# A Compendium of Unlikely Parameterizations for Prime Pairs Related to the Goldbach Conjecture

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## ABSTRACT
The long-standing Goldbach Conjecture states that every even integer greater than 4 is equal to the sum of two odd prime numbers. An interesting exercise would be to check if some simply stated rules on prime number candidates would generate solutions satisfying the Conjecture. A collection of rules which are insufficient are presented with examples. A final conjecture is proposed, yet unresolved, dealing with maximal prime products. The closing conjecture offers a parameterization which is a candidate solution for the Goldbach Conjecture.

## General Terms
Goldbach Conjecture, Maximal Prime Product Conjecture

## Keywords
Goldbach Conjecture; maximal prime products; composite numbers; Bertrand Conjecture; composite numbers; Golden Ratio

## 1. INTRODUCTION
One strategy to employ in resolving or studying the Goldbach Conjecture [4] is to offer rules, parameterizations or constraints for odd prime candidates whose sum equals the even integer 2n under investigation. Clearly many potential rules are likely to fail, since much work over the last few centuries has been inadequate to resolve the conjecture. What follows are some rules which appear to have a chance of being successful, but ultimately are inadequate, along with a closing conjecture which has not been resolved, and some related observations.

## 2. SOME PRIME RULES
In the following, the number 2n under investigation denotes the even integer greater than 4.

Example 1: **Let p = 3, then 2n – 3 is prime**. This option fails because if 2n = 12, then 12 – 3 = 9, and 9 is not prime.

Example 2: **Let p = the largest prime ≤ 2n – 3, then**

**2n – p is prime.**

This option fails because if 2n = 98, then 2n – 3 = 95. The largest prime less than 95 is equal to 89 and 2n – 89 = 98 - 89 = 9. And 9 is not prime.

Example 3: **Let p = the largest prime ≤ n, then 2n – p is prime**.

This option fails because if 2n =16 and p = 7, then 2n – p = 16 – 7 = 9. Again, 9 is not prime.

Example 4: **Let p = smallest prime ≥ n, then 2n – p is prime**.

This option fails because if 2n = 20, and p = 11, then 2n – p = 9.

There are clearly other simply stated "rules" which are inadequate to resolve the conjecture. One notes that the four examples above focus either on a prime "close to 2n" or close to the "middle dimension" n.

## 3. A GOLDEN RATIO NON-RULE
It would be interesting to determine if a rule regarding the Golden ratio applies in the context of Goldbach Conjecture solutions. It does not. For it 2n = p + q with p > q in the Golden Ratio. Then (2n)/p = p/q. This gives 2nq = $p^2$. And then 2 would divide the odd prime p, which is clearly a contradiction. Furthermore, constant ratio rules for Goldbach Conjecture solutions fail. In other words, there is no standard rectangle with sides of odd prime lengths p and q, respectively, with the quotient p/q being constant and for which p + q = 2n for all even integers 2n > 4. For if there were such a constant ratio then (3p)/(3q) would also satisfy this ratio, but 3p and 3q would not be odd primes.

## 4. FUNCTION MAXIMIZATION
Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = x(2n - x)$. It is well-known from introductory calculus [3] or the algebra of quadratic functions [5 ], that the maximum value

for $f$ occurs when $x = n$. Stated alternatively, the maximum area of a rectangle with fixed perimeter arises when the rectangle is a square. In the function $f$ above, the perimeter = $4n$.

Note that if n itself were prime, say $n = p$ for some odd prime $p$, then the maximum value of $f$ occurs at $x = p$. Furthermore, this value $p$ trivially coincides with a Goldbach Conjecture solution, stated simply as 2(odd prime $p$) = $p + p$. A natural example follows, leveraging the above. But this "rule," unfortunately, also fails.

Example 5: **Consider the set $A_n$ of all primes $p$ such that $3 \leq p \leq 2n - 3$. Then letting $\tilde{p}\,(2n - \tilde{p})$ = Max { $p(2n - p) \mid p \in A_n$ } is such that $(2n - \tilde{p})$ is a prime number.**

This option fails because for $2n = 16$, one has that 7(9) = 63 which is the maximum such product, but 9 is a composite (not prime) number.

The table below details the calculations for $2n = 16$.

**Table 1 Maximum Products**

| p | 2n - p | p(2n – p) |
|---|---|---|
| 3 | 13 | 39 |
| 5 | 11 | 55 |
| 7 | 9 | 63 |
| 11 | 5 | 55 |
| 13 | 3 | 39 |

## 5. A NEW MAXIMUM CONDITION

Notice from section 4, the product **Max** $\{\boldsymbol{p(2n-p \mid p \in An}\}$ does not necessarily generate a prime pair, ( $\tilde{p}$ , $(n - \tilde{p})$). If one modifies the construction slightly, one has for all $p \in A_n$, the factorization of the difference $\boldsymbol{2n-p}$ as $\boldsymbol{q \cdot o}$, where $\boldsymbol{q}$ is the largest prime factor of $(2n-p)$ and $\boldsymbol{o}$ is an odd integer ≥ 1. Consider the details of this construct in **Table 2** for 2n = 16:

So again, let $A_n = \{primes\ p | 3 \leq p \leq 2n-3\}$.

Also, let $\tilde{p} \cdot \tilde{q}$ = **Max{pq | q = maximal odd prime factor in 2n – p}.**

Notice that in **Table 2**, the number 55 = 5 (11) gives a maximal product with both 5 and 11 being primes. This construction appears to easily generate prime pairs satisfying the Goldbach Conjecture. It has been checked by computer ([1] Steve Beaty) for 2n ≤ 150,000 that this "maximal prime product construction" generates prime pairs satisfying the Goldbach Conjecture. One may then state the following ***CONJECTURE*** and ultimately try to prove it or detect the smallest positive even integer 2n for which the conjecture fails. Neither of these exercises has been completed to date.

**Table 2 Maximum Prime Products**

| p | 2n - p | q | o | p·q |
|---|---|---|---|---|
| 3 | 13 | 13 | 1 | 39 |
| 5 | 11 | 11 | 1 | 55 |
| 7 | 9 | 3 | 3 | 21 |
| 11 | 5 | 5 | 1 | 55 |
| 13 | 3 | 3 | 1 | 39 |

## 6. MAXIMUM PRIME PRODUCT CONSTRUCTION CONJECTURE:

For all integers $n \geq 3$, let $A_n = \{primes\ p \mid 3 \leq p \leq 2n-3\}$. For primes p in $A_n$, let $2n - p = q \cdot o$ where $q$ is the largest prime factor of $2n - p$ and $o$ is an odd integer ≥ 1. Then $\tilde{p} \cdot \tilde{q}$ defined as the maximum of $\{p \cdot q | \ p \in A_n\}$ is such that that $\tilde{p}, \tilde{q}$ is a pair of prime summands satisfying the Goldbach Conjecture.

It would be interesting to detect the first n for which this conjecture fails, or to offer an unequivocally correct proof.

## 7. CONCLUSION

This paper illustrates a few unlikely candidate parameterizations for solutions to the Goldbach conjecture by odd prime product pairs. It closes with the observation that one may want to consider the **Maximum Prime Product Conjecture** as a case study for a new candidate pair of primes.

## 8. REFERENCES

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