On Upper and Lower Faintly $\psi\alpha g$-Continuous Multifunctions

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ABSTRACT
The aim of this paper is to introduce and study upper and lower faintly $\psi\alpha g$-continuous multifunctions as a generalization of upper and lower $\psi\alpha g$-continuous multifunctions, respectively. The basic properties and characterizations of such functions are established.

Keywords
$\psi\alpha g$-open sets, $\psi\alpha g$-closed sets, faintly $\psi\alpha g$-continuous multifunctions, $\psi\alpha g$-θ-closed.

1. INTRODUCTION
In 1986, Neubrunn introduced and investigated the notion of upper (lower) α-continuous multifunctions. V. Kokilavani and P.R. Kavitha[5] introduced the concept of $\psi\alpha g$-closed sets in topological spaces. In this paper, we introduce and study upper and lower faintly $\psi\alpha g$-continuous multifunctions in topological spaces. The main purpose of this paper is to define faintly $\psi\alpha g$-continuous multifunctions and to obtain several characterizations and basic properties of such multifunctions. A subset A of X is called regular open(resp. regular closed)if and only if $A = int(cl(A))$ (resp. $A = cl(int(A))$). The family of all regular open subsets of (X, τ)form a base for a smaller topology τ_s on X.

2. PRELIMIERIES
Throughout the present paper, spaces (X, τ) and (Y, σ)(or simply X and Y) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset A of X, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively. A point $x \in X$ is called a θ-cluster point of A if $cl(V) \cap A = \phi$ for every open subset V of X containing x. The set of all θ-cluster point of A is called the θ-closure of A and is denoted by $cl_\theta(A)$. If $A = cl_\theta(A)$, then A is said to be θ-closed[4]. The complement of a θ-closed set is said to be θ-open. Clearly, A is θ-open if and only if for each $x \in A$, there exists an open set U such that $x \in U \subset cl(U) \subset A$. A subset A of (X, τ) is said to be $\psi\alpha g$-closed[5] if $\psi cl(A) \subset U$ whenever $A \subset U$ and U is αg-open. The complement of $\psi\alpha g$-closed is called $\psi\alpha g$-open. The family of all $\psi\alpha g$-open subsets of (X, τ) will be denoted by $\psi\alpha gO(X)$. By a multifunction $F: X \to Y$, we mean a point to set to-set correspondence from X into Y, also we always assume that $F(X) \neq \phi$ for all $x \in X$. For a multifunction $F: X \to Y$, the upper and loer inverse of any subset A of Y are denoted by $F^+(A)$ and $F^-(A)$ respectively, where $F^+(A) = \{x \in X: F(x) \subset A\}$ and $F^-(A) = \{x \in X: F(x) \cap A \neq \phi\}$. In particular, $F^-(y) = x \in X: y \in F(x)$ for each point $y \in Y$. A multifunction $F: X \to Y$ is said to be surjective if $F(X) = Y$. A multifunction $F: (X, \tau) \to (Y, \sigma)$ is said to be lower faintly $\psi\alpha g$-continuous(resp. upper faintly $\psi\alpha g$-continuous) multifunction if $F^-(V) \in \psi\alpha gO(X)$(resp. $F^+(V) \in \psi\alpha gO(X)$) for every $V \in \sigma$.

3. FAINTLY $\psi\alpha g$-CONTINUOUS MULTIFUNCTIONS

Definition:3.1 A multifunction is said to be
(i) upper faintly $\psi\alpha g$-continuous at $x \in X$ if for each θ-open subset V of Y containing $F(x)$, there exists $U \in \psi\alpha gO(X)$ containing x such that $F(U) \subset V$;
(ii)lower faintly $\psi\alpha g$-continuous at $x \in X$ if for each θ-open subset V of Y such that $F(x) \cap V \neq \phi$ for every $u \in U$;
(iii) upper(resp. lower) faintly $\psi\alpha g$-continuous if it is upper(resp. lower) faintly $\psi\alpha g$-continuous at each point of X.

Remark:3.2 Since every θ-open set is open, it is clear that every upper(lower) $\psi\alpha g$-continuous multifunction is upper(lower)faintly $\psi\alpha g$-continuous.

Theorem:3.3 For a multifunction $F: X \to Y$, the following are equivalent:
(i)F is upper faintly $\psi\alpha g$-continuous;
(ii)For each $x \in X$ and for each θ-open set V such that $x \in F^+(V)$, there exists a $\psi\alpha g$-open set Ucontaining x such that $U \subset F^+(V)$;
(iii)For each $x \in X$ and for each θ-closed set V such that $x \in F^+(Y - V)$, there exists a $\psi\alpha g$-closed set H such that $x \in X - H$ and $F^-(V) \subset H$;
(iv) $F^+(V)$ is $\psi\alpha g$-open for any θ-open subset V of Y;
(v) $F^-(V)$ is $\psi\alpha g$-closed for any θ-closed subset V of Y;
(vi) $F^-(Y - V)$ is $\psi\alpha g$-closed for any θ-open subset V of Y;
(vii) $F^+(Y - V)$ is $\psi\alpha g$-open for any θ-closed subset V of Y.
Proof: (i)⇔(ii):clear.
(ii)⇔(iii):Let $x \in X$ and V be a θ-closed subset of Y such that $x \in F^+(Y - V)$. By (ii), there exists a $\psi\alpha g$-open set Ucontaining x such that $U \subset F^+(Y - V)$. Thus $F^-(V) \subset X - U$. Take $H = X - U$. Then $x \in X - H$ and H is $\psi\alpha g$-closed. The converse is similar.
(i)⇔(iv): Let $x \in F^+(V)$ and V be a θ-open subset of Y. By (i), there exists a $\psi\alpha g$-open set U_x containing x such that $U_x \subset F^+(V)$. Thus, $F^+(V) = \cup_{x \in F^+(V)} U_x$. Since any union of $\psi\alpha g$-open set is $\psi\alpha g$-open. $F^+(V)$ is $\psi\alpha g$-open. The converse is clear.
(iv)⇔(vi):Follows from the fact that $F^-(V) = X - F^+(Y - V)$.
(iv)⇔(vii) and (v)⇔(vi): clear.

Theorem:3.4 For a multifunction $F: X \to Y$, the following are equivalent:
(i)F is lower faintly $\psi\alpha g$-continuous;
(ii)For each $x \in X$ and for each θ-open set V such that $x \in F^-(V)$, there exists a $\psi\alpha g$-open set Ucontaining x such that $U \subset F^-(V)$;
(iii)For each $x \in X$ and for each θ-closed set V such that $x \in F^-(Y - V)$, there exists a $\psi\alpha g$-closed set H such that $x \in X - H$ and $F^+(V) \subset H$;
(iv) $F^-(V)$ is $\psi\alpha g$-open for any θ-open subset V of Y;
(v) $F^+(V)$ is $\psi\alpha g$-closed for any θ-closed subset V of Y;
(vi) $F^+(Y - V)$ is $\psi\alpha g$-closed for any θ-open subset V of Y;
(vii) $F^-(Y - V)$ is $\psi\alpha g$-open for any θ-closed subset V of Y.
Proof: Proof is similar to that of Theorem 3.3.

Theorem:3.5 Suppose that (X, τ) and (X_i, τ_i) are topological spaces where $i \in I$. Let $F: X \to \prod_{i \in I} X_i$ be a multifunction from X to the product space $\prod_{i \in I} X_i$ and let $P_i: \prod_{i \in I} X_i \to X_i$ be a projection multifunction for each $i \in I$ which is defined by $P_i((x_i)) = \{x_i\}$. If F is upper(lower) faintly $\psi\alpha g$-continuous for each $i \in I$.
Proof: Let V_i be a θ-open set in (X_i, τ_i). Then $(P_i \circ$ *F+Vi=F+Pi+Vi=F+Vi×j≠iXj* (resp. $(P_i \circ F)^-(V_i) = F^-(P_i^-(V_i)) = F^-(V_i \times \prod_{j \neq i} X_j)$. Since F is upper(lower) faintly $\psi\alpha g$-continuous and since $V_i \times \prod_{j \neq i} X_j$ is a θ-open set, it follows from theorems 3.3 and 3.4 that $F^+(V_i \times \prod_{j \neq i} X_j)$(resp. $F^-(V_i \times \prod_{j \neq i} X_j)$) is a $\psi\alpha g$-open set in (X, τ). Hence again by theorems 3.3 and 3.4, $P_i \circ F$ is upper(lower) faintly $\psi\alpha g$-continuous for each $i \in I$.

Corollary:3.6 Let $F: X \to Y$ be a multifunction. If the graph multifunction G_F of F is upper(lower) faintly $\psi\alpha g$-continuous, then F is upper(lower) faintly $\psi\alpha g$-continuous, where $G_F: X \to X \times Y$, $G_F(x) = \{x\} \times F(x)$.
Proof: Let $x \in X$ and V be any θ-open subset of Y such that $x \in F^+(V)$. We obtain that $x \in G_F^+(X \times V)$ and that $X \times V$ is a θ-open set. Since the graph multifunction G_F is upper faintly $\psi\alpha g$-continuous, it follows that there exists an $\psi\alpha g$-open set U of X containing x such that $U \subset G_F^+(X \times V)$. Since $U \subset G_F^+(X \times V) = X \cap F^+(V)$. We obtain that $U \subset F^+(V)$. Thus F is upper faintly $\psi\alpha g$-continuous.

Suppose that G_F is lower faintly $\psi\alpha g$-continuous. Let $x \in X$ and V be any θ-open subset of Y such that $x \in F^-(V)$. Then $X \times V$ is θ-open in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \phi$. Since G_F is lower faintly $\psi\alpha g$-continuous, there exists an $\psi\alpha g$-open U containing x such that $U \subset G_F^-(X \times V)$; hence $U \subset F^-(V)$. This shows that F is lower faintly $\psi\alpha g$-continuous.

Theorem:3.7 Suppose that (X_i, τ_i) and (Y_i, σ_i) are topological spaces for each $i \in I$. Let $F_i: X_i \to Y_i$ be a multifunction for each $i \in I$ and let $F: \prod_{i \in I} X_i \to \prod_{i \in I} Y_i$ be the multifunction defined by $F((x_i)) = \prod_{i \in I} F_i(x_i)$. If F is upper faintly $\psi\alpha g$-continuous, then F_i is upper(lower) faintly $\psi\alpha g$-continuous for each $i \in I$.
Proof: Let V_i be a θ-open subset of Y_i. Then $V_i \times \prod_{j \neq i} X_j$ is a θ-open set. Since F is upper(lower) faintly $\psi\alpha g$-continuous, it follows from Theorem 3.3 and 3.4 that $F^+(V_i \times \prod_{j \neq i} Y_j) = F_i^+(V_i) \times \prod_{j \neq i} X_j$(resp. $F^-(V_i \times \prod_{j \neq i} Y_j) = F_i^-(V_i) \times \prod_{j \neq i} X_j$). Consequently, If $A \times B \in \psi\alpha gO(X \times Y)$, then $A \in \psi\alpha gO(X)$ and $B \in \psi\alpha gO(Y)$ that $F_i^+(V_i)$(resp. $F^-(V_i)$) is a $\psi\alpha g$-open set. Thus again by Theorem 3.3 and 3.4, F_i is upper(lower) faintly $\psi\alpha g$-continuous for each $i \in I$.

Definition:3.8 A topological space (X, τ) is said to be $\psi\alpha g$-T_2 (resp. θ-T_2) if for each pair of distinct points x and y of X, there exist disjoint $\psi\alpha g$-open (resp. θ-open) subsets U and V of Xcontaining x and y, respectively.

Theorem:3.9 Let $F: X \to Y$ be an upper faintly $\psi\alpha g$-continuous multifunction and punctually closed from a topological space X into a θ-normal space Y such that $F(x) \cap F(y) = \phi$ for each pair of distinct points x and y of X. Then X is $\psi\alpha g$-T_2.
Proof: Let x and y be any two distinct points of X. Then $F(x) \cap F(y) = \phi$. Since Y is θ-normal and F is punctually closed, there exists disjoint θ-open sets U and V containing $F(x)$ and $F(y)$, respectively. But F is upper faintly $\psi\alpha g$-continuous, so it follows from Theorem 3.3 that $F^+(U)$ and $F^+(V)$ are $\psi\alpha g$-open subsets of X containing x and y, respectively. Hence X is $\psi\alpha g$-T_2.

Definition:3.10 A topological space (X, τ) is said to be θ-compact[3] (resp. $\psi\alpha g$-compact) if every θ-open (resp. $\psi\alpha g$-open) cover of X has a finite subcover. A subset A of a topological space X is said to be θ-compact relative to X if every cover of A by θ-open subsets of X has a finite subcover of A.

Theorem:3.11 Let $F: X \to Y$ be an upper faintly $\psi\alpha g$-continuous surjective multifunction such that $F(x)$ is θ-compact relative to Y for each $x \in X$. If X is $\psi\alpha g$-compact, then Y is θ-compact.
Proof: Let $V_\alpha: \alpha \in \Lambda$ be a θ-open cover of Y. Since $F(x)$ is θ-compact relative to Y for each $x \in X$, there exists a finite subset $\Lambda(x)$ of A such that $F(x) \subset \bigcup_{\alpha \in \Lambda(x)} V_\alpha$. Put $V(x) = \bigcup_{\alpha \in \Lambda(x)} V_\alpha$. Then $V(x)$ is a θ-open subset of Y containing $F(x)$. Since F is upper faintly $\psi\alpha g$-continuous, it follows from Theorem 3.3 that $F^+(V(x))$ is a $\psi\alpha g$-open subset of X containing $\{x\}$. Thus the family $F^+(V(x)): x \in X$ is a $\psi\alpha g$-open cover of X. But X is $\psi\alpha g$-compact. So there exist $x_1, x_2, \ldots x_n \in X$ such that $X = \bigcup_{i=1}^{n} F^+(V(x_i))$. Hence $Y = F\left(\bigcup_{i=1}^{n} F^+(V(x_i))\right) = \bigcup_{i=1}^{n} F\left(F^+(V(x_i))\right) \subset \bigcup_{i=1}^{n} V(x_i) = \bigcup_{i=1}^{n} \bigcup_{\alpha \in \Lambda(x_i)} V_\alpha$. Hence Y is θ-compact.

For a given multifunction $F: X \to Y$, the graph multifunction $G_F: X \to X \times Y$ is defined as $G_F(x): \{x\} \times F(x)$ for every $x \in X$. In [2], it was shown that for a multifunction $F: X \to Y, G_F^+(A \times B) = A \cap F^+(B)$ and $G_F^-(A \times B) = A \cap F^-(B)$ where $A \subseteq X$ and $B \subseteq Y$. A multifunction $F: X \to Y$ is said to be an point closed if and only if for each $x \in X$, $F(x)$ is closed in Y.

Definition:3.12 Let $F: X \to Y$ be a multifunction. The multigraph $G(F) = \{(x, y): y \in F(x), x \in X\}$ of F is said to be $\psi\alpha g$-θ-closed if for each $(x, y) \in (X \times Y) - G(F)$, there exist a $\psi\alpha g$-open set U and a θ-open set V containing x and y respectively, such that $(U \times V) \cap G(F) = \phi$. That is, $F(U) \cap V = \phi$.

Theorem:3.13 If the graph multifunction $F: X \to Y$ is upper(lower) faintly $\psi\alpha g$-continuous, then F is upper(lower) faintly $\psi\alpha g$-continuous.

Proof: We shall only prove the case where F is upper faintly $\psi\alpha g$-continuous. Let $x \in X$ and V be a θ-open set in Y such that $x \in F^{+}(V)$. Then $G_F(x) \cap (X \times Y) = (\{x\} \times F(x)) \cap (X \times Y) = \{x\} \times (F(x) \cap V \neq \phi)$ and $X \times V$ is θ-open in $X \times Y$ by Theorem 5 in [1]. Since the graph multifunction G_F upper faintly $\psi\alpha g$-continuous, there exists an open set U containing x such that $z \in U$ implies that $G_F(z) \cap (X \times V) \neq \phi$. Therefore, we obtain $U \subseteq G_F^{+}(X \times V) = F^{+}(V) \in \psi\alpha gO(X)$ from the above equalities. Consequently, F is upper faintly $\psi\alpha g$-continuous.

Theorem:3.14 Let $F: X \to Y$ be a point closed multifunction. If F is upper faintly $\psi\alpha g$-continuous and assume that Y is regular, then $G(F)$ is θ-closed with respect to X.

Proof: Suppose $(X \times Y) \notin G(F)$. Then we have $Y \notin F(x)$. Since Y is regular, there exist disjoint open sets V_1, V_2 of Y such that $y \in V_1$ and $F(x) \in V_2$. By regularity of Y, V_2 is also θ-open in Y. Since F is upper faintly $\psi\alpha g$-continuous at x, there exists an $\psi\alpha g$-open set U in X containing x such that $F(U) \subseteq V_2$. Therefore, we obtain $x \in U, y \in V_1$ and $(x, y) \in U \times V1 \subseteq X \times Y - G(F)$. So $G(F)$ is θ-closed with respect to X.

Theorem:3.15 Let $F: (X, \tau) \to (Y, \sigma)$ be a point closed set and upper faintly $\psi\alpha g$-continuous multifunction. If F satisfies $x_1 \neq x_2 \Rightarrow F(x_1) \neq F(x_2)$ and Y is regular space, then X will be Hausdroff.

Proof: Let x_1, x_2 be two distinct points belong to X, then $F(x_1) \neq F(x_2)$. Since F is point closed and Y is regular, for all $y \in F(x_1)$ with $Y \notin F(x_2)$, there exists θ-open sets V_1, V_2 containing y and $F(x_2)$ respectively such that $V_1 \cap V_2 = \phi$. Since F is upper faintly $\psi\alpha g$-continuous and $F(x_2) \subseteq V_2$, there exists an open set U containing x, such that $F(U) \subseteq V_2$. Thus $x_1 \notin U$. Therefore U and $X - U$ are disjoint open sets separating x_1 and x_2.

Theorem:3.16 If a multifunction $F: X \to Y$ is upper faintly $\psi\alpha g$-continuous such that $F(x)$ is θ-compact relative to Y for each $x \in X$ and Y is θ-T_2, then the multigraph $G(F)$ of F is $\psi\alpha g$- θ-closed.

Proof: Let $(x, y) \in (X \times Y) - G(F)$. Then $y \in Y - F(x)$. Since Y is θ-T_2 for each $y \in F(x)$, there exist disjoint θ-open subsets $U(z)$ and $V(z)$ of Y containing z and y respectively. Thus $\{U(z): z \in F(x)\}$ is a θ-open cover of $F(x)$. But $F(x)$ is θ-compact relative to Y. So there exist$z_1, z_2, \ldots z_n \in F(x)$ such that $F(x) \subset \bigcup_{i=1}^{n} U(z_i)$. Put $U = \bigcup_{i=1}^{n} U(z_i)$ and $\bigcap_{i=1}^{n} V(z_i)$. Then U and V are open subsets of Y such that $F(x) \subset U, y \subset V$ and $U \cap V = \phi$. Since F is upper faintly $\psi\alpha g$-continuous, it follows from theorem3.4, that $F^{+}(U)$ is a $\psi\alpha g$-open subset of X. Also $x \in F^{+}(U)$. Since $F(x) \subset U$ and $F(F^{+}(U)) \cap V = \phi$. Since $U \cap V = \phi$. Hence, $G(F)$ is $\psi\alpha g$- θ-closed.

4. CONCLUSION

In this paper we introduced and studied upper and lower faintly $\psi\alpha g$-continuous multifunctions as a generalization of upper and lower $\psi\alpha g$-continuous multifunctions, respectively. The basic properties and characterizations of such functions are discussed.

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