Convex Hull of γ_{vct}-sets in Graphs

R.Vasanthi
Assistant Professor, Department of Mathematics
Alagappa Chettiar Government College of Engineering and Technology
Karaikudi,
Tamilnadu, India

K.Subramanian
Professor(Retired), Department of Mathematics
Alagappa Government Arts College, Karaikudi,
Tamilnadu, India

ABSTRACT

Let G = (V, E) be an undirected, simple and connnected graph. A set C $\subseteq$ V of vertices in G is called a *convex set* if I(C) = C where I(C) is the set of all vertices in the u-v geodesic path of G for all u, v $\in$ C. For any set C $\subseteq$ V, the *convex hull of C* denoted by [C] is defined as the smallest convex subset of V(G) containing C. Let S be a minimum vertex covering transversal dominating set viz. a γ_{vct}-set. Then the *convex hull of S* is defined as the smallest convex set containing S. We define the *convex hull number of G with respect to γ_{vct}-sets*, denoted by $CH_{\gamma_{vct}}$(G) as $CH_{\gamma_{vct}}$(G) = min.$\{|C|$: C = [S] is the convex hull of γ_{vct}-set S$\}$ where the minimum is taken over all the γ_{vct}-sets of G. If [S] = S, then S is called a *convex γ_{vct}-set.* If [S] = V(G), then S is called a *hull γ_{vct}-set*. In this paper, the convex hull of γ_{vct}-sets and the convex hull number with respect to γ_{vct}-sets in various graphs are analysed.

Keywords

minimum vertex covering transversal dominating set, convex hull number of G with respect to γ_{vct}-sets, convex γ_{vct}-set, hull γ_{vct}-set

1. INTRODUCTION

Independent transversal domination in graphs was introduced by Hamid [5]. Vasanthi and Subramanian [6] introduced vertex covering transversal domination in graphs. The vertex covering transversal domination number of some standard graphs and regular graphs are analysed in [6] and [7]. Further studies on vertex covering transversal domination number and vertex covering transversal dominating sets are carried out in [8]. In this paper, the convex hull of γ_{vct}-sets and hence the convex hull number with respect to γ_{vct}-sets in various graphs are analysed, based on the concept of convex sets in graphs. .

Let G = (V, E) be any graph and let S $\subset$ V be any subset of vertices of G. Then the *induced subgraph* $\langle S \rangle$ is the graph whose vertex set is S and whose edge set consists of all the edges in E that have both endpoints in S. The *degree of a vertex v $\in$ V of G* is the number of edges incident to the vertex v in G and is denoted by deg_G(v). The graph G^k called the *power of G* is obtained by taking the same vertex set as G and two vertices u ; v in G^k are adjacent if there exists a u-v path of length at most k in G for every positive integer k $\geq$ 2.

A set I $\subseteq$ V of vertices in G is called an independent set if no two vertices in I are adjacent. Also I is said to be a *maximum independent set* if there is no other independent set I' such that $|I'| > |I|$. The cardinality of a maximum independent set is called the *independence number* and is denoted by β_0(G). A set C $\subseteq$ V of vertices in G is called a *vertex covering set* (or simply *covering set*) if every edge of G is incident with at least one vertex in C. Also C is said to be a *minimum vertex covering set* if there is no other vertex covering set C' such that $|C'| < |C|$. The cardinality of a minimum vertex covering set is called the *vertex covering number* and is denoted by α_0(G).

A set D $\subseteq$ V of vertices in a simple connected graph G is called a *dominating set* if every vertex in V$-$D is adjacent to a vertex in D. A dominating set which intersects every minimum vertex covering set in G is called a *vertex covering transversal dominating set*. The minimum cardinality of a vertex covering transversal dominating set is called *vertex covering transversal domination number* of G and is denoted by γ_{vct}(G).

A dominating set of minimum cardinality is denoted by γ-set and a vertex covering transversal dominating set of minimum cardinality is denoted by γ_{vct}-set. Given a connected graph G and u; v are two vertices of G, the distance between u and v is the length of a shortest path between u and v, we denote it by d_G(u; v). A shortest path between u and v is called *a u-v geodesic*. A set C $\subseteq$ V of vertices in G is called a *convex set* if I(C) = C where I(C) is the set of all vertices in the u-v geodesic path of G for all u, v $\in$ C. For any set C $\subseteq$ V, the *convex hull of C* denoted by [C] is defined as the smallest convex subset of V(G) containing C. For other graph theoretic terminologies, refer to [2], [3] and [4].

2. DEFINITIONS WITH ILLUSTRATIONS

Using the concepts of convex sets and the convex hull of a set in graphs, the convex hull of a γ_{vct}-set and the convex hull number with respect to γ_{vct}-sets in a graph are defined in this section. Convex γ_{vct}-sets and hull γ_{vct}-sets are also defined accordingly. These concepts are explained with suitable illustrations.

DEFINITION 2.1. *Let G = (V, E) be an undirected, simple and connected graph. Let S $\subseteq$ V be a minimum vertex covering transversal dominating set viz. a γ_{vct}-set. Then the convex hull of S is defined as the smallest convex set containing S and is denoted by [S].*

DEFINITION 2.2. *The convex hull number of G with respect to γ_{vct}-sets, denoted by $CH_{\gamma_{vct}}$(G) is defined as $CH_{\gamma_{vct}}$(G) = min.$\{|C|$: C = [S] is the convex hull of γ_{vct}-set S$\}$ where the minimum is taken over all the γ_{vct}-sets of G.*

DEFINITION 2.3. *If [S] = S, then S is called a convex γ_{vct}-set.*

DEFINITION 2.4. *If [S] = V(G), then S is called a hull γ_{vct}-set.*

REMARK 2.5. *Any singleton set is a convex set in a graph G. So it follows that if G has a singleton set S as its γ_{vct}-set, then [S] = S which implies that S is a convex γ_{vct}-set and so $CH_{\gamma_{vct}}$(G) = 1.*

REMARK 2.6. *If S is a γ_{vct}-set of G, it is obvious that [S] $\supseteq$ S and so $CH_{\gamma_{vct}}$(G) $\geq \gamma_{vct}$(G).*

ILLUSTRATION 2.7. *For instance, consider the graph G shown in Figure 1.*

As illustrated in [6], S_1 = $\{$b, e$\}$, S_2 = $\{$b, f$\}$, S_3 = $\{$b, g$\}$ and S_4 = $\{$a, e$\}$ are the γ_{vct}-sets of G. The geodesic paths connecting the vertices of S_1 are (b, c, e) and (b, d, e). So the convex hull of S_1 is [S_1] = $\{$b, c, d, e$\}$.

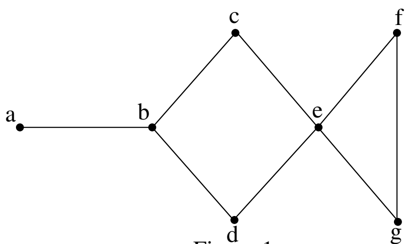


Figure 1

Similarly, the convex hull of S_2, S_3 and S_4 are [S_2] = $\{$b, c, d, e, f$\}$, [S_3] = $\{$b, c, d, e, g$\}$ and [S_4] = $\{$a, b, c, d, e$\}$. So the convex hull number with respect to the γ_{vct}-sets of G, $CH_{\gamma_{vct}}$(G) = min.$\{$4, 5$\}$ = 4.

ILLUSTRATION 2.8. *Consider the graph G shown in Figure 2.*

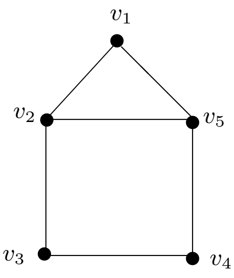


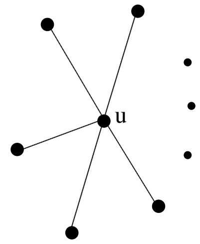
Figure 2

Here $C_1 = \{v_2, v_3, v_5\}$, $C_2 = \{v_2, v_4, v_5\}$, $C_3 = \{v_1, v_3, v_5\}$ and $C_4 = \{v_1, v_4, v_5\}$ are the α_0-sets of G.
Also $S_1 = \{v_2, v_3\}$, $S_2 = \{v_2, v_5\}$, $S_3 = \{v_3, v_4\}$ and $S_4 = \{v_4, v_5\}$ are the γ-sets which intersect all the α_0-sets of G. Therefore S_1, S_2, S_3 and S_4 are the γ_{vct}-sets of G. Then their convex hulls are [S_1] = $\{v_2, v_3\}$ = S_1, [S_2] = $\{v_2, v_5\}$ = S_2, [S_3] = $\{v_3, v_4\}$ = S_3 and [S_4] = $\{v_4, v_5\}$ = S_4.
Therefore $CH_{\gamma_{vct}}$(G) = 2.
Also each γ_{vct}-set S_i is a convex γ_{vct}-set as [S_i] = S_i for i = 1, 2, 3, 4.

3. CONVEX HULL OF γ_{VCT}-SETS IN SOME STANDARD GRAPHS

In this section, the convex hull of a γ_{vct}-sets and the convex hull number of G with respect to γ_{vct}-sets are analyzed for some standard graphs. Convex γ_{vct}-sets and hull γ_{vct}-sets are also examined in those graphs.

EXAMPLE 1. *If G is a star as shown in Figure 3, then CHγ_{vct}(G) = 1.*

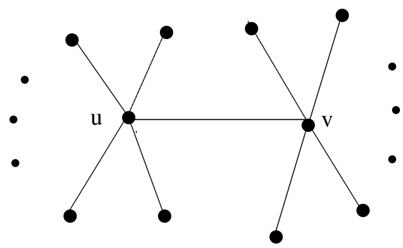


Star

Figure 3

Obviously S = $\{$u$\}$ is the unique γ_{vct}-set of G. Then [S] = S and so S is the convex γ_{vct}-set of G. Thus $CH_{\gamma_{vct}}$(G) = 1.

EXAMPLE 2. *If G is a bistar as shown in Figure 4, then $CH_{\gamma_{vct}}$(G) = 2.*

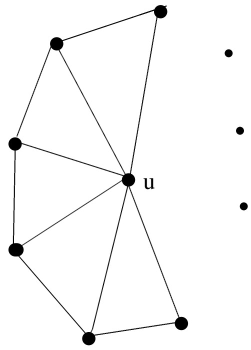


Bistar

Figure 4

It is obvious that S = $\{$u, v$\}$ is the unique γ_{vct}-set of G. Also [S] = S and so S is the convex γ_{vct}-set of G. Thus $CH_{\gamma_{vct}}$(G) = 2.

EXAMPLE 3. *If W_n is a wheel on n $\geq$ 5 vertices as shown in Figure 5, then $CH_{\gamma_{vct}}(W_n)$ = 1 since $\{u\}$ is the unique γ_{vct}-set of W_n and is convex also.*



Wheel

Figure 5

THEOREM 3.1. *Let G be a simple connected graph on n vertices.*
(i) If at least one γ_{vct}-set of G is convex, then $CH_{\gamma_{vct}}(G) = \gamma_{vct}(G)$.
(ii)If all the γ_{vct}-sets of G are hull, then $CH_{\gamma_{vct}}(G)$ = n.
(iii) If no γ_{vct}-set of G is convex or hull, then $\gamma_{vct}(G) < CH_{\gamma_{vct}}(G)$ < n.

PROOF. (i) Let S be a convex γ_{vct}-set of G. Then the convex hull of S is itself. So $CH_{\gamma_{vct}}$(G) = min.$\{|C|$: C is the convex hull of a γ_{vct}-set$\}$ = $|S|$ = γ_{vct}(G).
(ii) Assume that all the γ_{vct}-sets of G are hull. Then for any γ_{vct}-set S of G, the convex hull of S, [S] = V(G) as S is a hull γ_{vct}-set. Hence $CH_{\gamma_{vct}}$(G) = $|V(G)|$ = n.
(iii) It is obvious that $CH_{\gamma_{vct}}$(G) $\geq \gamma_{vct}$(G). Since no γ_{vct}-set of G is convex, it follows that $CH_{\gamma_{vct}}$(G) $\neq \gamma_{vct}$(G). Since no γ_{vct}-set of G is hull, it follows that $CH_{\gamma_{vct}}$(G) $\neq$ n. Thus γ_{vct}(G) < $CH_{\gamma_{vct}}$(G) < n.
□

THEOREM 3.2. *Let G be a simple connected graph and let S be a γ_{vct}-set of G. If $\langle S \rangle = K_2$, then S is a convex γ_{vct}-set of G and $CH_{\gamma_{vct}}(G)$ = 2.*

PROOF. Since $\langle S \rangle = K_2$, S contains only 2 vertices, say u and v. Also the only geodesic path connecting u and v is (u, v) which is of length 1. For, if P is any other path connecting u and v, it must include at least one more vertex so that the length of P is ≥ 2 and so is not geodesic.
Now the convex hull of S is itself and so S is a convex γ_{vct}-set of G. Therefore by theorem 3.1, $CH_{\gamma_{vct}}$(G) = γ_{vct}(G). But γ_{vct}(G) = $|S|$ = 2.
Hence $CH_{\gamma_{vct}}$(G) = 2.
□

THEOREM 3.3. *$CH_{\gamma_{vct}}(K_{m,n})$= 2 where $K_{m,n}$, $2 \leq m \leq n$ is a complete bipartite graph.*

PROOF. Let $K_{m,n}$, 2 $\leq$ m $\leq$ n be a complete bipartite graph with bipartition
(U, V) where U = $\{u_1, u_2, ..., u_m\}$ and V = $\{v_1, v_2, ..., v_n\}$. Then $S_{ij} = \{u_i, v_j\}$ for all i = 1, 2, 3, ..., m ; j = 1, 2, 3, ..., n is a γ_{vct}-set of $K_{m,n}$. It is clear that $\langle S_{ij} \rangle = K_2$. Therefore by theorem 3.2, each S_{ij} is a convex γ_{vct}-set for i = 1, 2, 3, ..., m ; j = 1, 2, 3, ..., n and $CH_{\gamma_{vct}}(K_{m,n})$= 2.
□

LEMMA 3.4. *$CH_{\gamma_{vct}}(P_n) = \begin{cases} 2 & if\ n = 2, 4 \\ 4 & if\ n = 5 \end{cases}$ $where P_n$ is a path on n vertices.*

PROOF. S = $\{v_1, v_2\}$ is the unique γ_{vct}-set of P_2. Obviously [S] = $\{v_1, v_2\}$ = S. So S is a convex γ_{vct}-set of P_2. Hence $CH_{\gamma_{vct}}(P_2)$ = 2.
If V(P_4) = $\{v_1, v_2, v_3, v_4\}$, then $S = \{v_2, v_3\}$ is the unique γ_{vct}-set of P_4. Since $\langle S \rangle = K_2$, by theorem 3.2, S is a convex γ_{vct}-set of P_4 and $CH_{\gamma_{vct}}(P_4)$ = 2.
If n = 5, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_2, v_5\}$ are the γ_{vct}-sets of P_5. Then their convex hulls are $[S_1] = \{v_1, v_2, v_3, v_4\}$ and $[S_2] = \{v_2, v_3, v_4, v_5\}$.
Hence $CH_{\gamma_{vct}}(P_5)$ = 4.
□

LEMMA 3.5. *If n $\equiv$ 0 (mod 3), then $CH_{\gamma_{vct}}(P_n)$ = n $-$ 2.*

PROOF. Let the vertex set of P_n be $\{v_1, v_2, v_3, v_4, ..., v_n\}$. In [6], it is proved that $\gamma_{vct}(P_n) = \lceil \frac{n}{3} \rceil$. It is clear that if n $\equiv$ 0 (mod 3), then S = $\{v_{3i-1}$: 1 $\leq$ i $\leq \frac{n}{3}$ $\}$ = $\{v_2, v_5, ..., v_{n-1}\}$ is the unique γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S are $v_2, v_3, v_4, v_5, ..., v_{n-1}$. Therefore [S] = V($P_n$) $-$ $\{v_1, v_n\}$. Hence $CH_{\gamma_{vct}}(P_n)$ = n $-$ 2.
□

LEMMA 3.6. *If n > 4 and n $\equiv$ 1 (mod 3), then $CH_{\gamma_{vct}}(P_n)$ = n $-$ 2.*

PROOF. In [6], it is proved that $\gamma_{vct}(P_n) = \lceil \frac{n}{3} \rceil$. So it is clear that if n $\equiv$ 1 (mod 3), then S_1 = $\{v_{3i+1}$: 0 $\leq$ i $\leq \frac{n-1}{3}$ $\}$ = $\{v_1, v_4, ..., v_n\}$ is a γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S_1 are $v_1, v_2, v_3, v_4, ..., v_n$. Therefore the convex hull of S_1 is $[S_1]$ = V(P_n). S_2 = $\{v_{3i+1}$: 1 $\leq$ i $\leq \frac{n-1}{3}$ $\}$ $\cup\{v_2\}$= $\{v_2, v_4, v_7, ..., v_n\}$ is also a γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S_2 are $v_2, v_3, v_4, ..., v_n$. Therefore $[S_2]$ = V(P_n)$-$ $\{v_1\}$. Also S_3 = $\{v_{3i-1}$: 1 $\leq$ i $\leq \frac{n-1}{3}$ $\}$ $\cup\{v_{n-1}\}$= $\{v_2, v_5, ..., v_{n-2}, v_{n-1}\}$ is a γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S_3 are $v_2, v_3, v_4, ..., v_{n-1}$. So $[S_3]$ = V(P_n) $-$ $\{v_1, v_n\}$. Also S_4 = $\{v_{3i-1}$: 1 $\leq$ i $\leq \frac{n-1}{3}$ $\}$ $\cup\{v_n\}$= $\{v_2, v_5, ..., v_{n-2}, v_n\}$ is another γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S_4 are $v_2, v_3, v_4, ..., v_n$. So $[S_4]$ = V(P_n) $-$ $\{v_1\}$. Similarly if S is any other γ_{vct}-set of P_n which necessarily excludes any one of the end vertices or both, then the convex hull of S contains n $-$ 1 or n $-$ 2 vertices.
Hence $CH_{\gamma_{vct}}(P_n)$ = min.$\{$n, n $-$ 1, n $-$ 2$\}$ = n $-$ 2.
□

LEMMA 3.7. *If n > 5 and n $\equiv$ 2 (mod 3), then $CH_{\gamma_{vct}}(P_n)$ = n $-$ 2.*

PROOF. In [6], it is proved that $\gamma_{vct}(P_n) = \lceil \frac{n}{3} \rceil$. So it is clear that if n $\equiv$ 2 (mod 3), then S_1 = $\{v_{3i+1}$: 0 $\leq$ i $\leq \frac{n-2}{3}$ $\}$ = $\{v_1, v_4, ..., v_{n-1}\}$ is a γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S_1 are $v_1, v_2, v_3, v_4, ..., v_{n-1}$. Therefore the convex hull of S_1 is $[S_1]$ = V(P_n)$-$ $\{v_n\}$. Also S_2 = $\{v_{3i-1}$: 1 $\leq$ i $\leq \frac{n+1}{3}$ $\}$ = $\{v_2, v_5, ..., v_n\}$ is another γ_{vct}-set of P_n. Then the vertices contained in the geodesic paths joining any two vertices of S_2 are $v_2, v_3, v_4, ..., v_n$. So $[S_2]$ = V(P_n) $-$ $\{v_1\}$. The other γ_{vct}-sets of P_n are S_3 = $\{v_{3i+1}$: 1 $\leq$ i $\leq \frac{n-2}{3}$ $\}$ $\cup$ $\{v_2\}$ = $\{v_2, v_4, v_7, ..., v_{n-1}\}$ and S_4 = $\{v_{3i-1}$: 1 $\leq$ i $\leq \frac{n-2}{3}$ $\}$ $\cup$ $\{v_{n-1}\}$= $\{v_2, v_5, ..., v_{n-3}, v_{n-1}\}$. Also $[S_3]$ = $[S_4]$ = V(P_n) $-$ $\{v_1, v_n\}$. Similarly if S is any other γ_{vct}-set of P_n which necessarily excludes any one of the end vertices or both, then the convex hull of S contains n $-$ 1 or n $-$ 2 vertices.
Hence $CH_{\gamma_{vct}}(P_n)$ = min.$\{$n $-$ 1, n $-$ 2$\}$ = n $-$ 2.
□

Thus we have the following theorem.

THEOREM 3.8. *If P_n is a path on n vertices, then*
$$CH_{\gamma_{vct}}(P_n) = \begin{cases} 2 & if\ n = 2, 4 \\ 4 & if\ n = 5 \\ n-2 & otherwise \end{cases}$$

REMARK 3.9. *As discussed in [1], P^k_{n+1} is the k^{th} power of P_{n+1} defined as follows.*
Let P_{n+1} be the path of order n + 1 on the vertices v_0, v_1, v_2, ..., v_n. Then P^k_{n+1} contains the same vertices v_0, v_1, v_2, ..., v_n and two vertices in P^k_{n+1} are adjacent if there exists a path of length at most k in P_{n+1} for every positive integer $k \geq 2$. When $k \geq n$, P^k_{n+1} is the complete graph.

REMARK 3.10. *The following theorem provides the convex hull number with respect to γ_{vct}-sets of P^{n-1}_{n+1}.*

THEOREM 3.11. *$CH_{\gamma_{vct}}(P^{n-1}_{n+1})$ = 1.*

PROOF. For n = 5, the graph P^4_6 is as shown in Figure 6.

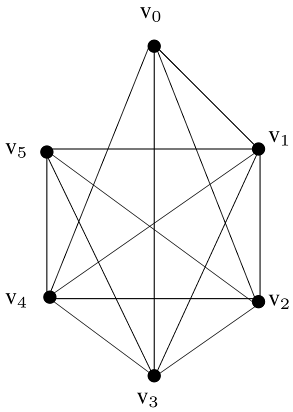


Figure 6

By the definition of P^{n-1}_{n+1}, it is clear that $\{v_0, v_n\}$ is the unique β_0-set. Therefore C = $\{v_1, v_2, ..., v_{n-1}\}$ is the unique α_0-set of P^{n-1}_{n+1}. So $S_i = \{v_i\}$ is a γ-set intersecting C for each i = 1, 2, ... n − 1. Hence each S_i is a γ_{vct}-set of P^{n-1}_{n+1} and so is a convex γ_{vct}-set of P^{n-1}_{n+1}. Therefore $CH_{\gamma_{vct}}(P^{n-1}_{n+1})$ = 1. □

4. CONCLUDING REMARKS

Domination theory is one of the most application-oriented area of research in the field of Graph theory. Several authors have introduced many new parameters in domination. The vertex covering transversal domination in graphs has been introduced in [6]. It influences one to restrict the family of γ-sets to the family of γ_{vct}-sets. It is observed that $\gamma_{vct} = \gamma$ in most of the graphs considered. But even though $\gamma_{vct} = \gamma$, there are graphs in which γ-sets do not become γ_{vct}-sets. Moreover, it is possible to concentrate on the convex hull of γ_{vct}-sets and hence the convex hull number with respect to γ_{vct}-sets in a graph. This concept paves the way to filter the γ_{vct}-sets which are responsible for producing the convex hull number with respect to γ_{vct}-sets.

5. REFERENCES

[1] ABUGHNEIM OA; AL-KHAMAISEH B; AL-EZEH H, *The geodetic, hull, and Steiner numbers of powers of paths* , Utilitas Mathematica, 2014, Vol.95, 289-294

[2] E.J. COCKANYE, S.T. HEDETNIEMI,*Towards a theory of domination in graphs*, Networks 7 (1977), 247-261.

[3] DOUGLAS B.WEST,*Introduction to Graph Theory*, Prentice-Hall of India Private Limited, New Delhi.

[4] GARY CHATRAND AND PING ZHANG,*Introduction to Graph Theory* , Eighth Reprint 2012, Tata McGraw Hill Education Private Limited, New Delhi.

[5] ISMAIL SAHUL HAMID,*Independent Transversal Domination in Graphs*, Discussiones Mathematicae Graph Theory 32(1) (2012) (5-17)

[6] R.VASANTHI , K.SUBRAMANIAN, *Vertex covering transversal domination in graphs*, International Journal of Mathematics and Soft Computing, Vol.5, No.2 (2015), 01 - 07.

[7] R.VASANTHI , K.SUBRAMANIAN, *On vertex covering transversal domination number of regular graphs*, The Scientific World Journal, Vol 2016, Article ID 1029024, 7 pages.

[8] R.VASANTHI , K.SUBRAMANIAN, *On the minimum vertex covering transversal dominating sets in graphs and their classification*, Discrete Mathematics, Algorithms and Applications Vol. 9, No. 5 (2017) 1750069 (15 pages).