values are tabulated for sample size $n = 10(5)100$. The power of the test is estimated for some commonly used distributions in reliability. Finally, medical applications for real data are proposed to illustrate the theoretical results.

Keywords

UBAC(2) classes of life distributions; Survival function; Exponentiality; Total time on test (TTT)-transform; Monte Carlo method.

1. INTRODUCTION

Nonparametric test for testing exponentiality versus some classes of life distributions has received a good deal of attention. For testing against UBAC(2), we refer to IFR, DMRL, UBA, UBAE, DVRL see Bryson and Siddiqui [1], Barlow and Proschan [2], Abu-Youssef [3] and Ahmad [4].

The scaled TTT-transform and the TTT-plot were introduced by Barlow and Campo [5]. These concepts have proved to be very useful in the statistical analysis of life data.

Applications, properties and interpretations of the TTT-transform order in the statistical theory of reliability and economic can be found in Klefsjo[6,7,8] Bergman and Klefsjo [9], Pham and Turkkan [10], Bartoszewicz [11], Bartoszewicz [12], Haupt and Schabe [13], Kochar et al. [14], Li and Zou [15], Ahmed et al. [16], Li and Shaked [17], Al-Nachawati [18], Nanda and Shaked [19], Abu-Youssef et al [20], M. M. Mohie El-Din et al [21].

Let X denote the life time of a specific item with distribution function $F(x) = P(X \leq t)$, survival function $\bar{F} = 1 - F$ and has finite mean μ and variance σ^2

Definition(1.1): The distribution function F is said to be UBAC(2) if

$$\int_t^\infty \bar{F}(s)ds - \int_{x+t}^\infty \bar{F}(s)ds \geq (1 - e^{-x})\bar{F}(t). \quad (1.1)$$

which can be written as :

$$v(t) - v(x + t) \geq (1 - e^{-x})\bar{F}(t). \quad (1.2)$$

See Ali [22].

Our proposed class UBAC(2) includes many classes of life distributions. Willmot and Cai [23] showed that the UBA (used better than aged) class includes the DMRL (decreasing mean residual life) class. While Di Crescenzo [24] has shown that UBAE (used better than aged in expectation) class is contained in the HUBAE (harmonic used better than aged in

expectation) class. And from Mohi el Din et al [25]. So we have:-

$$IFR \Rightarrow UBA \Rightarrow UBAC(2)$$

Thus we have

$$IFRIFR \subset DMRL \subset UBA \subset UBAC(2)$$

U

$$HUBAE \subset UBAE$$

For definition and properties of these classes, you can see Deshpande et al. [26] and Deshpande and Purohit [27].

The main theme of this paper is dealing with the problem of testing $H_0: \bar{F}$ is exponential against $H_1: \bar{F}$ is UBAC(2) and isn't exponential based on TTT Transform.

In sec (2), the concept of total time on test transform is given. In sec (3), we derive the empirical test statistic for the UBAC (2) class of life distribution based on TTT-transform. The critical values via monte carlo method and the power of the test are tabulated in sec (4). Finally, applications using real data are also presented in sec (5).

2. THE CONCEPT OF TORAL TIME ON TEST TRANSFORM (TTT-TRANSFORM)

We present the following definitions of Barlow and Campo [5].

Definition (2.1): The total time on test transform (TTT-Transform) of F is defined as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(s)ds. \quad (2.1)$$

where $F^{-1}(t) = \inf\{x: F(x) \geq t\}$ and $0 \leq t \leq 1$.

This transform was first studied by Barlow and Doksum [28] and Barlow et al [29]. Since the mean of F is given by

$$\mu = H_F^{-1}(1) = \int_0^{F^{-1}(1)} \bar{F}(s)ds ,$$

then the transform

$$\phi_F(t) = \frac{1}{\mu} \int_0^{F^{-1}(t)} \bar{F}(s)ds. 0 \leq t \leq 1. \quad (2.2)$$

is scale invariant and is named the scaled TTT-transform. The scaled TTT-transform and an empirical counterpart, named the TTT-plot, were presented by Barlow and Campo [5] as a tool for identification of failure distribution models. Scaled TTT-transforms for some families of life distributions are given by Barlow and Campo [5], Barlow [30], Bergman and Klefsjo [9], here we present the following theorem to find the scaled TTT-transforms for UBAC(2) class of life

distributions.

Definition (2.2): The function

$$\phi_F(t) = \frac{H_{\bar{F}^{-1}(t)}}{H_{\bar{F}^{-1}(1)}} \quad (2.3)$$

is called the scaled TTT-transform. if F is the exponential distribution then TTT-transform is given by

$$\phi_F(t) = \frac{\int_0^{F^{-1}(t)} e^{-u} du}{\int_0^{F^{-1}(1)} e^{-u} du} = t, 0 \leq t \leq 1. \quad (2.4)$$

Theorem 2.1. Let F and $\phi_F(t)$ be a distribution function and scaled TTT-transform, then F is UBAC(2) if

$$\frac{1}{\mu} \int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \bar{F}(s) ds - \phi_F(w_2) \geq \frac{1}{\mu} [1 - w_2] (1 - e^{-F^{-1}(w_1)}) \quad (2.5)$$

Proof: Since F is UBAC(2), then

$$v(t) - v(x+t) \geq (1 - e^{-x}) \bar{F}(t), \quad (2.6)$$

$$\int_t^\infty \bar{F}(s) ds - \int_{x+t}^\infty \bar{F}(s) ds \geq (1 - e^{-x}) \bar{F}(t), \quad (2.7)$$

$$\int_0^{x+t} \bar{F}(s) ds - \int_0^t \bar{F}(s) ds \geq (1 - e^{-x}) \bar{F}(t), \quad (2.8)$$

Let $x = F^{-1}(w_1)$, $t = F^{-1}(w_2)$, $0 \leq w_1, w_2 \leq 1$.

Then we can write (2.8) as

$$\int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \bar{F}(s) ds - \int_0^{F^{-1}(w_2)} \bar{F}(s) ds \geq [1 - w_2] (1 - e^{-F^{-1}(w_1)}) \quad (2.9)$$

After dividing (2.9) by μ , hence the theorem is proved.

3. TEST STATISTIC BASED ON THE SCALED TTT- TRANSFORM

In this section we present a test statistic using the scaled TTT-transform for testing $H_0: \bar{F}$ is exponential against $H_1: \bar{F}$ is UBAC(2) and isn't exponential is studied for a random sample X_1, X_2, \dots, X_n from a population with distribution function F .

We proposed the following measure of departure

$$\delta_{U_T} = \frac{1}{\mu} \int_0^{F^{-1}(w_1)+F^{-1}(w_2)} \bar{F}(s) ds$$

$$-\phi_F(w_2) - \frac{1}{\mu} [1 - w_2] (1 - e^{-F^{-1}(w_1)}) \quad (3.1)$$

Note that under $H_0: \delta_{U_T} = 0$, while under $H_1: \delta_{U_T} \geq 0$.

Let $X_{(1)} \leq \dots \leq X_{(n)}$ be the ordered statistics of the independent random sample X_1, \dots, X_n and \bar{X} be the sample mean where $X_{(0)} = 0$. Then the empirical distribution of the distribution function F is $\hat{F}(x) = F_n(x) = \frac{1}{n} (X_{(i)} \leq x)$, $i = 1, 2, \dots, n$, $\bar{F}(x)$ the survival function is estimated by $\bar{F}_n(x) = 1 - F_n(x)$, $H_{\bar{F}^{-1}(x)}$ the TTT-transform is estimated by $H_{\bar{F}_n^{-1}(x)} = \frac{S_j}{n}$, where $S_j = \sum_{i=1}^j D_i$, $D_i = (n - i + 1)(X_{(i)} - X_{(i-1)})$ "the normalized sample spacing" and $\phi_F(x)$ the scaled TTT-transform is estimated by $\phi_{F_n}(x) = \frac{S_j}{n\bar{X}}$. Hence, the estimate of δ_{U_T} is proposed by

$$\hat{\delta}_{U_T} = \frac{1}{n\bar{X}} \sum_{m=1}^L (n - m + 1) [X_{(m)} - X_{(m-1)}] - \frac{1}{n\bar{X}} \sum_{j=1}^n (n - j + 1) [X_{(j)} - X_{(j-1)}] - \frac{1}{\bar{X}} \left[1 - \frac{j-1}{n} \right] [1 - e^{-X_{(i)}}]$$

where $i = 1, 2, \dots, n, j = 1, 2, \dots, n$ and $X_{(1)} \leq \dots \leq X_{(n)}$ are the ordered statistics of the independent random sample X_1, \dots, X_n .

Note that $L = i+j$ if $i + j < n$ but $L = n$ if $i + j \geq n$

To reduce the size of the test statistic we use:

$$\hat{\Delta}_{U_T}^* = \frac{\hat{\delta}_{U_T}}{n^2} \quad (3.2)$$

4. MONTE CARLO NULL DISTRIBUTION CRITICAL POINTS

We calculate, via Monte Carlo method, the empirical critical points of $\hat{\delta}_{U_T}^*$ in Eq. (3.2) for samples 10(5)100.

Tables (4.1) gives the lower and the upper percentile points for 1%, 5%, 10%, 90%, 95%, 99% The calculations are based on 10000 simulated samples of sizes $n = 10(5)100$.

Table (4.1) Critical Values of $\hat{\Delta}_{U_T}^*$

n	1%	5%	10%	90%	95%	99%
10	-5.29544	-5.23303	-5.19978	-4.96655	-4.9333	-4.87089
15	-3.08288	-3.03193	-3.00478	-2.81435	-2.7872	-2.73624
20	-1.96965	-1.92552	-1.90201	-1.73709	-1.71358	-1.66945
25	-1.36901	-1.32954	-1.30851	-1.161	-1.13997	-1.1005
30	-1.01254	-0.976512	-0.957313	-0.822659	-0.803461	-0.76743
35	-0.784398	-0.75104	-0.733265	-0.6086	-0.590826	-0.557468
40	-0.629619	-0.598416	-0.581789	-0.465175	-0.448549	-0.417346
45	-0.519704	-0.490285	-0.47461	-0.364665	-0.34899	-0.319571
50	-0.43874	-0.410831	-0.39596	-0.291658	-0.276786	-0.248877
55	-0.377292	-0.350681	-0.336502	-0.237054	-0.222875	-0.196265
60	-0.329481	-0.304004	-0.290428	-0.195214	-0.181638	-0.156161
65	-0.291493	-0.267015	-0.253972	-0.162493	-0.14945	-0.124972
70	-0.260764	-0.237177	-0.224608	-0.136456	-0.123888	-0.100301
75	-0.235518	-0.21273	-0.200588	-0.115426	-0.103283	0.0804958

80	-0.214495	-0.192431	-0.180674	-0.0982158	-0.086459	-0.064395
85	-0.196778	-0.175373	-0.163967	-0.0839708	-0.0725651	-0.0511598
90	-0.181689	-0.160887	-0.149803	-0.0720603	-0.0609759	-0.0401737
95	-0.168717	-0.148469	-0.137681	-0.0620118	-0.051223	-0.0309757
100	-0.157469	-0.137734	-0.127219	-0.0534658	-0.0429503	-0.0232156

It is shown from the table that the values of the percentiles decreases when the sample size increases.

Finally, the power of the test statistics $\hat{\Delta}_{U_T}$ is considered for 95% percentiles in Table (4.2) for three of the most commonly used alternatives [see Hollander and Proschan [31]], they are

(i) Linear failure rate: $\bar{F}_\theta = e^{-x - \frac{\theta x^2}{2}}, x > 0, \theta > 0$

(ii) Makeham: $\bar{F}_\theta = e^{-x - \theta(x + e^{-x} - 1)}, x \geq 0, \theta > 0$

(iii) Weibull: $\bar{F}_\theta = e^{-x^\theta}, x \geq 0, \theta > 0$

These distributions are reduced to exponential distribution for appropriate values of θ

Power estimates increase when the parameter θ is far from exponentiality and when the size of the sample n increases.

5. APPLYING THE TEST

Example 1

Consider real data representing 40 patients suffering from blood cancer. We use the data as given in Abu-Youssef [32]. The ordered life times (in day) are:

115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1169, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1578, 1599, 1603, 1604, 1696, 1735, 1799, 1815, 1852. Using equation (3.2), the value of test statistics, based on the above data is $\hat{\Delta}_{M_T} = -0.870536$. the critical value at $\alpha = 0.05$ is -0.448549 . This value leads to the rejection of H_0 at the significance level $\alpha = 0.05$. Therefore the data has UBAC(2) Property.

Example 2

Table (4.2) Power Estimate of $\hat{\Delta}_{U_T}$

Distribution	θ	Sample Size		
		n = 10	n = 20	n = 30
F_1 Linear failure rate	2	0.99974	1	1
	3	0.99896	1	1
	4	0.99702	1	1
F_2 Makham	2	0.99978	1	1
	3	0.99942	1	1
	4	0.99846	1	1
F_3 Weibull	2	1	1	1
	3	1	1	1
	4	1	1	1

In an experiment at Florida state university to study the effect of methyl mercury poisoning on the life lengths of fish goldfish were subjected to various dosages of methyl mercury (Kochar [33]). At one dosage level the ordered times to death in week are:

6, 6.143, 7.286, 8.714, 9.429, 9.857, 10.143, 11.571, 11.714, 11.714

The value of test statistics, based on the above data is $\hat{\Delta}_{U_T} = -0.875165$. the critical value at $\alpha = 0.05$ is -4.9333 . Then H_0 at the significance level $\alpha = 0.05$ is rejected. Therefore the data has UBAC(2) Property.

Example 3

The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt Attia et al [34]. The ordered life times (in days) are:

107, 18, 74, 20, 23, 20, 23, 24, 52, 105, 60, 31, 75, 107, 71, 107, 14, 49, 10, 15, 30, 26, 14, 87, 51, 17, 116, 67, 20, 14, 40, 14, 30, 96, 20, 20, 61, 150, 14

The value of test statistics, based on the above data is $\hat{\Delta}_{M_T} = -0.930386$. the critical value at $\alpha = 0.05$ is -0.472933 , then we reject H_0 at the significance level $\alpha = 0.05$. Therefore the data has UBAC(2) Property.

6. CONCLUSION

In this paper, Testing exponentiality versus used better than aged in increasing concave ordering UBAC(2) based on total time on test (TTT)-Transform is proposed. The percentiles of proposed test are tabulated in table (4.1), which clearly show the critical values of the estimate of departure from measure is decreasing as the sample size increasing. This test is more efficient than the test for (UBAC). It gives a very good power for most alternatives.

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8. AUTHORS' CONTRIBUTIONS

The authors read and approved the final manuscript.

9. COMPETING INTERESTS

The authors declare that they have no competing interests.

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