A New Numerical Approach for Solving Fractional Bagley-Torvik Equation

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ABSTRACT
In this paper, we present a method which is based on Bernoulli Collocation Method to give approximate solution of the Bagley-Torvik equation. The Bagley-Torvik equation is transformed into a system of algebraic equations by this method and this algebraic equations are solved through by assistance of Maple 2016. Further some numerical examples are given to illustrate and establish the accuracy and reliability of the proposed algorithm.

General Terms
Your general terms must be any term which can be used for general classification of the submitted material such as Pattern Recognition, Security, Algorithms et. al.

Keywords
Bagley-Torvik equation, fractional differantial equation, Bernoulli Collocation, approximate solution.

1. INTRODUCTION
In recent years, fractional calculus has become most popular in different areas of science and engineering such as viscoelasticity, heat conduction, electrode–electrolyte polarization, electromagnetic waves, diffusion wave, control theory, acoustic, mathematical biology etc [1-5]. Altough fractional calculus arises many areas, it is difficult to find solutions to fractional equations, even impossible. So many researchers developed different methods to solve fractional equations such as the operational matrix method[6-9], Adomian decomposition method [10], homotopy-perturbation method[11], collocation method[12-13], and others[14-16].

The Bagley-Torvik equation is in fractional calculus,

$$\left(AD^2 + A_3 D^{3/2} + A_0 D^0\right) y(x) = f(x) \qquad (1)$$

with the initial conditions

$$y(0) = a,\ y'(0) = b \qquad (2)$$

where $A = m$, $A_3 = 2A\sqrt{\mu\rho}$, $A_0 = k$ and where μ is the viscosity, ρ is the fluid density. This equation arises in the modelling of the motion of a rigid plate immersed in a Newtonian fluid. The motion of a rigid plate of mass m and area A connected by a mass less spring of stiffness k, immersed in a Newtonian fluid.

The questions of existence and uniqueness of the solution to this initial value problem have been discussed in [5-6]. An analytical solution is possible and can be given in the form [2]

$$y(x) = \int_0^x G(x-u) f(u) du$$

with

$$G(x) = \frac{1}{A} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{A_0}{A}\right)^k x^{2k+1} E^{(k)}_{1/2, 2+3k/2}\left(-\frac{A_3}{A}\sqrt{t}\right)$$

where $E^{(k)}_{\lambda,\mu}$ is the kth derivative of the Mittag-Leffler function with parameters λ and μ given by

$$E^{(k)}_{\lambda,\mu} = \sum_{j=0}^{\infty} \frac{(j+k)!\, t^j}{j!\,\Gamma(\lambda j + \lambda k + \mu)}.$$

Note that this analytical solution involves the evaluation of a convolution integral, containing a Green’s function expressed as an infinite sum of derivatives of Mittag–Leffler functions, and for general functions f this cannot be evaluated conveniently. For inhomogeneous initial conditions even more complicated expressions arise. An analytical expression for the inhomogeneous case is given in [5]. It involves multivariate generalizations of Mittag–Leffler functions and is also quite cumbersome to handle. We are motivated by the difficulty of obtaining an analytical solution to investigate numerical schemes for the solution of (1) with initial conditions (2) that can be relied upon to perform well. We seek the approximate solution of Eq.(1) under the conditions Eq.(2) with the fractional truncated Bernoulli series as,

$$y_N(x) = \sum_{n=0}^{N} a_n \mathbf{B}_n^{\alpha}(x) \qquad (3)$$

where $0 < \alpha \le 1$.

2. FUNDEMENTAL RELATIONS
In this section, we consider the fractional differential equations

$$\sum_{k=0}^{m} P_k(x) D_*^{k\alpha} y(x) = f(x),$$

$$a \le x \le b,\ n-1 \le m\alpha < n \quad (4)$$

with initial conditions

$$D_*^i y(c) = \lambda_i,\ i = 0,1,\ldots,n-1,\ a \le c \le b \quad (5)$$

which $P_k(x)$ and $f(x)$ are functions defined on $a \le x \le b$, λ_i is a appropriate constant.

We use the fractional truncated Bernoulli series expansions of each term in expression and their matrix representations for solving $(k\alpha)^{th}$ order linear fractional differential equation with variable coefficients. We first consider the solution $y(x)$ of Eq. (1) defined by a fractional truncated Bernoulli series (3). Then, we have the matrix form of the solution $y(x)$

$$[y\ (x)] = \mathbf{B}^{\alpha}(x)\mathbf{A} \quad (6)$$

where

$$\mathbf{B}^{\alpha}(x) = \begin{bmatrix} B_0^{\alpha}(x) & B_1^{\alpha}(x) & B_2^{\alpha}(x) & \cdots & B_N^{\alpha}(x) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

On the other hand, fractional Bernoulli polynomials are,

$$B_N^{\alpha}(x) = \sum_{i=0}^{N} \binom{N}{i} b_{N-i} x^{i\alpha},\ \alpha > 0,$$

$b_{N-i} = B_{N-i}(0)$ Bernoulli numbers. (7)

Matrix representation of Eq.(7) is,

$$\mathbf{B}^{\alpha}(x) = \mathbf{X}^{\alpha}(x)\mathbf{S} \quad (8)$$

where

$$\mathbf{X}^{\alpha}(x) = \begin{bmatrix} 1 & x^{\alpha} & x^{2\alpha} & \cdots & x^{N\alpha} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} \binom{0}{0}b_0 & \binom{1}{0}b_1 & \binom{2}{0}b_2 & \cdots & \binom{N}{0}b_N \\ 0 & \binom{1}{1}b_0 & \binom{2}{1}b_1 & \cdots & \binom{N}{1}b_{N-1} \\ 0 & 0 & \binom{2}{2}b_0 & \cdots & \binom{N}{2}b_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{N}{N}b_0 \end{bmatrix}$$

By substituting (6) into (8), we obtain

$$[y\ (x)] = \mathbf{X}^{\alpha}(x)\mathbf{S}\mathbf{A} \quad (9)$$

Similarly, the matrix representation of the function $D_*^{\alpha} y(x)$ become

$$D_*^{\alpha} y(x) = D_*^{\alpha}\mathbf{X}^{\alpha}\mathbf{S}\mathbf{A} \quad (10)$$

where, we compute the $D_*^{\alpha}\mathbf{X}^{\alpha}$, then

$$D_*^{\alpha}\mathbf{X}^{\alpha} = \begin{bmatrix} D_*^{\alpha}1 & D_*^{\alpha}x^{\alpha} & D_*^{\alpha}x^{2\alpha} & \cdots & D_*^{\alpha}x^{N\alpha} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}x^{\alpha} & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)}x^{(N-1)\alpha} \end{bmatrix}$$
$$= \mathbf{X}^{\alpha}\mathbf{R}_1$$

(11)

where

$$\mathbf{R_1} = \begin{bmatrix} 0 & \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

substituting (11) into (10) we obtain the matrix represantation of the fractional derivative by,

$$D_*^{\alpha} y(x) = \mathbf{X}^{\alpha}\mathbf{R}_1\mathbf{S}\mathbf{A}$$

(12)

In a similar way for any i, it can be written by

$$D_*^{k\alpha} y(x) = \mathbf{X}^{\alpha}\mathbf{R}_k\mathbf{S}\mathbf{A} \quad (13)$$

where

$\mathbf{R}_k =$

$$\begin{bmatrix} 0 & 0 & \dots & \frac{\Gamma(k\alpha+1)}{\Gamma(1)} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma(\alpha+1)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \frac{\Gamma(N\alpha+1)}{\Gamma((N-k)\alpha+1)} \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}$$

Thus, we obtain the fundemental matrix form of Eq.(1) and

$$\sum_{k=0}^{m} \mathbf{P}_k \mathbf{X}^{\alpha} \mathbf{R}_k \mathbf{S} \mathbf{A} = \mathbf{F} \tag{14}$$

the matrix representation of the condition in given Eq.(3) respectively by

$$\mathbf{U}_i = \mathbf{X}^{\alpha}(c)\mathbf{R}_k = \begin{bmatrix} u_{i0} & u_{i1} & u_{i2} & \cdots & u_{iN} \end{bmatrix} = \begin{bmatrix} \lambda_i \end{bmatrix} \tag{15}$$

3. METHOD OF SOLUTION

We can write Eq. (14) in the form

$$\mathbf{W}\mathbf{A} = \mathbf{F} \tag{16}$$

where

$$\mathbf{W} = \begin{bmatrix} w_{ij} \end{bmatrix} = \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{m}} \mathbf{P}_k \mathbf{X}^{\alpha} \mathbf{R}_k \mathbf{S},\ i,j = 0.1,...N\ .$$

Consequently, to find the unknown Bernoulli coefficients a_k, $k = 0,1,\dots,N$, related with the approximate solution of the problem consisting of Eq. (1) and conditions (2), by replacing the m row matrices (15) by the last *m* rows of the matrix (16), we have augmented matrix

$$\begin{bmatrix} \mathbf{W}^*; \mathbf{F}^* \end{bmatrix} = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{N-m0} & w_{N-m1} & \cdots & w_{N-mN} & ; & f(x_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ u_{m-10} & u_{m-11} & \cdots & u_{m-1N} & ; & \lambda_{m-1} \end{bmatrix}$$

or the corresponding matrix equation

$$\mathbf{W}^* \mathbf{A} = \mathbf{F}^* \tag{17}$$

So, we obtained to a system of $(N+1)\times(N+1)$ linear algebraic equations with $(N+1)$ unknown coefficients. If $rank\mathbf{W}^* = rank[\mathbf{W}^*;\mathbf{F}^*] = N+1$, the we can be write $\mathbf{A} = (\mathbf{W}^*)^{-1}\mathbf{F}^*$. Thus, the matrix $\mathbf{A}$ is uniquely determined. Also the Eq.(1) with conditions (2) has a unique solution. On the other hand, when $\left|\mathbf{W}^*\right| = 0$, if $rank\mathbf{W}^* = rank[\mathbf{W}^*;\mathbf{F}^*] < N+1$, then we may find a particular solution. Otherwise if $rank\mathbf{W}^* \neq rank[\mathbf{W}^*;\mathbf{F}^*]$, then it is not a solution.

Therefore, the approximate solution is given by the truncated fractional Bernoulli series

$$\left[y(x)\right] = \mathbf{X}^{\alpha}\mathbf{R}_0\mathbf{S}\mathbf{A}. \tag{18}$$

Moreover, we can easily check the accuracy of the method. Since the truncated fractional Bernoulli series (3) is an approximate solution of Eq.(1), when the solution $y(x)$ and its fractional derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is , for $x = x_q \in [a,b],\ q = 0,1,2,...$

$$R_N(x_q) = \left|\left(AD^2 + A_3D^{3/2} + A_0D^0\right)y(x_q) - f(x_q)\right| \cong 0$$

4. EXAMPLES

In order to illustrate the effectiveness of the method proposed in this paper, several numerical examples are carried out in this section. All algorithms are implemented in Dell inspiron 15r on a Intel core i5-3337U, 1.80 GHz CPU machine with 8GB RAM.

4.1 Example

Let us consider the fractional integro-differential equation

$$a_2 D_*^2 y(x) + a_1 D_*^{3/2} y(x) + a_0 y(x) = f(x)$$

with the initial conditions

$y(0) = 1,\ y'(0) = 1$.

Then $f(x) = x+1,\ a_2 = 1,\ a_1 = 1,\ a_0 = 1$. We seek the approximate solutions y_4 by truncated fractional Bernoulli series, for $\alpha = \frac{1}{2}$,

$$y_4(x) = \sum_{n=0}^{4} a_n B^{n\alpha}(x)$$

Fundamental matrix relation of this is

$$\left(\mathbf{P}_4 + \mathbf{P}_3 + \mathbf{P}_0\right)\mathbf{A} = \mathbf{G}$$

where

$$\mathbf{P}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 & \frac{3\sqrt{\pi}}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{\sqrt{\pi}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, we have the matrix representation of conditions,

$$y(0) = \begin{bmatrix} 1 & -1/2 & 1/6 & 0 & -1/30 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 1 \end{bmatrix}$$
$$y'(0) = \begin{bmatrix} 0 & 0 & 1 & -3/2 & 1 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 1 \end{bmatrix}$$

and so we solve the this equation, we obtained the coefficients of the truncated fractional Bernoulli series

$$\mathbf{A} = \begin{bmatrix} 1.3333333333333 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Hence, for $N = 4$, the approximate solution of example 1 is given $y_4 = 1 + x$ which is the exact solution of this equation.

4.2 Example

Let us consider the fractional integro-differential equation

$$D_*^2 y(x) + D_*^{3/2} y(x) + y(x) = f(x)$$

with the initial conditions

$$y(0) = 0,\ y(1) = 1.$$

Then $f(x) = x^2 + 2 + 4\sqrt{x/\pi}$. We seek the approximate solutions y_4 by truncated fractional Bernoulli series, for $\alpha = \frac{1}{2}$,

$$y_4(x) = \sum_{n=0}^{4} a_n B^{n\alpha}(x)$$

Fundamental matrix relation of this is

$$(\mathbf{P}_4 + \mathbf{P}_3 + \mathbf{P}_0)\mathbf{A} = \mathbf{G}$$

where

$$y(0) = 0,\ y'(0) = 0.$$

Numerical results with comparison to Ref. [16] are given in Table 4.

Table 4: Comparison of numerical results.

x	Exact solution	Adomian method	Taylor method	Present method
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.125221	0.140640	0.125254	-0.41366
0.4	0.455435	0.533284	0.455468	-0.07387
0.6	0.950392	1.148840	0.950398	0.563342
0.8	1.579557	1.963033	1.579689	1.411317
1.0	2.315526	2.952567	2.315589	2.414368

5. CONCLUSION

In this study, we present a Bernoulli collocation method for the numerical solutions of Bagley-Torvik equation. This method transform Bagley-Torvik equation into a system of linear algebraic equation. The approximate solutions can be obtained by solving the resulting system, which can be effectively computed using symbolic computing codes on Maple 2016. This method has been given to find the analytical

$$\mathbf{P}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{P}_3 = \begin{bmatrix} 0 & 0 & 0 & \frac{3\sqrt{\pi}}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{\sqrt{\pi}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Also, we have the matrix representation of conditions,

$$y(0) = \begin{bmatrix} 1 & -1/2 & 1/6 & 0 & -1/30 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 0 \end{bmatrix}$$
$$y(1) = \begin{bmatrix} 1 & 1/2 & 1/6 & 0 & -1/30 \end{bmatrix}\mathbf{A} = \begin{bmatrix} 1 \end{bmatrix}$$

and so we solve the this equation, we obtained the coefficients of the Taylor series

$$\mathbf{A} = \begin{bmatrix} 0.19999999999 & 1 & 2 & 2 & 1 \end{bmatrix}.$$

Hence, for $N = 4$, the approximate solution of example 2 is given $y_4 = x^2$ which is the exact solution of this equation.

4.3 Example

Consider the problem [12,16]

$$D_*^2 y(x) + D_*^{3/2} y(x) + y(x) = 8,\ x \in [0,1]$$

subject to the initial conditions

solutions if the system has exact solutions that are polynomial functions. If the exact solutions of problem are not polynomial functions, then a good approximation can be gained by using the proposed method . The method can also extended to the system of linear Fractional differential equations with variable coefficients, but some modifications are required.

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