L-Quadratic Distribution

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ABSTRACT
In this paper, the Generalization of the U-Quadratic Distribution using the quadratic rank transmutation map is developed called L-Quadratic (***LQ***) distribution with some important related integrations. Most of the mathematical properties are studied and the model parameters are estimated by the maximum likelihood method. Finally, an application to generated data sets is illustrated.

Keywords
U-Quadratic distribution, Transmutation Map, Maximum Likelihood Estimation, Survivor Function, Cumulative Hazard Function, Harmonic Mean, Moments, Moment Generating Function.

1. INTRODUCTION
The U-quadratic distribution is one of the types of continuous probability distributions with two parameters α and β. The distribution is often abbreviated ***UQ(a,b)***, and defined by the former two parameters as follows

$$g(x)=\alpha(x-\beta)^2\ ,\quad a<x<b\quad ,\qquad a,b>0 \qquad [1]$$

with distribution function (cdf)

$$G(x)=\frac{\alpha}{3}[(x-\beta)^3+(\beta-a)^3]\ ,\quad a<x<b\quad ,\qquad a,b>0 \quad [2]$$

where $\alpha=\frac{12}{(b-a)^3}$ (vertical scale), and $\beta=\frac{b+a}{2}$ (gravitational balance center).

In applied probability theory, the ***UQ*** distribution is one of the kinds of bimodal distributions. It is easily traceable to the modeling of symmetric bimodal processes with expected value and median : β, two Modes: a, b and standard deviation: 0.387 $(b-a)$, [1].

In this paper, transmutation map approach suggested by Shaw and Buckley to define a new model which generalizes the ***UQ*** model is used. It is called the generalized distribution as the L-Quadratic Distribution**,** because the pdf of the ***LQ*** distribution takes the form of the small letter "*l*", as shown in Figure (1). It is denoted by ***LQ*** distribution and it is abbreviated ***LQ*(a,b,λ).** In the rest of this paper, mathematical formulations with some important related integrations and properties of the ***LQ*** distribution are provided, [2].

2. L- QUADRATIC DISTRIBUTION
According to the Quadratic Rank Transmutation Map (QRTM) approach, the cumulative distribution function (cdf) satisfy the relationship

$$F(x)=(1+\lambda)G(x)-\lambda G^2(x) \qquad [3]$$

where $G(x)$ is the cumulative distribution function (*cdf*) of the base distribution, which on differentiation yields, $f(x)$, such that

$$f(x)=(1+\lambda)g(x)-2\lambda g(x)G(x) \qquad [4]$$

If $\lambda=0$ then the distribution of the base random variable is obtained. By using Eq.(2) and Eq.(3), the *cdf* of ***LQ*** distribution has the following form

$$F(x)=\frac{\alpha}{3}(x-\beta)^3\left[1-\frac{\alpha\lambda}{3}(x-\beta)^3\right]+\frac{1}{4}(\lambda+2) \qquad [5]$$

where λ is the transmuted parameter. The corresponding pdf of Eq.(5) is given as follows

$$f(x)=\alpha(x-\beta)^2\left[1-\frac{2\alpha\lambda}{3}(x-\beta)^3\right] \qquad [6].$$

3. STATISTICAL PROPERTIES
Some statistical properties of the new generalization are provided, as follows, [3], [4]:

3.1 Survivor Function
There is a relation between the *cdf* and the reliability function, i.e., $\boldsymbol{RF=1-F(x)}$. Therefore, the Reliability Function (RF) of the *LQ* distribution ($\boldsymbol{RF_{LQ}}$) is defined as:

$$RF_{LQ}(x)=\frac{\alpha}{3}(x-\beta)^3\left[\frac{\alpha\lambda}{3}(x-\beta)^3-1\right]+\frac{1}{4}(2-\lambda)$$

3.2 Hazard Function
There is a relation between the *pdf*, reliability and hazard function, i.e., $\boldsymbol{h(x)=\frac{f(x)}{R(x)}}$ Therefore, the Hazard Function (HF) of the *LQ* distribution ($\boldsymbol{HF_{LQ}}$) is defined as:

$$HF_{LQ}(x)=\frac{\alpha(x-\beta)^2\left[1-\frac{2\alpha\lambda}{3}(x-\beta)^3\right]}{\frac{\alpha}{3}(x-\beta)^3\left[\frac{\alpha\lambda}{3}(x-\beta)^3-1\right]+\frac{1}{4}(2-\lambda)}$$

3.3 Cumulative Hazard Function
There is a relation between the *cdf* and the cumulative hazard function, i.e., $CHF(x)=-\ln F(x)$. Therefore, the Cumulative Hazard Function (CHF) of the ***LQ*** distribution **(**$\boldsymbol{CHF_{LQ}}$**)** is defined as :

$$\boldsymbol{CHF_{LQ}}(x)=-ln\left[\frac{\alpha}{3}(x-\beta)^3\left[1-\frac{\alpha\lambda}{3}(x-\beta)^3\right]+\frac{1}{4}(\lambda+2)\ \right].$$

As shown from Figure (1), the ***LQ*** distribution is an extended model to analyze data from complex situations. Also, it is observed that, The pdf and the cumulative hazard function of the ***LQ*** distribution takes the form of the small letter "*l*", but the cdf takes the form of the inverted "∩" quadratic distribution. The ***LQ*** distribution has highest reliability at the lower limit "a", and then it is begins decreasing until the median, after that it is increasing again.

3.4 Random Number Generation
To generate random numbers when the parameters *a* and *b* are known, the method of inversion can be used from the ***LQ*** distribution as

$$X_{rv}=\beta+\left[\frac{3u-0.75(2+\lambda)}{\alpha-0.5\theta(X_{rv}-\beta)^3}\right]^{\frac{1}{3}},\ \ \theta=\frac{2\alpha^2\lambda}{3} \qquad [7]$$

Eq.(7) doesn't have a closed form solution, so "u" will be generated as uniform random variables from $U(0,1)$, and then solve for X_{rv} in order to generate random numbers from ***LQ*** distribution. From Eq.(7), the quantile X_q of the ***LQ*** distribution is given by

$$X_q = \beta + \left[\frac{3q - 0.75(2+\lambda)}{\alpha - 0.5\theta\left(X_q - \beta\right)^3}\right]^{\frac{1}{3}} \quad ,\theta = \frac{2\alpha^2\lambda}{3} \qquad [8]$$

Put $q = 0.5$, the median of the ***LQ*** distribution is obtained as $X_{0.5} = \beta + \left[\frac{0.75\,\lambda}{0.5\theta(X_{0.5}-\beta)^3-\alpha}\right]^{\frac{1}{3}} \quad ,\theta = \frac{2\alpha^2\lambda}{3}$

Also, the percentiles and quartiles can be obtained, by putting different values of q in Eq.(8), e.g., the 4th quartiles, 90th percentiles of the ***LQ*** distribution is obtained when

$q = 0.25,\ q = 0.90$ respectively.

3.5 Useful Important Integrations

In this paper, the following integrations are developed by two forms for $r = 1, 2, 3, \ldots$ as follows:

The First Form (1):

$$A(1) = \int_a^b x^r\ (x-\beta)^2\, dx$$

$$= \sum_{j=0}^{2} \frac{(-1)^j \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial\beta^j}\ \beta^2\right] \qquad [9]$$

$$B(1) = \int_a^b x^r\ (x-\beta)^5\, dx$$

$$= \sum_{j=0}^{5} \frac{(-1)^{j+1} \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial\beta^j}\ \beta^5\right] \qquad [10]$$

Proof

$$A(1) = \int_a^b x^r\ (x-\beta)^2\, dx = \int_a^b \{x^{r+2} - 2\beta x^{r+1} + \beta^2 x^r\}\, dx$$

$$= \frac{(b^{r+3} - a^{r+3})}{r+3} - 2\beta\frac{(b^{r+2} - a^{r+2})}{r+2} + \beta^2\frac{(b^{r+1} - a^{r+1})}{r+1}$$

$$= \sum_{j=0}^{2} \frac{(-1)^j \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial\beta^j}\ \beta^2\right]$$

and

$$B(1) = \int_a^b x^r\ (x-\beta)^5\, dx$$

$$= \int_a^b \{x^{r+5} - 5\beta x^{r+4} + 10\beta^2 x^{r+3} - 10\beta^3 x^{r+2} + 5\beta^4 x^{r+1} - \beta^5 x^r\}\, dx$$

$$= \frac{(b^{r+6} - a^{r+6})}{r+6} - 5\beta\frac{(b^{r+5} - a^{r+5})}{r+5} + 10\beta^2\frac{(b^{r+4} - a^{r+4})}{r+4} - 10\beta^3\frac{(b^{r+3} - a^{r+3})}{r+3} + 5\beta^4\frac{(b^{r+2} - a^{r+2})}{r+2} + \beta^5\frac{(b^{r+1} - a^{r+1})}{r+1}$$

$$= \sum_{j=0}^{5} \frac{(-1)^{j+1} \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial\beta^j}\ \beta^5\right]$$

In general for $r = 1, 2, 3, \ldots$

$$\int_a^b x^r\ (x-\beta)^s\, dx$$

$$= \begin{cases} \displaystyle\sum_{j=0}^{s} \frac{(-1)^j \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial\beta^j}\ \beta^s\right], & s = 0, 2, 4, \ldots \\ \displaystyle\sum_{j=0}^{s} \frac{(-1)^{j+1} \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial\beta^j}\ \beta^s\right], & s = 1, 3, 5, \ldots \end{cases}$$

The Second Form (2):

$$A(2) = \int_a^b x^r\ (x-\beta)^2\, dx$$

$$= \sum_{j=0}^{2} \frac{(-1)^j \left(b^{r+j+1} - (-1)^j\ a^{r+j+1}\right)}{2^{2-j}\ \prod_{i=1}^{j+1}(r+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^2\right] \qquad [11]$$

$$B(2) = \int_a^b x^r\ (x-\beta)^5\, dx$$

$$= \sum_{j=0}^{5} \frac{(-1)^j \left(b^{r+j+1} - (-1)^{j+1}\ a^{r+j+1}\right)}{2^{5-j}\ \prod_{i=1}^{j+1}(r+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^5\right] \qquad [12]$$

where $Q = (b - a)$

Proof: $A(2) = \int_a^b x^r\ (x-\beta)^2\, dx = \int_{a-\beta}^{b-\beta} y^2\ (y+\beta)^r dy$

using differentiation by parts, it is proved that

$$A(2) = \int_a^b x^r\ (x-\beta)^2\, dx$$

$$= \sum_{j=0}^{2} \frac{(-1)^j \left[b^{r+j+1} - (-1)^j\ a^{r+j+1}\right]}{2^{2-j}\ \prod_{i=1}^{j+1}(r+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^2\right]$$

and

$B(2) = \int_a^b x^r\ (x-\beta)^5\, dx =$
$\sum_{j=0}^{5} \frac{(-1)^j \left[b^{r+j+1} - (-1)^{j+1}\ a^{r+j+1}\right]}{2^{5-j}\ \prod_{i=1}^{j+1}(r+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^5\right]$

In general for $r = 1, 2, 3, \ldots$, $\int_a^b x^r\ (x-\beta)^s\, dx =$

$$\begin{cases} \sum_{j=0}^{s} \frac{(-1)^j \left(b^{r+j+1} - (-1)^j\ a^{r+j+1}\right)}{2^{s-j}\ \prod_{i=1}^{j+1}(r+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^s\right], & s = 0, 2, 4, \ldots \\ \sum_{j=0}^{s} \frac{(-1)^j \left(b^{r+j+1} - (-1)^{j+1}\ a^{r+j+1}\right)}{2^{s-j}\ \prod_{i=1}^{j+1}(r+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^s\right], & s = 1, 3, 5, \ldots \end{cases}$$

For given samples with different choices of a, b and λ, it is proved that, the integrations of Eq.9 and Eq.11 gave the same results, Table 1 summarizes the results.

Table 1. The Numerical Results of the integrations A(1) and A(2)

$n = 25,\ a = 1.038,\ b = 7.274,\ \lambda = 0.151$		$n = 30,\ a = 1.326,\ b = 7.428,\ \lambda = 0.100$	
A(1)	A(2)	A(1)	A(2)
83.998	83.998	82.889	82.889
467.012	467.012	468.599	468.599
2921.015	2921.015	2977.149	2977.149
19068.861	19068.861	19814.846	19814.846
126723.993	126723.993	134532.567	134532.567
850219.148	850219.148	922883.188	922883.188
5741058.791	5741058.791	6372979.704	6372979.704
38961370.633	38961370.633	44228833.864	44228833.864
265532964.483	265532964.483	308220461.918	308220461.918

Also, for given samples with different choices of a, b and λ, it is proved that, the integrations of Eq.10 and Eq. gave the same results, Table 2 summarizes the results

Table 2. The Numerical Results of the integrations B(1) and B(2)

$n = 25,\ a = 1.038,\ b = 7.274,\ \lambda = 0.151$		$n = 30,\ a = 1.326,\ b = 7.428,\ \lambda = 0.100$	
B(1)	B(2)	B(1)	B(2)
818.725	818.725	703.379	703.379
6805.652	6805.652	6157.646	6157.646
48620.070	48620.070	45522.784	45522.783
338054.080	338054.080	325130.688	325130.684
2340281.552	2340281.552	2305646.766	2305646.738
16210654.402	16210654.402	16342813.696	16342813.506
112473810.571	112473810.571	115981914.625	115981913.316
781800350.335	781800350.335	824421395.565	824421386.519
5443934588.051	5443934588.051	5869711034.899	5869710972.075

3.6 Central Tendency

The mean, median, mode of the ***LQ*** distribution can be obtained as follows:

Theorem* 1**: If X a r.v has ***LQ distribution then the *mean* is: $\mu = \acute{\mu}_1 = \beta - \frac{3\lambda}{14}\,(b-a)$

Proof: By using the integrations of Eq.9 to Eq.12, the mean of the ***LQ*** distribution is:

$$\mu = \acute{\mu}_1 = \int_a^b x\, f(x)dx = \int_a^b x[\alpha(x-\beta)^2 - \theta(x-\beta)^5]\ dx$$

Then,

$$\mu = \mu(A) = \sum_{j=0}^{2} \frac{(-1)^j\left(b^{j+2} - a^{j+2}\right)}{j!\ (j+2)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\alpha\beta^2)\right] - \sum_{j=0}^{5} \frac{(-1)^{j+1}\left(b^{j+2} - a^{j+2}\right)}{j!\ (j+2)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\theta\beta^5)\right] \qquad [13]$$

$$\mu = \mu(B) = \alpha \sum_{j=0}^{2} \frac{(-1)^j\left(b^{j+2} - (-1)^j a^{j+2}\right)}{2^{2-j}\ \prod_{i=1}^{j+1}(1+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^2\right] - \theta \sum_{j=0}^{5} \frac{(-1)^j\left(b^{j+2} - (-1)^{j+1} a^{j+2}\right)}{2^{5-j}\ \prod_{i=1}^{j+1}(1+i)} \cdot \left[\frac{\partial^j}{\partial Q^j}\ Q^5\right] \qquad [14]$$

So, $\mu = \mu(A) = \mu(B) = \beta - \frac{3\lambda}{14}\,(b-a)$ [15]

For different choices of a, b and λ, it is proved that, the forms of the arithmetic mean in Eq.13, Eq.14 and Eq.15 gave the same results, Table 3 summarizes the results.

Table 3. The Numerical Solution of the mean

a	b	λ	μ	$\mu(A)$	$\mu(B)$
0.05	0.10	0.50	0.0696	0.0696	0.0696
0.05	0.20	0.75	0.1009	0.1009	0.1009
0.10	1.00	1.00	0.3571	0.3571	0.3571
0.25	1.50	1.50	0.4286	0.4286	0.4286
0.50	1.50	2.00	0.5714	0.5714	0.5714
1.00	2.00	2.50	1.1786	1.1786	1.1786
2.00	3.00	2.75	1.9107	1.9107	1.9107
2.50	3.25	3.00	2.3929	2.3929	2.3929
3.00	4.00	3.50	2.7500	2.7500	2.7500
5.00	10.0	6.00	1.0714	1.0714	1.0714

Theorem* 2**: If X a r.v has ***LQ distribution then the *median* is: $m = \beta + \left[\frac{9\lambda}{4\alpha[\alpha\lambda(m-\beta)^3 - 3]}\right]^{1/3}$

Proof:

$$0.5 = \int_a^m f(x)dx = \int_a^m [\alpha(x-\beta)^2 - \theta(x-\beta)^5]\ dx = \frac{\alpha}{3}\,(m-\beta)^3\left[1 - \frac{\alpha\lambda}{3}(m-\beta)^3\right] + \left(\frac{\lambda+2}{4}\right)$$

Then, $m = \beta + \left[\frac{9\lambda}{4\alpha[\alpha\lambda(m-\beta)^3 - 3]}\right]^{1/3}$ and it is equivalents of Eq.(8).

Theorem* 3**: If X r.v has ***LQ distribution then the *mode* is: $mode = mo = \beta + \left[\frac{0.6}{\alpha\lambda}\right]^{1/3}$

***Proof*:** By taking the 1st derivative of Eq.(6) with respect to the r.v X**,**

$$\acute{f}(x) = 2\alpha\,(x-\beta)^3\left[1 - \frac{5\alpha\lambda}{3}(x-\beta)^3\right]$$

Then, $mo = \beta + \left[\frac{0.6}{\alpha\lambda}\right]^{1/3}$**,** where

$$\acute{\acute{f}}(x) = 2\alpha - \frac{40\,\alpha^2\,\lambda}{3}\,(x-\beta)^3 < 0$$

Theorem* 4**: If X a r.v has ***LQ distribution then the Harmonic mean (*Hm*) is:

$$Hm = \left\{\alpha\beta^2\left(1 + \frac{2\alpha\lambda\beta^3}{3}\right)\ln\left(\frac{b}{a}\right) - \alpha\beta(b-a) - \sum_{j=0}^{5} \frac{(-1)^{j+1}\left(b^j - a^j\right)}{j!\ (j)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\theta\beta^5)\right]\right\}^{-1}$$

***Proof* :**

$$\frac{1}{Hm} = E\left(\frac{1}{x}\right) = \int_a^b \frac{1}{x}\, f(x)dx = \int_a^b \frac{1}{x}[\alpha(x-\beta)^2 - \theta(x-\beta)^5]\ dx = \alpha\beta^2\left(1 + \frac{2\alpha\lambda\beta^3}{3}\right)\ln\left(\frac{b}{a}\right) - \alpha\beta(b-a) - \sum_{j=0}^{5} \frac{(-1)^{j+1}\,(b^j - a^j)}{j!\,(j)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\theta\beta^5)\right]$$

Then, $Hm = \left\{\alpha\beta^2\left(1 + \frac{2\alpha\lambda\beta^3}{3}\right)\ln\left(\frac{b}{a}\right) - \alpha\beta(b-a) - \sum_{j=0}^{5} \frac{(-1)^{j+1}\,(b^j - a^j)}{j!\,(j)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\theta\beta^5)\right]\right\}^{-1}$

3.7 Moments

Theorem* 5:** If X a r.v has ***LQ distribution then the r^{th} moments are :

$$\acute{\mu}_r = \mu(A_r) = \sum_{j=0}^{2} \frac{(-1)^j\left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\alpha\beta^2)\right] - \sum_{j=0}^{5} \frac{(-1)^j\left(b^{r+j+1} - a^{r+j+1}\right)}{j!\ (r+j+1)} \cdot \left[\frac{\partial^j}{\partial \beta^j}\ (\theta\beta^5)\right] \qquad [16]$$

or

$$\acute{\mu}_r = \mu(B_r)$$
$$= \alpha \sum_{j=0}^{2} \frac{(-1)^j \left(b^{r+j+1} - (-1)^j a^{r+j+1}\right)}{2^{s-j} \prod_{i=1}^{j+1}(r+i)} . \left[\frac{\partial^j}{\partial Q^j} Q^s\right]$$
$$- \theta \sum_{j=0}^{s} \frac{(-1)^j \left(b^{r+j+1} - (-1)^{j+1} a^{r+j+1}\right)}{2^{s-j} \prod_{i=1}^{j+1}(r+i)} . \left[\frac{\partial^j}{\partial Q^j} Q^s\right] \quad [17]$$

Proof*:** Using the integrations of Eq.9 to Eq.12, The r^{th} ordinary moment of the ***LQ distribution is given by **:** $\acute{\mu}_r = \int_a^b x^r f(x)dx = \int_a^b x^r[\alpha(x-\beta)^2 - \theta(x-\beta)^3]\ dx$

Then,

$$\acute{\mu}_r = \begin{cases} \sum_{j=0}^{2} \frac{(-1)^j \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\,(r+j+1)} . \left[\frac{\partial^j}{\partial \beta^j} (\alpha\beta^2)\right] - \sum_{j=0}^{5} \frac{(-1)^{j+1} \left(b^{r+j+1} - a^{r+j+1}\right)}{j!\,(r+j+1)} . \left[\frac{\partial^j}{\partial \beta^j} (\theta\beta^5)\right], & r = 1, 2, \ldots \\ \alpha \sum_{j=0}^{2} \frac{(-1)^j \left(b^{r+j+1} - (-1)^j a^{r+j+1}\right)}{2^{2-j} \prod_{i=1}^{j+1}(r+i)} . \left[\frac{\partial^j}{\partial Q^j} Q^2\right] - \theta \sum_{j=0}^{5} \frac{(-1)^j \left(b^{r+j+1} - (-1)^{j+1} a^{r+j+1}\right)}{2^{5-j} \prod_{i=1}^{j+1}(r+i)} . \left[\frac{\partial^j}{\partial Q^j} Q^5\right] & r = 1, 2, \ldots \end{cases}$$

so,

$\acute{\mu}_1 = \beta - \frac{3\lambda}{14}(b-a)$, $\acute{\mu}_2 = \frac{(b^2+a^2)}{2} - \frac{(b-a)^2}{10} - \frac{3\lambda(b^2-a^2)}{14}$

$\acute{\mu}_3 = \frac{(b^3+a^3)}{2} - \frac{3(b+a)(b-a)^2}{20} - \frac{\lambda(b^3-a^3)}{4} + \frac{3\lambda(b^2+a^2)(b-a)}{56} - \frac{3\lambda(b-a)^3}{7(2^8)}$, $\sigma^2 = \frac{3(b-a)^2}{4}\left[\frac{1}{5} - \frac{3\lambda^2}{49}\right]$**.**

If $\lambda = 0$ then the first three moments of the base random variable are obtained:

$\acute{\mu}_1 = \beta = \mu$**,** $\acute{\mu}_2 = \frac{(b^2+a^2)}{2} - \frac{(b-a)^2}{10}$, $\acute{\mu}_3 = \frac{(b^3+a^3)}{2} - \frac{3(b+a)(b-a)^2}{20}$ and $\sigma^2 = 0.15\,(b-a)^2$

3.8 Moment Generating Function

The moment generating function (mgf) is important especially if it is existing. Then the moment generating function of ***LQ*** distribution is derived**.**

Theorem* 6:** If X a r.v has the ***LQ distribution then the mgf is: $\boldsymbol{mgf_{LQ}} = m_x(t) = \sum_{r=1}^{\infty} \frac{t^r}{r!} \acute{\mu}_r$

Proof:

$\boldsymbol{mgf_{LQ}} = m_x(t) = E(e^{tx}) = \int_a^b e^{tx} f(x)dx = \int_a^b e^{tx}[\alpha(x-\beta)^2 - \theta(x-\beta)^3]\ dx$

Then

$\boldsymbol{mgf_{LQ}} = \sum_{r=1}^{\infty} \frac{t^r}{r!}\left[\int_a^b [\alpha x^r (x-\beta)^2 - \theta x^r (x-\beta)^3]\ dx\right] = \sum_{r=1}^{\infty} \frac{t^r}{r!} \acute{\mu}_r$ [15]

Put $t = it$ in Eq.(15), then characteristic function (Q_{LQ}) of the ***LQ*** distribution is:

$$\boldsymbol{Q_{LQ}} = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} \left[\int_a^b [\alpha x^r (x-\beta)^2 - \theta x^r (x-\beta)^3]\ dx\right]$$
$$= \sum_{r=1}^{\infty} \frac{(it)^r}{r!} \acute{\mu}_r$$

From Eq.(15), notice that, the r[th] moment is the coefficient of $\frac{t^r}{r!}$, i.e., $\acute{\mu}_r = coef.of\ \frac{t^r}{r!}$, and if $\lambda = 0$ then

$\boldsymbol{mgf_{LQD}} = \sum_{r=1}^{\infty} \frac{t^r}{r!}\left\{\int_a^b [\alpha x^r (x-\beta)^2]\ dx\right\} = \sum_{r=1}^{\infty} \frac{t^r}{r!} \acute{\mu}_r$
which is the mgf of the ***UQ*** distribution.

4. PARAMETER ESTIMATION

The maximum likelihood estimates, MLEs, of the parameters that are inherent within the ***LQ*** distribution function is given by the following: Let $x_1, x_2, \ldots, x_n$ be a sample of size n from ***LQ*** distribution, then the likelihood function is given by

$$L = \alpha^n \left\{\prod_{i=1}^{n}\left[(x_i-\beta)^2\left(1 - \tfrac{2\alpha\lambda}{3}(x_i-\beta)^3\right)\right]\right\}$$

Put $w_i(\beta) = (x_i - \beta)$ and $w_i(\underline{\Phi}) = \left(1 - \frac{2\alpha\lambda}{3} w_i^3(\beta)\right)$, $\underline{\Phi} = (\alpha, \lambda, \beta)$

Then $L = \alpha^n\left\{\prod_{i=1}^{n} w_i^2(\beta)\ w_i(\underline{\Phi})\right\}$ [16]

The log-likelihood function of Eq.(16) is given by

$$l = \ln L = n\ln\alpha + 2\sum_{i=1}^{n} \ln w_i(\beta) + \sum_{i=1}^{n} \ln w_i(\underline{\Phi}) \quad [17]$$

The log-likelihood can be maximized by differentiating Eq.(17) to obtain the maximum likelihood estimate (*MLE*) of the unknown parameter (α, λ, β). Therefore, The 1[st] partial derivatives of Eq.(17) with respect to the unknown parameters (α, λ, β) are given by:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - \frac{2\lambda}{3} \sum_{i=1}^{n}\left[w_i^3(\beta)\, w_i^{-1}(\underline{\Phi})\right]$$
$$\frac{\partial l}{\partial \beta} = 2\sum_{i=1}^{n} w_i^{-1}(\beta)\left[\alpha\lambda\, w_i^3(\beta)\, w_i^{-1}(\underline{\Phi}) - 1\right]$$
$$\frac{\partial l}{\partial \lambda} = -\frac{2\alpha}{3} \sum_{i=1}^{n}\left[w_i^3(\beta)\, w_i^{-1}(\underline{\Phi})\right]$$

By solving the last three non-linear equations simultaneously, then $\hat{\underline{\Phi}} = (\hat{\alpha}, \hat{\lambda}, \hat{\beta})$ will be obtained as shown in Section (6).

5. FISHER'S INFORMATION MATRIX

The 2[nd] partial derivatives of the 1[th] partial derivatives of Eq.(17) with respect to the unknown parameters (α, λ, β) are given as follows:

$$I_1 = \frac{\partial^2 l}{\partial \alpha^2} = -\left\{\frac{n}{\alpha^2} + \frac{4\lambda^2}{9}\sum_{i=1}^{n} w_i^6(\beta)\, w_i^{-2}(\underline{\Phi})\right\}$$

$$I_2 = \frac{\partial^2 l}{\partial \alpha \partial \beta} = \frac{2\lambda}{3}\sum_{i=1}^{n} w_i^2(\beta)\, w_i^{-1}(\underline{\Phi})\left[3 + 2\alpha\lambda\, w_i^3(\beta)\, w_i^{-1}(\underline{\Phi})\right]$$

$$I_3 = \frac{\partial^2 l}{\partial \alpha \partial \lambda} = \frac{-2}{3}\sum_{i=1}^{n} w_i^3(\beta)\, w_i^{-1}(\underline{\Phi})\left[1 + \frac{2\alpha\lambda}{3} w_i^3(\beta)\, w_i^{-1}(\underline{\Phi})\right]$$

$$I_4 = \frac{\partial^2 l}{\partial \beta^2} = -2\sum_{i=1}^{n} w_i^{-2}(\beta)\left\{2\alpha\lambda\, w_i^3(\beta) w_i^{-1}(\underline{\Phi})\left[1 + \alpha\lambda w_i^3(\beta) w_i^{-1}(\underline{\Phi})\right] + 1\right\}$$

$$I_5 = \frac{\partial^2 l}{\partial \beta \partial \lambda} = 2\alpha\sum_{i=1}^{n} w_i^2(\beta)\, w_i^{-1}(\underline{\Phi})\left[1 + \frac{2\alpha\lambda}{3} w_i^3(\beta)\, w_i^{-1}(\underline{\Phi})\right]$$

$$I_6 = \frac{\partial^2 l}{\partial \lambda^2} = -\frac{4\alpha^2}{9}\sum_{i=1}^{n}\left[w_i^6(\beta)\, w_i^{-2}(\underline{\Phi})\right]$$

Therefore, the Fisher's information matrix (*I*), is obtained as follows:

$$I = -\begin{bmatrix} I_1 & I_2 & I_3 \\ I_2 & I_4 & I_5 \\ I_3 & I_5 & I_6 \end{bmatrix}$$

The approximate $100(1-\gamma)\%$ confidence intervals (***C.I***) for the unknown parameters (α, λ, β) are given by: $B < \underline{\widehat{\Phi}} < A, \quad where\ A = \underline{\widehat{\Phi}} + Z_{\frac{\gamma}{2}} \sqrt{var(\underline{\widehat{\Phi}})}\ , \quad B = \underline{\widehat{\Phi}} - Z_{\frac{\gamma}{2}} \sqrt{var(\underline{\widehat{\Phi}})}$

6. APPLICATION OF *LQ* DISTRIBUTION

The estimators and the corresponding summary statistics are obtained by the proposed model using MathCAD program. For a given samples with different choices of a, b and λ the maximum likelihood estimators (MLEs), the mean squared error (MSE), relative absolute bias (RAB) and the confidence interval are obtained, Table 1, summarizes the results.

Table 4. Estimates the Unknown Parameters with Corresponding Summary Statistics

		Initial values		*MLEs*	*MSE*	*RAB*	*variance*	*Lower limit*	*Upper Limit*
n	11	α	0.0492	0.0497	0.4E-6	0.0006	0.0001	0.0496	0.0499
a	1.25	β	4.375	3.9268	0.2009	-0.4482	0.0386	3.8511	4.0025
b	7.5	λ	0.15	0.1518	0.3E-5	0.0018	0.0238	0.1052	0.1984
n	20	α	0.0492	0.0495	0.1E-6	0.0003	0.2E-6	0.0494	0.0497
a	1.25	β	4.375	4.1563	0.0479	-0.2188	0.0027	4.1509	4.1616
b	7.5	λ	0.15	0.1509	0.9E-6	0.0009	0.0075	0.1362	0.1657
n	25	α	0.0492	0.0495	0.1E-6	0.0003	0.2E-6	0.0493	0.0495
a	1.25	β	4.375	4.1563	0.0479	-0.2188	0.0033	4.1497	4.1628
b	7.5	λ	0.15	0.1510	0.1E-5	0.0010	0.0207	0.1105	0.1915
n	30	α	0.0529	0.0528	0.4E-7	-0.6E-4	-0.1E-5	0.0520	0.0529
a	1.3	β	4.35	4.3772	0.0007	0.0272	0.0007	4.3759	4.3785
b	7.4	λ	0.1	0.0999	0.2E-7	-0.9E-4	0.0019	0.0962	0.1036

From Table 4, Estimate the true parameters α, β, λ well with relatively small MSEs and RAB. Also it is noticed that, the coverage probabilities of the asymptotic confidence interval are close to the nominal level. These results indicate that the proposed model and the asymptotic approximation work well under the situation. Table 5 summarizes the results of some measures of central tendency and dispersion of the ***LQ*** distribution for a given samples with different choices of a, b and λ.

Table 5. Some Measures of Central Tendency and Dispersion of the LQ Distribution

n	11	20	25	30
Mean	3.7243	3.9545	3.9545	4.2466
Median	5.1490	5.1490	5.1490	5.1490
Mode	8.2256	8.4718	8.4706	9.2221
Harmonic mean	2.1492	2.4811	2.4815	2.9068
Variance	33.3094	33.5757	33.5556	31.0103
Kurtosis	0.0343	0.0339	0.0339	0.0370
Skewness	0.0112	0.0110	0.0110	0.0077
Pearson1	-0.7799	-0.7796	-0.7796	-0.8935
Pearson2	-2.3398	-2.3388	-2.3389	-2.6805

Also, Table 6 summarizes the results of the 3^{th} non central moment about zero at different values of the sample size distribution.

Table 6. The 3^{th} non central moment about zero at different values of n

n	r	$\acute{\mu}_r$		$\mu(A_r)$	$\mu(B_r)$
11	1	$\acute{\mu}_1$	3.724	3.724	3.724
	2	$\acute{\mu}_2$	19.642	19.642	19.642
	3	$\acute{\mu}_3$	118.286	118.13	118.13
20	1	$\acute{\mu}_1$	3.955	3.955	3.955
	2	$\acute{\mu}_2$	21.433	21.433	21.433
	3	$\acute{\mu}_3$	132.73	132.574	132.574
25	1	$\acute{\mu}_1$	3.9545	3.9545	3.9545
	2	$\acute{\mu}_2$	21.4305	21.4305	21.4305
	3	$\acute{\mu}_3$	132.706	132.5493	132.5493
30	1	$\acute{\mu}_1$	4.2465	4.2465	4.2465
	2	$\acute{\mu}_2$	23.6018	23.6018	23.6018
	3	$\acute{\mu}_3$	148.8577	148.7605	148.7605

7. CONCLUSION

In this article, a new model called the LQ distribution was obtained, which extends the UQ distribution. It is observed that the proposed LQ distribution has several desirable properties, such as: expectation, harmonic mean, variance, moments, reliability function, hazard rate function, cumulative hazard function, the moment generating function, the characteristic function, the MLE of the unknown parameter with its variance. Some important related integrations were developed which can be useful for other researches. Therefore, the new LQ distribution to generated data is an extended model to analyze data from complex situations, then it will be important in applied probability, and can be used quite effectively to provide better fits of modeling of symmetric bimodal processes than the UQ distribution.

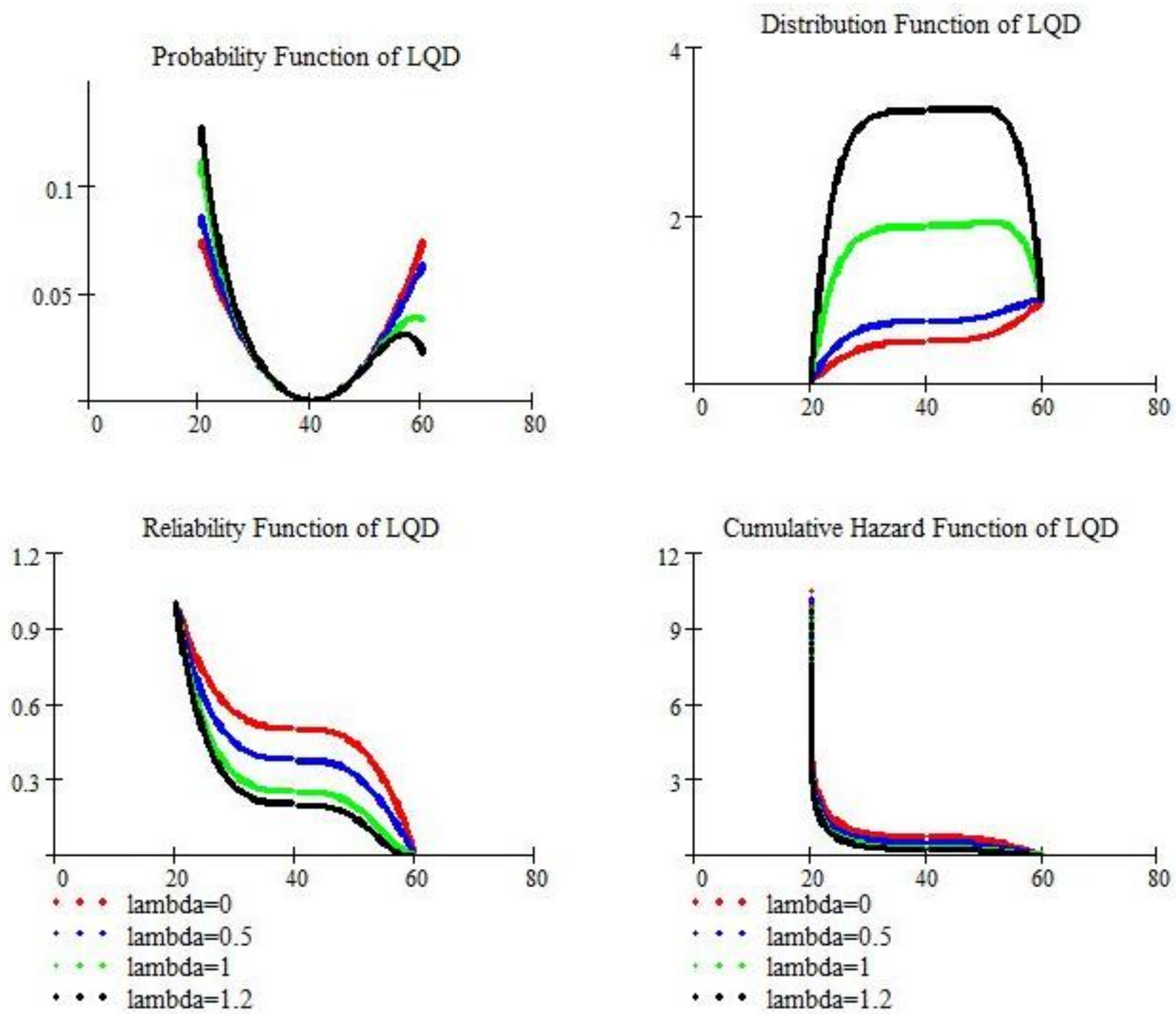

Fig. 1. The *pdf, cdf,* reliability and cumulative hazard function of *LQ* distribution

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