Minimal Dominating Functions of Corona Product Graph of a Path with a Cycle

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ABSTRACT
Graph Theory has been realized as one of the most useful branches of Mathematics or recent origin, finding widest applications in all most all branches of Sciences, Engineering and Computer Science. An introduction and an extensive overview on domination in graph and related topics are given by Haynes et al. [1, 2].

Recently, dominating functions in domination theory have received much attention. They give rise to important classes of graphs and deep structural problems. In this paper it is a discussion on some results on minimal dominating functions of corona product graph of a Path with a Cycle.

Keywords
Corona Product, Path, Cycle, Dominating Function

1. INTRODUCTION
The Domination theory is an important branch of Graph Theory and the concept of domination number of a graph is first introduced by Berge [9] in his book on graph theory. The concept of Total dominating sets are introduced by C.J.Cockayane, and Hedetniemi,S.T[4] and the concept of dominating functions introduced by Hedetniemi et al.[5].

Fruch and Harary [7] introduced a new product on two graphs \( G_1 \) and \( G_2 \), called corona product denoted by \( G_1 \circ G_2 \). The object is to construct a new and simple operation on two graphs \( G_1 \) and \( G_2 \) called their corona, with the property that the group of the new graph is in general isomorphic with the wreath product of the groups of \( G_1 \) and \( G_2 \).

Here some basic properties of corona product graph of a Path with a Cycle and some results on minimal dominating functions are presented.

2. CORONA PRODUCT OF \( P_n \circ C_m \)
The corona product of a Path \( P_n \) with a Cycle \( C_m \), is a graph obtained by taking one copy of a n-vertex graph \( P_n \) and \( m \) copies of \( C_m \) and then joining the \( i^{th} \) vertex of \( P_n \) to all vertices of \( i^{th} \) copy of \( C_m \) and it is denoted by \( P_n \circ C_m \). Now we present some of the properties of corona product graph \( P_n \circ C_m \) without proofs.

Theorem 2.1: The graph \( G = P_n \circ C_m \) is a connected graph.

Theorem 2.2: The degree of a vertex \( v \) in \( G = P_n \circ C_m \) is given by

\[
d(v) = \begin{cases} 
    m + 2, & \text{if } v \in P_n \text{ and } 2 \leq i \leq (n - 1), \\
    m + 1, & \text{if } v \in P_n \text{ and } i = 1 \text{ or } n, \\
    3, & \text{if } v \in C_m .
\end{cases}
\]

Theorem 2.3: The number of vertices and edges in \( G = P_n \circ C_m \) is given respectively by

1. \( |V(G)| = n(m + 1) \),
2. \( |E(G)| = (2mn + n - 1) \).

Theorem 2.4: The graph \( G = P_n \circ C_m \) is non hamiltonian.

Theorem 2.5: The graph \( G = P_n \circ C_m \) is not eulerian.

Theorem 2.6: The graph \( G = P_n \circ C_m \) is not bipartite.

3. DOMINATING SETS AND DOMINATING FUNCTIONS
In this section we study dominating sets, dominating functions of the graph \( G = P_n \circ C_m \) and we present some results related to minimal dominating functions of this graph.

Theorem 3.1: The domination number of \( G = P_n \circ C_m \) is \( n \).

Proof: Let \( D \) denote a dominating set of \( G \).

Case 1: Suppose \( D \) contains the vertices of \( P_n \) in \( G \).

By the definition of \( G \), the \( i^{th} \) vertex in \( P_n \) is adjacent to all vertices of \( i^{th} \) copy of \( P_m \). That is the vertices in \( P_n \) dominate the vertices in all copies of \( C_m \) respectively. Therefore the vertices of \( D \) dominate all vertices of \( G \). Thus \( D \) becomes a DS of \( G \). This set is also minimal, because, if we delete one vertex say \( v \) from \( D \), then the vertices in the \( i^{th} \) copy of \( C_m \) are not dominated by any vertex in \( D \). Hence \( \gamma(G) = n \).

Case 2: Suppose \( D \) contains any one vertex of \( C_m \) in each copy of \( G \).

That is \( |D| = n \). Obviously every vertex in \( C_m \) dominates every other vertex in \( C_m \) and also a single vertex of \( P_n \) to which it is associated.

Therefore the vertices in \( D \) dominate all vertices of \( G \). Further this set is also minimal. Therefore \( \gamma(G) = n \).

Theorem 3.2: Let \( D \) be a MDS of \( G = P_n \circ C_m \). Then a function \( f: V \rightarrow [0, 1] \) defined by

\[
f(v) = \begin{cases} 
1, & \text{if } v \in D, \\
0, & \text{otherwise}.
\end{cases}
\]

becomes a MDF of \( G = P_n \circ C_m \).

Proof: we have seen in Theorem 3.1, that the DS of \( G \) contains all the vertices of \( P_n \) and this set is also minimum. Also the set of vertices whose degree is \( m \) in each copy of \( C_m \)
form a minimal DS of G. Let D be a MDS of G. For definiteness let D contain the vertices of P_n in G.

In P_n, there are two end vertices of degree m+1 and there are n-2 intermediate vertices of degree m + 2 respectively in G.

In C_m, there are m vertices of degree 3 respectively in G.

The summation value taken over N[v] of v ∈ V is as follows:
Case 1: Let v ∈ P_n be such that d(v) = m + 2 in G.
Then N[v] contains m vertices of C_m and three vertices of P_n in G.

So \( \sum_{u \in N[v]} f(u) = 1 + 1 + 1 + 0 + \ldots + 0 = 3 \).

Case 2: Let v ∈ P_n be such that d(v) = m+1 in G.
Then N[v] contains m vertices of C_m and two vertices of P_n in G.

So \( \sum_{u \in N[v]} f(u) = 1 + 1 + 0 + \ldots + 0 = 2 \).

Case 3: Let v ∈ C_m be such that d(v) = 3 in G.
Then N[v] contains 3 vertices of C_m and one vertex of P_n in G.

So \( \sum_{u \in N[v]} f(u) = 1 + 0 + 0 + 0 = 1 \).

Therefore for all possibilities, we get \( \sum_{u \in N[v]} f(u) \geq 1 \),
\( \forall \ v \in V \).

This implies that f is a DF.

Now we check for the minimality of f.

Define \( g : V \to [0, 1] \) by
\( g(v) = \begin{cases} r, & \text{if } v = v_k \in D \text{ with } d(v_k) = m+1, \\ 1, & \text{if } v \in D - \{v_k\}, \\ 0, & \text{otherwise}. \end{cases} \)

where 0 < r < 1.

Since strict inequality holds at the vertex \( v_k \in D \), it follows that \( g < f \).

Now the following cases arise.

Case (i): Let v ∈ P_n be such that d(v) = m + 2 in G.

Sub case 1: Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + 1 + 1 + 0 + \ldots + 0 = r + 2 > 1 \).

Sub case 2: Let \( v_k \not\in N[v] \).

Case (ii): Let v ∈ P_n be such that d(v) = m+1 in G.

Sub case 1: Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + 1 + 0 + \ldots + 0 = r + 1 > 1 \).

Sub case 2: Let \( v_k \not\in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = 1 + 1 + 0 + \ldots + 0 = 2 \).

Case (iii): Let v ∈ C_m be such that d(v) = 3 in G.

Sub case 1: Let \( v_k \in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = r + 0 + 0 + 0 = r < 1 \).

Sub case 2: Let \( v_k \not\in N[v] \).

Then \( \sum_{u \in N[v]} g(u) = 1 + 0 + 0 + 0 = 1 \).

This implies that \( \sum_{u \in N[v]} g(u) < 1 \), for some \( v \in V \).

So g is not a DF.
Since g is taken arbitrarily, it follows that there exists no \( g < f \) such that g is a DF.

Thus f is a MDF.

**Theorem 3.3:** A function \( f : V \to [0, 1] \) defined by \( f(v) = \frac{1}{q} \), \( \forall v \in V \) is a DF of G = \( P_n \cup C_m \) if \( q \leq 4 \). It is a MDF if \( q = 4 \).

**Proof:** Let f be a function defined as in the hypothesis.

We know that in \( P_n \), there are two end vertices of degree m+1 and there are \( n - 2 \) intermediate vertices of degree m + 2 respectively in G. In C_m, there are m vertices of degree 3 respectively in G.

Case 1: Suppose \( 0 < q < 4 \).

The summation value taken over N[v] of v ∈ V is as follows:

Case 1: Let v ∈ P_n be such that d(v) = m + 2 in G.

Then N[v] contains m vertices of C_m and three vertices of P_n in G.

So \( \sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m + 3}{q} \).

Since \( q < 4 \), it follows that \( \frac{m + 3}{q} > 1 \).

Case 2: Let v ∈ P_n be such that d(v) = m+1 in G.
Then $N[v]$ contains $m$ vertices of $C_m$ and two vertices of $P_n$ in $G$.

So $\sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m + 2}{q}$.

Since $q < 4$, it follows that $\frac{m + 2}{q} > 1$.

Case 3: Let $v \in C_m$ be such that $d(v) = 3$ in $G$.

Then $N[v]$ contains $3$ vertices of $C_m$ and one vertex of $P_n$ in $G$.

So $\sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \frac{1}{q} = 4$.

Since $q < 4$, it follows that $\frac{4}{q} > 1$.

Therefore for all possibilities, we get $\sum_{u \in N[v]} f(u) > 1$, $\forall v \in V$.

This implies that $f$ is a DF.

Now we check for the minimality of $f$.

Define $g : V \rightarrow [0, 1]$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_k \in D \text{ with } d(v_k) = m + 1, \\ \frac{1}{q}, & \text{otherwise.} \end{cases}$$

where $0 < r < \frac{1}{q}$.

Since strict inequality holds at a vertex $v_k$ of $V$, it follows that $g < f$.

Case (i): Let $v \in P_n$ be such that $d(v) = m + 2$ in $G$.

Sub case 1: Let $v_k \in N[v]$.

Then $\sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m + 3}{q}$.

Thus $g$ is a DF.

This implies that $f$ is not a MDF.

Case II: Suppose $q = 4$.

The summation value taken over $N[v]$ of $v \in V$ is as follows:

Case 1: Let $v \in P_n$ be such that $d(v) = m + 2$ in $G$.

Then $N[v]$ contains $m$ vertices of $C_m$ and two vertices of $P_n$ in $G$.
Case 2: Let \( v \in P_n \) be such that \( d(v) = m+1 \) in \( G \).
Then \( N[v] \) contains \( m \) vertices of \( C_m \) and two vertices of \( P_n \) in \( G \).
So
\[
\sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{4} > 1, \text{ since } q = 4.
\]

Case 3: Let \( v \in C_m \) be such that \( d(v) = 3 \) in \( G \).
Then \( N[v] \) contains 3 vertices of \( C_m \) and one vertex of \( P_n \) in \( G \).
So
\[
\sum_{u \in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \frac{1}{q} = \frac{4}{4} = 1, \text{ since } q = 4.
\]
Therefore for all possibilities, we get \( \sum_{u \in N[v]} f(u) \geq 1 \), \( \forall v \in V \).
This implies that \( f \) is a DF.

Now we check for the minimality of \( f \).
Define \( g: V \rightarrow [0, 1] \) by
\[
g(v) = \begin{cases} r, & \text{if } v = v_k \in D \text{ with } d(v_k) = m + 1, \\ \frac{1}{q}, & \text{otherwise.} \end{cases}
\]
where \( 0 < r < \frac{1}{q} \).
Since strict inequality holds at a vertex \( v_k \) of \( V \), it follows that \( g < f \).

Case (i): Let \( v \in P_n \) be such that \( d(v) = m + 2 \) in \( G \).
Sub case 1: Let \( v_k \in N[v] \).
Then
\[
\sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{4} > 1.
\]

Sub case 2: Let \( v_k \not\in N[v] \).
Then
\[
\sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{4} > 1.
\]

Case (ii): Let \( v \in P_n \) be such that \( d(v) = m+1 \) in \( G \).
Sub case 1: Let \( v_k \in N[v] \).
Then
\[
\sum_{u \in N[v]} g(u) = r + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{4} > 1.
\]

Sub case 2: Let \( v_k \not\in N[v] \).
Then
\[
\sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{4} > 1.
\]

Case (iii): Let \( v \in C_m \) be such that \( d(v) = 3 \) in \( G \).
Sub case 1: Let \( v_k \in N[v] \).
Then
\[
\sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+2}{4} = 1.
\]

Sub case 2: Let \( v_k \not\in N[v] \).
Then
\[
\sum_{u \in N[v]} g(u) = \frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q} = \frac{m+3}{4} = 1.
\]
This implies that \( \sum_{u \in N[v]} g(u) < 1 \), for some \( v \in V \).
So \( g \) is not a DF.
Since \( g \) is defined arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a DF.
Thus \( f \) is a MDF.
**Theorem 3.4:** A function $f : V \rightarrow [0, 1]$ defined by $f(v) = \frac{p}{q}$, $\forall v \in V$ where $p = \min (m, n)$ and $q = \max (m, n)$ is a DF of $G = \mathbb{P}_n \cap C_m$ if $\frac{p}{q} \geq \frac{1}{4}$. Otherwise it is not a DF. Also it becomes a MDF $q^2 = \frac{1}{4}$.

**Proof:** Let $f : V \rightarrow [0, 1]$ be defined by $f(v) = \frac{p}{q}$, $\forall v \in V$, where $p = \min (m, n)$ and $q = \max (m, n)$.

Clearly $\frac{p}{q} > 0$.

The summation value taken over $N[v]$ of $v \in V$ is as follows:

**Case 1:** Let $v \in P_n$ be such that $d(v) = m + 2$ in $G$.

Then $N[v]$ contains $m$ vertices of $C_m$ and three vertices of $P_n$ in $G$.

$$\sum_{u \in N[v]} f(u) = \frac{P}{q} + \frac{P}{q} + \ldots + \frac{P}{q} = (m + 3) \frac{P}{q}.$$ **Case 2:** Let $v \in P_n$ be such that $d(v) = m + 1$ in $G$.

Then $N[v]$ contains $m$ vertices of $C_m$ and two vertices of $P_n$ in $G$.

$$\sum_{u \in N[v]} f(u) = \frac{P}{q} + \frac{P}{q} + \ldots + \frac{P}{q} = (m + 2) \frac{P}{q}.$$ **Case 3:** Let $v \in C_m$ be such that $d(v) = 3$ in $G$.

Then $N[v]$ contains $3$ vertices of $C_m$ and one vertex of $P_n$ in $G$.

$$\sum_{u \in N[v]} f(u) = \frac{P}{q} + \frac{P}{q} + \frac{P}{q} + \frac{P}{q} = 4 \left( \frac{P}{q} \right).$$

From the above three cases, we observe that $f$ is a DF if $\frac{p}{q} \geq \frac{1}{4}$.

Otherwise $f$ is not a DF.

**Case 4:** Suppose $\frac{p}{q} \geq \frac{1}{4}$.

Clearly $f$ is a DF.

Now we check for the minimality of $f$.

Define $g : V \rightarrow [0, 1]$ by

$$g(v) = \begin{cases} r, & \text{if } v = v_k \in D \text{ with } d(v_k) = m + 1, \\ \frac{P}{q}, & \text{otherwise.} \end{cases}$$

where $0 < r < \frac{P}{q}$.

Since strict inequality holds at a vertex $v_k$ of $V$, it follows that $g < f$.

**Case (i):** Let $v \in P_n$ be such that $d(v) = m + 2$ in $G$.

Then

$$\sum_{u \in N[v]} g(u) = r + \frac{P}{q} + \frac{P}{q} + \ldots + \frac{P}{q} = (m + 3) \frac{P}{q} > 1,$$

since $\frac{p}{q} > \frac{1}{4}$.

**Sub case 1:** Let $v_k \in \mathbb{N}^n[v]$.

Then

$$\sum_{u \in \mathbb{N}^n[v]} g(u) = r + \frac{P}{q} + \frac{P}{q} + \ldots + \frac{P}{q} = (m + 3) \frac{P}{q} > 1.$$
Hence, it follows that \[ \sum_{u \in N[v]} g(u) > 1, \quad \forall \ v \in V. \]

Thus \( g \) is a DF.

This implies that \( f \) is not a MDF.

**Case 5:** Suppose \( \frac{p}{q} = \frac{1}{4} \).

As in Case 1 and 2, we have that

\[
\sum_{u \in N[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \ldots + \frac{p}{q} = (m + 3) \frac{p}{q} = (m + 3) \frac{1}{4} > 1,
\]

if \( v \in P_n \).

and

\[
\sum_{u \in N[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \frac{p}{q} = (m + 2) \frac{p}{q} = (m + 2) \frac{1}{4} > 1,
\]

if \( v \in P_n \).

Again as in Case 3, we have if \( v \in C_m \) then

\[
\sum_{u \in N[v]} f(u) = \frac{p}{q} + \frac{p}{q} + \frac{p}{q} + \frac{p}{q} = 4 \left( \frac{p}{q} \right) = 1.
\]

Therefore for all possibilities, we get \( \sum_{u \in N[v]} f(u) \geq 1, \quad \forall \ v \in V. \)

This implies that \( f \) is a DF.

Now we check for the minimality of \( f \).

Define \( g : V \to [0, 1] \) by

\[
g(v) = \begin{cases} r, & \text{if } v = v_k \in D \text{ with } d(v_k) = m + 1, \\ \frac{p}{q}, & \text{otherwise.} \end{cases}
\]

where \( 0 < r < \frac{p}{q} \).

Since strict inequality holds at a vertex \( v_k \) of \( V \), it follows that \( g < f \).

Then we can show as in Case (i) of Case 4 that

\[
\sum_{u \in N[v]} g(u) = \frac{r + \frac{p}{q} + \frac{p}{q} + \ldots + \frac{p}{q}}{(m+3)\text{-times}} > 1,
\]

if \( v \in P_n \) and \( v_k \in N[v] \).

And

\[
\sum_{u \in N[v]} g(u) = \frac{\frac{p}{q} + \frac{p}{q} + \ldots + \frac{p}{q}}{(m+2)\text{-times}} > 1,
\]

if \( v \in P_n \) and \( v_k \in N[v] \).

Again as in Case (ii) of Case 4, we can show that

\[
\sum_{u \in N[v]} g(u) = r + \frac{p}{q} + \frac{p}{q} + \ldots + \frac{p}{q} > 1,
\]

if \( v \in P_n \) and \( v_k \in N[v] \).

And

\[
\sum_{u \in N[v]} g(u) = \frac{p}{q} + \frac{p}{q} + \ldots + \frac{p}{q} > 1,
\]

if \( v \in P_n \) and \( v_k \in N[v] \).

This implies that \( \sum_{u \in N[v]} g(u) < 1 \), for some \( v \in V. \)

So \( g \) is not a DF.

Since \( g \) is defined arbitrarily, it follows that there exists no \( g < f \) such that \( g \) is a DF.

Thus \( f \) is a MDF. \( \blacksquare \)
4. CONCLUSION

It is interesting to study the minimal dominating functions of the corona product graph of a path with a cycle. This work gives the scope for the study of convexity of these minimal dominating functions and the authors have also studied this concept.

5. REFERENCES