# Weak Domination in LICT Graphs

M. H. Muddebihal
Professor
Department of Mathematics,
Gulbarga University,
Kalaburagi-585106,
Karnataka, India

Geetadevi Baburao
Research Scholar
Department of Mathematics,
Gulbarga University,
Kalaburagi-585106,
Karnataka, India

## ABSTRACT
The lict graph n(G) of a graph G is the graph whose set of vertices is the union of set of edges and the set of cut vertices of G in which two vertices are adjacent if and only if the corresponding edges are adjacent or the corresponding members of G are incident.In this paper, we initiate the study of variation of standard domination, namely weak lict domination. A weak dominating set D is a weak dominating set of n(G), if for every vertex $y \in V[n(G)] - D$ there is a vertex $x \in D$ with $\deg(x) \leq \deg(y)$ and y is adjacent to x. A weak domination number of $n(G)$ is denoted by $\gamma_{wn}(G)$, is the smallest cardinality of a weak dominating set of $n(G)$. We determine best possible upper and lower bounds for $\gamma_{wn}(G)$, in terms of elements of $G$.

## Keywords
Domination, Double domination, restrained domination, weak domination, weak lict domination.

## Subject classification number: AMS-05C69, 05C70.

## 1. INTRODUCTION
In this note, we will use the notation and terminology in [1].Let $G = (V, E)$ be a simple graph with $|V| = p$ and $|E| = q$. The degree, neighborhood and closed neighborhood of a vertex $v$ in a graph $G$ are denoted as $\deg(v)$, $N(v)$ *and* $N[v] = N(v) \cup v$ respectively. We use $\Delta(G)$ $[\Delta'(G)]$ denotes the maximum degree of a vertex (edge)in $G$ and $\delta(G)$ $\delta'(G)$ denotes the minimum degree of a vertex (edge) in $G$. The notation $\alpha_0(G)$ $[\alpha_1(G)]$ is the minimum number of vertices (edges) in a vertex (edge) covers of $G$. The notation $\beta_0(G)[(\beta_1)]$ is the minimum number of vertices (edges) in a maximal independent set of a vertex (edge) of $G$.

We say a set $S$ is a dominating set of $G$, if for every vertex $u \epsilon V - S$ , there exists a vertex $v \in S$ such that $uv \in E$. The domination number of $G$, denoted by $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. For compressive work on the subject has been done in [3],[4].A dominating set $S$ of $G$ is called a connected dominating set of $G$, if the induced subgraph $< S >$ is connected. The minimum cardinality taken over all the minimal connected domination number and is denoted by $\gamma_c(G)$. A dominating set $S \subseteq V(G)$ is called the total dominating set, if for every vertex $v \epsilon V$, there exist a vertex $u \in S$, $u \neq v$ such that $u$ is adjacent to $v$. The total domination number of $G$ is denoted by $\gamma_t(G)$ is the minimum cardinality of total dominating set of $G$.A set $S$ of vertices in a graph $G$ is an independent dominating set of $G$, if $S$ is an independent set and every vertex not in $S$ is adjacent to a vertex in $S$. The smallest cardinality of an independent dominating set is called an independent domination number of $G$ and is denoted by $i(G)$.A dominating set $D$ of a graph $G$ is a split dominating set if the induced subgraph $< V - D >$ is disconnected. The split domination number $\gamma_s(G)$ is the least cardinality of a split dominating set of $G$.

In [5], the strong slit domination was given as follows. A set $S \subseteq G$ is called the strong slit dominating set of $G$, if $< V - S >$ is totally disconnected with at least two isolated vertices. The strong slit domination number $\gamma_{ss}(G)$ is the lowest level cardinality of minimal strong split dominating set of $G$.

A dominating set $D$ of a graph $G = (V, E)$ is a cotatal dominating set if, every vertex $v \in V - D$ is not an isolated vertex in the induced subgraph $< V - D >$. The cototal domination number $\gamma_{cot}(G)$ of $G$ is the smallest cardinality of a cototal dominating set of $G$.A set $S \subseteq V(G)$ is a Restrained dominating set of $G$, if every vertex in $V - S$ is adjacent to a vertex in $S$ as well as another vertex in $V - S$.The Restrained domination number of a graph $G$ is denoted by $\gamma_{Res}(G)$ is the least cardinality of a Restrained dominating set of $G$.Given two adjacent vertices $u$ and $v$ of $G$. We say $v$ weakly dominates $u$, if $\deg(v) \leq \deg(u)$. A set $S \subseteq V(G)$ is a weakly dominating set, if every vertex in $< V - S >$ is weakly dominated by at least one vertex in $S$. Weak domination number $\gamma_w(G)$ is the minimum cardinality of a weak dominating set of $G$.

In [7], the author showed that a weak dominating set $S$ is a weak dominating set of $L(G)$, if for every vertex $u \in V[L(G)] - S$ there is a vertex $v \in S$ with $\deg(v) \leq \deg(u)$ and $u$ is adjacent to $v$. A weak line domination number $\gamma_{wl}(G)$ is the least cardinality of a weak line dominating set $S$ of $L(G)$.

In [6], the author has defined that the lict graph $n(G)$ of a graph $G$ is the graph whose set of vertices is the union of set of edges and the set of cutvertices of $G$ in which two vertices are adjacent if and only if the corresponding edges are adjacent or corresponding members of $G$ are incident

In [8], Sampathkumar and Pushpa Latha have shown weak and strong domination number.

In this paper, we study the theoretic properties of $\gamma_{wn}(G)$ and many bounds are obtained in terms of elements of $G$ and its relationship with other domination parameter were found.

In section 2 we determine this parameter for some standard graphs. We obtained best possible upper and lower bound for $\gamma_{wn}(G)$.

## 2. RESULTS
First we list out the exact values of $\gamma_{wn}(G)$ for some standard graphs.

*Theorem 2.1:* **i].**For any path $P_n$ with $n \geq 3$ vertices

$\gamma_{wn}(G) = P - 2$ .

**ii]**.For any star $K_{1,n}$ with $n \geq 2$ vertices

$$\gamma_{wn}[K_{1,n}] = \gamma(G) = \gamma_n(G) = \gamma_s(G) = \gamma_w(G) = \gamma_{wl} = 1.$$

**iii].** For any cycle $C_p$with p≥ 3 vertices.

$$\gamma_{wn}(C_p) = \begin{cases} \frac{p}{3} if\ p \equiv O(mod3) \\ \left\lceil \frac{p}{3} \right\rceil\ otherwise. \end{cases}$$

**iv].** For any wheel $W_p$ with $p \geq 4$ vertices.

$$\gamma_{wn}(W_p) = \begin{cases} \frac{p}{2}\ if\ p\ is\ even \\ \left\lfloor \frac{p}{2} \right\rfloor if\ p\ is\ odd \end{cases}$$

**v].** For any complete graph $K_p$ with $p \geq 3$ vertices.

$$\gamma_{wn}(K_p) = \begin{cases} \frac{p}{3}\ if\ p \equiv 0(mod3) \\ \left\lceil \frac{p}{3} \right\rceil\ Otherwise. \end{cases}$$

**vi].** For any star $K_{1,p}$ with $p \geq 2$ vertics $\gamma_{wn}(K_{1,p}) = \alpha_0$ .

## Upper bounds for $\gamma_{wn}(G)$.

We obtained an upper bound for $\gamma_{wn}(G)$ in terms of $\beta_1(G)$ and weak line domination $\gamma_{wl}(G)$ .

*Theorem 2.2:* For any graph $G, \gamma_{wn}(G) \leq \gamma_{wl}(G) + \beta_1 + 1$.

*Proof:* Let $B = \{e_1, e_2, e_3, \ldots, e_m\} \subseteq E(G)$ with $N(e_i) \cap N(e_j) = e$ for every $e_i,\ e_j \in B,\quad 1 \leq i \leq m,\ 1 \leq j \leq m$ and $e \in E(G) - B$. Clearly $B$ forms a maximal independent edge set in $G$.Let $A = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq V[L(G)]$ be the set of vertices with $\deg(u_i) \geq 1,\ 1 \leq i \leq n$. Such that $N[A] = V[L(G)]$. Hence $A$ forms a $\gamma - set$ of $L(G)$.

Suppose there exists a set $A_1 \subseteq V[L(G)] - A$ such that $\forall\ u_i \in A_1, \deg(u_i) \geq \deg(u_k),\ \ \forall\ u_k \in A$. Then $A \cup A_1$ forms a minimal weak dominating set of $L(G)$.Suppose $S = \{u_1 u_2 u_3, \ldots, u_n\} \subseteq V[n(G)]$ and $\deg(u_m) \geq \deg(u_n)$, $\forall\ u_m \in V[n(G)] - S$ and $\forall\ u_n \in S$ such that $N[S] = V[n(G)]$. Then $S$ forms a dominating set of $n(G)$.Suppose there exists a set $S_1 \subseteq V[n(G)] - S$ such that $u_i \in S\ deg(u_i) \geq \deg(u_k)$, $\forall\ u_k \in S_1$. Then $S \cup S_1$ forms a minimal weak dominating set of $n(G)$. Hence it gives $|S \cup S_1| \leq |A \cup A_1| + |B| + 1$. Clearly $\gamma_{wn}(G) \leq \gamma_{wl}(G) + \beta_1 + 1$.

*A Roman domination* function of a graph $G = (V, E)$ is a fuction $f: V \to \{0,1,2\}$ satisfying the condition that every vertex $u$ for which $f(u) = 0$ is adjacent to at least one vertex $v$ for which $f(v) = 2$. The weight of a *Roman dominating function* in $G$ is the value of $f(v) = \sum_{u \epsilon v} f(u)$.The *Roman domination number* of $G$ is denoted by $\gamma_R(G)$, equals the smallest weight of a *Roman dominating* function on $G$. Farther we relates our concept to$\gamma_R(G)\ and\ \alpha_1(G)$.

*Theorem 2.3:* For any connected $(p, q)$ graph $G, with\ (p \geq 2)$, then $\gamma_{wn}(G) \leq \alpha_1(G) + \gamma_R(G) + 1$.

*Proof:* Let $f: V_i \to \{0, 1, 2\}$ ; $i = 0, 1, 2$ be a Roman domination function of $D'$ where $D'$ is a Roman dominating set of $G$. Let $A = \{e_j\},\ j = 1, 2, \ldots, m$ be the minimal edge caver of $G$, so that $|A| = \alpha_1(G)$. Now in $n(G)$, let $B = \{u_1, u_2, u_3, \ldots, u_n\} \subseteq V[n(G)]$ be the minimal set of vertices where $N[B] = V[n(G)]$. Then $B$ is a dominating set of $n(G)$. Suppose $\forall\ v_i \in V[n(G)] - B$, $deg(v_i) \leq \deg(v_k)$ where $\forall\ v_k \in B$. Then $B \cup \{v_i\}$ forms a minimal weak lict $\gamma - set$ of $G$. Thus $|B \cup \{v_i\}| \leq |A| + |D'| + 1$ gives $\gamma_{wn}(G) \leq \alpha_1(G) + \gamma_R(G) + 1$.

*Theorem 2.4:* For any nontrivial tree $T$, $\gamma_{wn}(T) \leq i(T) + \gamma_{Res}(T) + 1$.

*Proof:* Suppose $D = \{v_1, v_2, \ldots \ldots \ldots, v_n\} \subseteq V(T)$ be the set of vertices such that $N[D] = V(T)$. Then $D$ is a minimal $\gamma - set$ of $T$.If $< D >$ is a set of isolates, then $D$ itself is an independent dominating set of $T$. Otherwise, if there exists an edge $e$ in $D$ then $D - \{e\}$ forms an independent dominating set of $T$. Since $D$ is dominating set of $T$. If the set of vertices in $V - D$ is adjacent to at least one vertex of $D$ and at least one vertex of $V - D$, then $D$ itself is a $\gamma_{Res} - set$ of $T$.In lict graph $n(T)$, let $M = \{v_1, v_2, \ldots \ldots, v_n\} \subseteq V[n(T)]$ be the set of minimum degree vertices such that $N[M] = V[n(T)]$.Then $M$ forms a minimal $\gamma_{wn} - set$ of $T$. Hence $|M| \leq |D - \{e\}| + |D|$, gives $\gamma_{wn}(T) \leq i(T)$+$\gamma_{Res}(T) + 1$.

## Lower Bounds for $\gamma_{wn}(G)$:

Here we have found lower bounds for $\gamma_{wn}(G)$ in terms of elements of $G$ and other domination parameters of $G$.

*Theorem 2.5:* For any connected $(p, q)$ graph G, $\gamma_{wn}(G) \geq \gamma(G)$.

*Proof:* Since the $V[n(G)] = E(G) \cup C(G)$, C (G) a cutvertex set of $G$.Now we deal with two cases.

*Case1:* Suppose $G$ is a nontrival tree. Then one can see that $\gamma_{wn}(G) \geq \gamma(G)$.

*Case2:* Suppose G is not a tree. Then again it is true that $\gamma_{wn}(G) \geq \gamma(G)$.

*Theorem 2.6:* For any connected$(p, q)$ graph $G$, $\gamma_{wn}(G) \geq diam(G) - 1$.

*Proof:* Let $D = \{e_1, e_2, e_3, \ldots, e_m\} \subseteq E(G)$ be the set of edges which constitute the diametral path in $G$. Then $|D| = diam(G)$. Farther consider $E = \{e_1, e_2, e_3, \ldots \ldots, e_m\}$ ; $C = \{c_1, c_2, \ldots \ldots, c_n\}$ be the set edges and cutvertices in G. Then $V[n(G)] = E(G) \cup C(G)$. Let $S = \{u_1, u_2, u_3, \ldots, u_n\}$ be the minimum degree vertices in $n(G)$, which covers all the vertices in $n(G)$ such that $N[S] = V[n(G)]$. Thus $S$ forms a minimal weak dominating set of $n(G)$. It follows that $|S| \geq |D| - 1$. Hence $\gamma_{wn}(G) \geq diam(G) - 1$.

In [2 ],the author has defined the *double domination*. A dominating set $D \subseteq V(G)$ is said to be *double dominating* set of $G$, if every vertex of $V$ is dominated by at least two verities in $D$. The *double domination number* $\gamma_{dd}(G)$ of $G$ is the least cardinality of a *double dominating set of* $G$.

*Theorem 2.7:* For any connected$(p, q)$ graph G, then $\gamma_{wn}(G) \geq \left\lfloor \frac{\gamma_{dd}(G)}{\Delta(G)} \right\rfloor$.

*Proof:* Suppose $A = \{v_1, v_2, v_3, \ldots, v_j\} \subseteq V(G)$ be the set vertices with $\deg(v_i) \geq 2, 1 \leq i \leq j$ such that $N[A] = V(G)$. Hence $A$ forms a minimal dominating set of $G$. $A$ set $A \subseteq V(G)$ is a double dominating set, if for every vertex $v \in V(G)$, $A' = |N[v] \cap A| \geq 2$.Now we consider the lict graph

$n(G)$ of a graph $G$. Such that $S = \{v_1, v_2, v_3, \dots, v_k\} \subseteq V[n(G)]$ be the set of vertices with $\deg v_l \geq 2; 1 \leq l \leq k$ and if $N[S] = V[n(G)]$. Then $S$ forms a $\gamma - set$ of $n(G)$.Suppose there exists a set $S_1 \subseteq V[n(G)] - S$ such that $\forall\, v_i \in S_1$, $deg\,(v_i) \leq deg\,(v_j)$, $\forall\, v_j \in S$. Then $S \cup S_1$ forms a minimal weak dominating set of $n(G)$. Suppose there exists at least one vertex $v \in A$ of maximum degree such that $\deg(v) = \Delta(G)$.Hence $|S \cup S_1|.\Delta(G) \geq |A|$. Clearly, it follows that $\gamma_{wn}(G) \geq \left\lfloor \frac{\gamma_{dd}(G)}{\Delta(G)} \right\rfloor$.

*Theorem 2.8:* For any tree $T$ with $K$ number of cut vertices then $\gamma_{wn}(T) \geq K$.

*Proof:* Let $B = \{c_1, c_2, c_3, \dots, c_n\}$ be the set of all cutvertices in tree $T$, with $|B| = K$. Further let $J = \{e_1, e_2, e_3, \dots, e_i\}$ be the set of all end edges in $T$, also $J' = \{e_1', e_2', e_3', \dots, e_i'\}$ be the set of all nonend edges in $T$. Then in lict graph $n(T), V[n(T)] = J \cup J' \cup B$. Suppose $D \subseteq V[n(T)]$ and $N[D] = V[n(T)]$ and $\deg(v_i) < deg\,(v_j)$ where $v_i \in V[n(T)] - D$ and $v_j \in D$. Since $v_i \in D_1 = V[n(T)] - D$, then $|D \cup D_1| \geq |K|$ which gives $\gamma_{wn}(T) \geq K$.

*Theorem 2.9:* For any non trivial tree $T$, vertices and $l$ end vertices and $S$ the number of support vertices, then $\gamma_{wn}(T) \geq \left\lceil \frac{q-2-l+s}{3} \right\rceil$.

*Proof:* Let $I = \{u_1, u_2, u_3, \dots, u_m\} \subseteq V(T)$ be the set of all end vertices in $T$ with $|I| = l$ and let $J = \{v_1, v_2, v_3, \dots, v_n\}$ be the support vertices in $T$ with $|J| = S$. In lict graph $n(T)$, $D = \{v_1, v_2, \dots, v_i\} = V[n(T)]$. Suppose $D_1 \subseteq D$, $\forall\, v_j \in D_1$, $\deg\,(v_j) \geq 2$ and $D_2 \subseteq D$ be the set of minimum degree vertices which are adjacent to a cutvertex of $n(T)$, since each block of $n(T)$ is complete and covers all the vertices of $n(T)$. Then $\{D_1 \cup D_2\}$ is a minimal weak dominating set of $n(T)$.Thus $3|D_1 \cup D_2| \geq |E(T)| - 2 - |I| + |J|$, gives $\gamma_{wn}(T) \geq \left\lceil \frac{q-2-l+s}{3} \right\rceil$.

A dominating set $S \subseteq V(G)$ is a *strong nonsplit dominating set*, if the induced subgraph $< V - S >$ is complete. The *strong nonsplit domination* number $\gamma_{sns}(G)$ of $G$ is minimum cardinality of a *strong nonsplit dominating* set of $G$.

*Theorem 2.10:* For any connected $(p, q)$ graph $G$, with $p \geq 2$ then $\gamma_{wn}(G) \geq \left\lfloor \frac{\gamma_{sns}(G)+2}{3} \right\rfloor$ and $G \neq K_{1,n}(n \geq 4)$.

*Proof:* Suppose $G = K_{1,n}$, $n \geq 4, \gamma_{wn}(G) = 1 < \left\lfloor \frac{n+2}{3} \right\rfloor$, a contradiction to the fact.

Lat $D = \{v_1, v_2, \dots \dots \dots, v_n\}$ be the vertex set of G. Suppose $D_1 = \{v_1, v_2, \dots \dots \dots, v_m\} \subseteq D$ and if $N[D_1] = V(G)$. Then $D_1$ is a minimal dominating set of$G$. If the induced subgraph $< V - D_1 >$ is complete, then $D_1$ itself is a $\gamma_{sns} - set$ of $G$.Since the $V[n(G)] = E(G) \cup C(G)$, C(G) a cutvertex set of $G$. Suppose $F = \{v_1, v_2, \dots \dots \dots, v_k\} \subseteq V[n(G)]$ if $N[F] = V[n(G)]$ and $|deg\,(x) - deg\,(y)| \geq 2$, $\forall\, x \in V[n(G)] - F, \forall\, y \in F$. Then $F$ forms a weak dominating set of $n(G)$. Otherewise there exists at least one vertex $\{w\} \subseteq F$ such that $F \cup \{w\}$ forms a minimal $\gamma_{wn} - set$. Thus $3|F| \geq |D_1| + 2$. Hence $\gamma_{wn}(G) \geq \left\lfloor \frac{\gamma_{sns}(G)+2}{3} \right\rfloor$.

*Theorem 11:* For any connected $(p, q)$ graph $G$, then $\gamma_{wn}(G) + \alpha_0(G) \geq \gamma(G) + \gamma_{ss}(G)$.

*Proof:* Let $M = \{v_1, v_2, \dots \dots \dots, v_n\} \subseteq V(G)$ and $\forall\, e_i \in E(G)$ is incident to at least one vertex of $M$. Then $|M| = \alpha_0$. Let $M_1 \subseteq M$ Such that $N[M_1] = V(G)$ then $M_1$ is a minimal dominating set of G and $M_2 \subseteq V(G) - M_1$. Suppose $M_2' \subseteq M_2$ and $H = [V(G) - (M_1 \cup M_2')]$ where $N[M_1 \cup M_2'] = V(G)$ and $< H >$ is totally disconnected. Clearly $M_1 \cup M_2' = \gamma_{ss} - set\ of\ G$. Now in $n(G)$, let $R = \{u_1 u_2, \dots \dots \dots, u_n\} \subseteq V[n(G)]$ be the minimal set of vertices where $N[R] = V[n(G)]$. Then $R$ is a dominating set of $n(G)$. Suppose $\forall\, v_i \in R_1 = V[n(G)] - R, \deg\,(v_i) \leq deg(v_k)$ where $\forall\, v_k \in R$, then $\{R \cup R_1\}$ forms a minimal weak dominating set of $n(G)$.Clearly $|R \cup R_1| + |M| \geq |M_1| + |M_1 \cup M_2'|$, gives $\gamma_{wn}(G) + \alpha_0(G) \geq \gamma(G) + \gamma_{ss}(G)$.

Next, we give a bound of $\gamma_{wn}(G)$of G with minimum edge degree $\delta'\, and\ \gamma_w(G)$.

*Theorem 2.12:* If $G$ is a $(p, q)$ connected graph then $\gamma_{wn}(G) \geq \left\lfloor \frac{\gamma_w(G)+\delta'(G)}{3} \right\rfloor - 1$ and $G \neq K_{1,n}$ $(n > 5)$

*Proof:* Suppose $G = K_{1,n}$ $n > 5, \gamma_{wn}(G) = 1 < \left\lfloor \frac{\gamma_w(G)+\delta'(G)}{3} \right\rfloor - 1$, a contradiction to the fact.

Let $S \subseteq V(G)$ where $\forall\, v_i \in S$ is adjacent to at least one vertex of $V(G) - S$. Then $S$ is a dominating set of $G$. Suppose there exists a set $H \subseteq V(G) - S$ such that $\forall\, v_j \in H$, $\deg\,(v_j) \leq \deg\,(v_k)$, $\forall\, v_k \in S$. Then $S \cup H$ forms a minimal $\gamma_w - set$ of $G$.In $n(G), V[n(G)] = [E(G) \cup C(G)]$. Let $D = \{u_1, u_2, \dots \dots \dots, u_n\} \subseteq V[n(G)]$ be the set of vertices in $n(G)$.Suppose $D' \subseteq D$ be the set of vertices with $\deg(w) \geq 3$ ,$\forall\, w \in D'$ such that $N[D'] = V[n(G)]$. If $\forall\, v_i \in V[n(G)] - D'$ with $\deg(v_i) < 3$. $Then\ \{D'\} \cup \{v_i\}$ forms a $\gamma_{wn} - set\ of\ G$.Sience for any graph $G$ there exists at least one edge $e$ with $|dege| = \delta'(G)$. Hence $3|D' \cup \{v_i\}| + 1 \geq |S \cup H| + |dege|$. Therefore $\gamma_{wn}(G) \geq \left\lfloor \frac{\gamma_{wn}(G)+\delta'(G)}{3} \right\rfloor - 1$.

The minimum number of color in any coloring of a graph $G$ such that no two adjacent vertices have same color is called the *chromatic number* of $G$ and is denoted by $\chi(G)$.

*Theorem 2.13:* If T is a tree$, with\ p \geq 2$, then $\gamma_{wn}(T) \geq \gamma_s(T) + \chi(T) - 2$.

*Proof:* Let $B_1 = \{v_1, v_2, \dots \dots \dots, v_k\} \subseteq V(T)$ be the set of all nonend vertices in $T$. Such that $N[B_1] = V(T)$ then $B_1$ forms a $\gamma - set\ of\ T$. Suppose the induced subgraph $< V - B_1 >$ is have more than one component then $B_1$ itself is a split dominating set of $T$. In $n(T)$, let $A = \{v_1, v_2, \dots \dots \dots, v_m\} \subseteq V[n(T)]$ be the $\gamma_{wn} - set$. For any tree, $\chi(T) = 2$ and $\gamma_{wn}(T) \geq \gamma_s(T) + 2 - 2$. Therefore $|A| \geq |B_1| + \chi(T) - 2$, gives $\gamma_{wn}(T) \geq \gamma_s(T) + \chi(T) - 2$.

A set $F \subset E(G)$ is called an *edge dominating set* of $G$, if every edge not in $F$ is adjacent to at least one edge in $F$. The minimum cardinality taken over all the *edge dominating* set of $G$, denoted as $\gamma'(G)$ is called an *edge domination number of* $G$.

*Theorem 2.14:* For any connected $(p, q\,)$graph $G, with\ p \geq 2$, then $\gamma_{wn}(G) + \Delta'(G) \geq \gamma'(G) + \left\lfloor \frac{\beta_0}{2} \right\rfloor - 1$.

*Proof:* Let $E = \{e_1, e_2, \dots \dots \dots, e_k\}$ be the edge set of $G$. Then $E_n \subseteq E$ such that $N[E_n] = E(G)$, then $E_n$ is an edge dominating set of $G$. Suppose $e \in E_n$ is an edge have maximum degree in $G$ such that $deg(e) = \Delta'(G)$. Let $A =$

$\{v_1, v_2, \dots\dots\dots, v_n\} \subseteq V(G)$ be a maximal independent vertex set of $G$ such that $N(v_i) \cap N(v_j) = u, 1 \le i \le n$ and $1 \le j \le n$. Also $u \in V(G) - A$. Hence $|A| = \beta_0(G.)$.In lict graph $n(G)$, let $L = \{v_1, v_2, \dots\dots\dots, v_n\} \subseteq V[n(G)]$ such that $N[L] = V[n(G)]$ and if $\forall v_l \in L$ has degree at least 2 and $v_k \in V[n(G)] - L$ and $\deg(v_k) \le \deg(v_l)$. Then $L \cup \{v_l\}$ forms $\gamma_{wn} - set$. Hence $|L \cup \{v_l\}| + |\deg(e)| \ge |E_n| + \left\lfloor \frac{A}{2} \right\rfloor - 1$, gives $\gamma_{wn}(G) + \Delta'(G) \ge \gamma'(G) + \left\lfloor \frac{\beta_0}{2} \right\rfloor - 1$.

*Theorem 2.15:* For any connected $(p, q)$ graph $G$,then $\gamma_{wn}(G) \ge \left\lfloor \frac{\gamma_t(G) + \gamma_c(G)}{2} \right\rfloor$ and $G \ne C_p$ $(p \ge 5)$.

*Proof:* Let $H$ be a dominating set of $G$.If the induced subgraph $< H >$ has exactly one component then $H$ itself is a connected dominating set of $G$.Otherwise, if $H$ has more than one component then attach the minimum number of vertices $\{v_i\} \in V(G) - H, \forall v_i, \deg(v_i) \ge 2$, so that $S = H \cup \{v_i\}$ forms exactly one component. Clearly $S = \gamma_c - set$ of $G$. Let $V_1 = V(G) - H$ and $if\ J \subseteq V_1$ be the minimum set of vertices which are adjacent to at least one vertex of $H$.Then $< H \cup J >$ is a $\gamma_t - set\ of\ G$.Suppose $R$ be the vertex set of lict graph $n(G)$. Let $R_1 \subseteq R$ be the set of vertices with minimum degree and $N[R_1]= V[n(G)]$.Then $R_1$ is a minimal $\gamma_{wn} - set$. Hence $2|R_1| \ge |H \cup J| \cup |S|$, resulting in $\gamma_{wn}(G) \ge \left\lfloor \frac{\gamma_t(G) + \gamma_c(G)}{2} \right\rfloor$.

One can easily check for $C_p\ p \ge 5$, which gives a contradiction.

*Theorem 2.16:* Let $T$ be a tree which is not a star, then $\gamma_{wn}(T) \ge \frac{\gamma_{\cot}(T)}{\gamma_{Res}(T)} + \delta$.

*Proof:* Suppose $T = K_{1,n}$.Then $\gamma_{wn}(T) = 1 < \left| \frac{\gamma_{cot}(T)}{\gamma_{Res}(T)} + \delta \right|$. Hence $T \ne K_{1,n}$. Let $S$ be a minimal dominating set of $T$. Suppose the subgraph $< V - S >$ has no isolate then S itself is a $\gamma_{cot} - set\ of\ T$. Otherwise there exist a set $R \subseteq V(T) - S$ with $\deg(v_j) = 0$, $\forall v_j \in R$. Add $\{v_j\} \subseteq V(T) - S\ to\ \{R\}$, so that $S \cup R$ forms a $\gamma_{cot} - set$.Suppose $V_1 = \{v_1, v_2, \dots\dots\dots, v_n\}$ be the set of end vertices in $T$.Now $\forall v_i \in V(T) - \{S \cup V_1\}$ is adjacent to at least one vertex of $S \cup V_1$ and at least one vertex of $V(T) - \{S \cup V_1\}$. Then $\{S \cup V_1\}$ is a $\gamma_{Res} - set$ of T. Suppose $D = \{v_1, v_2, \dots\dots\dots, v_m\} \subseteq V[n(T)]$ and $\deg(v_m) \le \deg(v_k)$, $\forall v_k \in V[n(T)] - D$ and $\forall v_m \in D$ such that $N[v_m] = V[n(T)]$. Then $D$ forms a $\gamma_{wn} - set$ of $T$. Clearly $|D|.|S \cup V_1| - \delta \ge |S \cup R|$ resulting into $\gamma_{wn}(T) \ge \frac{\gamma_{cot}(T)}{\gamma_{Res}(T)} + \delta$.

## 3. REFERENCES


[1] F. Harary, Graph Theory, Addison-Wesley, Reading Mass, 1974.

[2] F. Harary and T. W. Haynes, Double domination in graphs, Ars combin, Vol-55, 2000, 201- 213

[3] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graph, Marcell Dekker, INC-1998.

[4] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds), Domination in graph: Advanced Topics, Marcel Dekker, INC, New York, 1998.

[5] V.R. Kulli, Theory of domination in Graphs, Viswa international publications, 2010.

[6] V R. Kulli and M. H. Muddebihal Lict and Litact graph of a graphs, journal of analysis and computation, Vol-2, 2006, 133-143.

[7] M. H. Muddebihal and Geetadevi Baburao, weak line domination in graph theory, IJRAR Vol-6, 2019, 129-133.

[8] E. Sampathkumar and Puspa Latha, strong weak domination number and domination balance in graph, Discrete Math, Vol-161, 1996, 235-242.